Mathematical Methods

MTH303

Virtual University of Pakistan

Knowledge beyond the boundaries

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Lecture 1 Introduction

Background
Linear *y=mx+c*
Quadratic *ax2+bx+c=0*
Cubic *ax3+bx2+cx+d=0*
Systems of Linear equations

ax+by+c=0
lx+my+n=0

Solution ?
Equation

Differential Operator

$$\frac{dy}{dx}=\frac{1}{x}$$

Taking anti derivative on both sides
y=ln x

From the past

- *Algebra*
- *Trigonometry*
- *Calculus*
- *Differentiation*
- *Integration*

- *Differentiation*
  - *Algebraic Functions*
  - *Trigonometric Functions*
  - *Logarithmic Functions*
  - *Exponential Functions*
  - *Inverse Trigonometric Functions*

- More Differentiation
  - Successive Differentiation
  - Higher Order
  - Leibnitz Theorem
- Applications
  - Maxima and Minima
  - Tangent and Normal
- Partial Derivatives

y=f(x)

f(x,y)=0

z=f(x,y)

Integration

- Reverse of Differentiation
- By parts
- By substitution
- By Partial Fractions
- Reduction Formula

Frequently required

- Standard Differentiation formulae
- Standard Integration Formulae

Differential Equations

- Something New
- Mostly old stuff
  - Presented differently
  - Analyzed differently
  - Applied Differently

$$\frac{dy}{dx} - 5y = 1$$

$$(y - x)dx + 4xdy = 0$$

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial y} = u$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} = 0$$

Lecture 2 Fundamentals of Differential Equation

Fundamentals

- Definition of a differential equation.
- Classification of differential equations.
- Solution of a differential equation.
- Initial value problems associated to DE.
- Existence and uniqueness of solutions

Elements of the Theory

- Applicable to:
  - Chemistry
  - Physics
  - Engineering
  - Medicine
  - Biology
  - Anthropology
- Differential Equation – involves an unknown function with one or more of its derivatives
- Ordinary D.E. – a function where the unknown is dependent upon only one independent variable

Examples of DEs

$$\frac{dy}{dx}-5y=1$$

$$(y-x)dx+4xdy=0$$

$$\frac{d^2y}{dx^2}+5\left(\frac{dy}{dx}\right)^3-4y=e^x$$

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial v}{\partial y}=u$$

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}+2\frac{\partial u}{\partial t}=0$$

Specific Examples of ODE’s

$$\frac{du}{dt} = F(t)G(u), \qquad \text{the growth equation;}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = F(t), \qquad \text{the pendulum equation;}$$

$$\frac{d^2y}{dt^2} + \varepsilon(y^2+1)\frac{dy}{dt} + y = 0, \qquad \text{the van der Pol equation;}$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t), \qquad \text{the LCR oscillator equation;}$$

$$\frac{dp}{dt} = -2a(t)p + \frac{b(t)^2}{u(t)}p^2 - v(t), \qquad \text{a Riccati equation.}$$

- The order of an equation:
  - The order of the highest derivative appearing in the equation

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

$$a^2\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} = 0$$

Ordinary Differential Equation

If an equation contains only ordinary derivatives of one or more dependent variables, *w.r.t* a single variable, then it is said to be an Ordinary Differential Equation **(ODE)**. For example the differential equation

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

is an ordinary differential equation.

Partial Differential Equation

Similarly an equation that involves partial derivatives of one or more dependent variables *w.r.t* two or more independent variables is called a Partial Differential Equation **(PDE)**. For example the equation

$$a^2\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial x^2} = 0$$

is a partial differential equation.

Results from ODE data

- The solution of a general differential equation:
  - *f*(*t*, *y*, *y*', . . . , *y*(*n*)) = 0
  - is defined over some interval *I* having the following properties:
    - *y*(*t*) and its first *n* derivatives exist for all *t* in *I* so that *y*(*t*) and its first *n* - 1 derivates must be continuous in *I*
    - *y*(*t*) satisfies the differential equation for all t in *I*

- General Solution – all solutions to the differential equation can be represented in this form for all constants
- Particular Solution – contains no arbitrary constants
- Initial Condition
- Boundary Condition
- Initial Value Problem (IVP)
- Boundary Value Problem (BVP)

IVP Examples

- The Logistic Equation
  - *p*' = *ap* – *bp*2
  - with initial condition *p*(*t*0) = *p*0; for *p*0 = 10 the solution is:
  - *p*(*t*) = 10*a* / (10*b* + (*a* – 10*b*)*e*-*a*(*t*-*t*0))
- The mass-spring system equation
  - *x*'' + (*a* / *m*) *x*' + (*k* / *m*)*x* = *g* + (F(*t*) / *m*)

BVP Examples

- Differential equations
  - *y*'' + 9*y* = sin(*t*)
    - with initial conditions *y*(0) = 1, *y*'(2p) = -1
    - *y*(*t*) = (1/8) sin(*t*) + cos(3*t*) + sin (3*t*)
  - *y*'' + p2*y* = 0
    - with initial conditions *y*(0) = 2, *y*(1) = -2
    - *y*(*t*) = 2cos(p*t*) + (*c*)sin(p*t*)

Properties of ODE's

- Linear – if the nth-order differential equation can be written:

- *an*(*t*)*y*(*n*) + *an*-1(*t*)*y*(*n*-1) + . . . + *a*1*y*' + *a*0(*t*)*y* = *h*(*t*)

- Nonlinear – not linear

  *x**3**(y''')**3**-x2y(y'')**2**+3xy'+5y=e**x***

Superposition

- Superposition – allows us to decompose a problem into smaller, simpler parts and then combine them to find a solution to the original problem.

Explicit Solution

A solution of a differential equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^2y}{dx^2}\right) = 0$$

that can be written as *y = f(x)* is known as an explicit solution .

Example: The solution ***y = xex*** is an explicit solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

Implicit Solution

A relation ***G(x,y)*** is known as an implicit solution of a differential equation, if it defines one or more explicit solution on *I*.

Example: The solution *x2 + y2 - 4=0* is an implicit solution of the equation *y' = - x/y* as it defines two explicit solutions *y=+(4-x2)1/2*

Lecture 3 Separable Equations

The differential equation of the form

$$\frac{dy}{dx} = f(x, y)$$

is called **separable** if it can be written in the form

$$\frac{dy}{dx} = h(x)g(y)$$

To solve a separable equation, we perform the following steps:

1. We solve the equation $g(y) = 0$ to find the constant solutions of the equation.

2. For non-constant solutions we write the equation in the form.

$$\frac{dy}{g(y)} = h(x)dx$$

Then integrate $$\int \frac{1}{g(y)} dy = \int h(x)dx$$

to obtain a solution of the form

$$G(y) = H(x) + C$$

3. We list the entire constant and the non-constant solutions to avoid repetition..

4. If you are given an IVP, use the initial condition to find the particular solution.

Note that:

(a) No need to use two constants of integration because $C_1 - C_2 = C$.

(b) The constants of integration may be relabeled in a convenient way.

(c) Since a particular solution may coincide with a constant solution, **step 3 is important.**

Example 1:

Find the particular solution of

$$\frac{dy}{dx} = \frac{y^2 - 1}{x}, \quad y(1) = 2$$

Solution:

1. By solving the equation

$$y^2 - 1 = 0$$

We obtain the constant solutions

$$y = \pm 1$$

2. Rewrite the equation as

$$\frac{dy}{y^2 - 1} = \frac{dx}{x}$$

Resolving into partial fractions and integrating, we obtain

$$\frac{1}{2}\int\left[\frac{1}{y-1} - \frac{1}{y+1}\right]dy = \int \frac{1}{x}dx$$

Integration of rational functions, we get

$$\frac{1}{2}\ln\frac{|y-1|}{|y+1|} = \ln|x| + C$$

3. The solutions to the given differential equation are

$$\begin{cases} \frac{1}{2}\ln\frac{|y-1|}{|y+1|} = \ln|x| + C \\ y = \pm 1 \end{cases}$$

4. Since the constant solutions do not satisfy the initial condition, we plug in the condition

$y = 2$ When $x = 1$ in the solution found in step 2 to find the value of C.

$$\frac{1}{2}\ln\left(\frac{1}{3}\right) = C$$

The above implicit solution can be rewritten in an explicit form as**:**

$$y = \frac{3 + x^2}{3 - x^2}$$

Example 2:

Solve the differential equation

$$\frac{dy}{dt} = 1 + \frac{1}{y^2}$$

Solution:

1. We find roots of the equation to find constant solutions

$$1 + \frac{1}{y^2} = 0$$

No constant solutions exist because the equation has no real roots.

2. For non-constant solutions, we separate the variables and integrate

$$\int \frac{dy}{1 + 1/y^2} = \int dt$$

Since $$\frac{1}{1 + 1/y^2} = \frac{y^2}{y^2 + 1} = 1 - \frac{1}{y^2 + 1}$$

Thus $$\int \frac{dy}{1+1/y^2} = y - \tan^{-1}(y)$$

So that $$y - \tan^{-1}(y) = t + C$$

It is **not easy** to find the **solution in an explicit form** i.e. y as a function of t.

3. Since $\exists$ no constant solutions, all solutions are given by the implicit equation found
in step 2.

Example 3:
Solve the initial value problem

$$\frac{dy}{dt} = 1 + t^2 + y^2 + t^2y^2, \quad y(0) = 1$$

Solution:

1. Since $$1 + t^2 + y^2 + t^2y^2 = (1+t^2)(1+y^2)$$
The equation is separable & has no constant solutions because $\exists$ no real roots of
$$1 + y^2 = 0.$$

2. For non-constant solutions we separate the variables and integrate.
$$\frac{dy}{1+y^2} = (1+t^2)dt$$
$$\int \frac{dy}{1+y^2} = \int (1+t^2)dt$$
$$\tan^{-1}(y) = t + \frac{t^3}{3} + C$$
Which can be written as
$$y = \tan\left(t + \frac{t^3}{3} + C\right)$$

3. Since $\exists$ no constant solutions, all solutions are given by the implicit or explicit equation.

4. The initial condition $y(0) = 1$ gives
$$C = \tan^{-1}(1) = \frac{\pi}{4}$$
The particular solution to the initial value problem is
$$\tan^{-1}(y) = t + \frac{t^3}{3} + \frac{\pi}{4}$$
or in the explicit form $$y = \tan\left(t + \frac{t^3}{3} + \frac{\pi}{4}\right)$$

Example 4:

Solve

$$(1+x)dy - ydx = 0$$

Solution:

Dividing with $(1+x)y$, we can write the given equation as

$$\frac{dy}{dx} = \frac{y}{(1+x)}$$

1. The only constant solution is $y = 0$

2. For non-constant solution we separate the variables

$$\frac{dy}{y} = \frac{dx}{1+x}$$

Integrating both sides, we have

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln|y| = \ln|1+x| + c_1$$

$$y = e^{\ln|1+x|+c_1} = e^{\ln|1+x|}.e^{c_1}$$

or $$y = |1+x|\, e^{c_1} = \pm e^{c_1}(1+x)$$

$$y = C(1+x), \quad C = \pm e^{c_1}$$

If we use $\ln|c|$ instead of c_1 then the solution can be written as

$$\ln|y| = \ln|1+x| + \ln|c|$$

or $$\ln|y| = \ln|c(1+x)|$$

So that $$y = c(1+x).$$

3. The solutions to the given equation are

$$y = c(1+x)$$

$$y = 0$$

Example 5

Solve

$$xy^4dx + (y^2+2)e^{-3x}dy = 0.$$

Solution:

The differential equation can be written as

$$\frac{dy}{dx} = \left(-xe^{3x}\right)\left(\frac{y^4}{y^2+2}\right)$$

1. Since $\frac{y^4}{y^2+2} \Rightarrow y = 0$. Therefore, the only constant solution is $y = 0$.

2. We separate the variables

$$xe^{3x}dx + \frac{y^2+2}{y^4}dy = 0 \quad \text{or} \quad xe^{3x}dx + \left(y^{-2}+2y^{-4}\right)dy = 0$$

Integrating, with use integration by parts by parts on the first term, yields

$$\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} - y^{-1} - \frac{2}{3}y^{-3} = c_1$$

$$e^{3x}(3x-1) = \frac{9}{y} + \frac{6}{y^3} + c \quad \text{where} \quad 9c_1 = c$$

3. All the solutions are

$$e^{3x}(3x-1) = \frac{9}{y} + \frac{6}{y^3} + c$$

$$y = 0$$

Example 6:

Solve the initial value problems

(a) $\frac{dy}{dx} = (y-1)^2, \quad y(0) = 1$ (b) $\frac{dy}{dx} = (y-1)^2, \quad y(0) = 1.01$

and compare the solutions.

Solutions**:**

1. Since $(y-1)^2 = 0 \Rightarrow y = 1$. Therefore, the only constant solution is $y = 0$.

2. We separate the variables

$$\frac{dy}{(y-1)^2} = dx \quad \text{or} \quad (y\text{-}1)^{-2}dy = dx$$

Integrating both sides we have

$$\int (y-1)^{-2}\,dy = \int dx$$

$$\frac{(y-1)^{-2+1}}{-2+1} = x + c$$

or $$-\frac{1}{y-1} = x + c$$

3. All the solutions of the equation are

$$-\frac{1}{y-1} = x + c$$
$$y = 1$$

4. We plug in the conditions to find particular solutions of both the problems

(a) $y(0)=1 \Rightarrow y=1$ when $x=0$. So we have

$$-\frac{1}{1-1} = 0 + c \Rightarrow c = -\frac{1}{0} \Rightarrow c = -\infty$$

The particular solution is

$$-\frac{1}{y-1} = -\infty \Rightarrow y - 1 = 0$$

So that the solution is $y=1$, which is same as constant solution.

(b) $y(0)=1.01 \Rightarrow y=1.01$ when $x=0$. So we have

$$-\frac{1}{1.01-1} = 0 + c \Rightarrow c = -100$$

So that solution of the problem is

$$-\frac{1}{y-1} = x - 100 \Rightarrow y = 1 + \frac{1}{100 - x}$$

5. Comparison: A radical change in the solutions of the differential equation has Occurred corresponding to a very small change in the condition**!!**

Example 7:

Solve the initial value problems

(a) $\dfrac{dy}{dx} = (y-1)^2 + 0.01, \quad y(0)=1$ (b) $\dfrac{dy}{dx} = (y-1)^2 - 0.01, \quad y(0)=1.$

Solution:

(a) First consider the problem

$$\frac{dy}{dx} = (y-1)^2 + 0.01, \qquad y(0)=1$$

We separate the variables to find the non-constant solutions

$$\frac{dy}{\left(\sqrt{0.01}\right)^2 + (y-1)^2} = dx$$

Integrate both sides

$$\int \frac{d(y-1)}{\left(\sqrt{0.01}\right)^2 + (y-1)^2} = \int dx$$

So that $$\frac{1}{\sqrt{0.01}} \tan^{-1} \frac{y-1}{\sqrt{0.01}} = x + c$$

$$\tan^{-1}\left(\frac{y-1}{\sqrt{0.01}}\right) = \sqrt{0.01}(x+c)$$

$$\frac{y-1}{\sqrt{0.01}} = \tan\left[\sqrt{0.01}(x+c)\right]$$

or $$y = 1 + \sqrt{0.01}\tan\left[\sqrt{0.01}(x+c)\right]$$

Applying $y(0) = 1 \Rightarrow y = 1$ when $x = 0$, we have

$$\tan^{-1}(0) = \sqrt{0.01}(0+c) \Rightarrow 0 = c$$

Thus the solution of the problem is

$$y = 1 + \sqrt{0.01}\tan\left(\sqrt{0.01}\, x\right)$$

(b) Now consider the problem

$$\frac{dy}{dx} = (y-1)^2 - 0.01, \qquad y(0) = 1.$$

We separate the variables to find the non-constant solutions

$$\frac{d\,y}{(y-1)^2 - \left(\sqrt{0.01}\right)^2} = dx$$

$$\int \frac{d\,(y-1)}{(y-1)^2 - \left(\sqrt{0.01}\right)^2} = \int dx$$

$$\frac{1}{2\sqrt{0.01}} \ln\left|\frac{y-1-\sqrt{0.01}}{y-1+\sqrt{0.01}}\right| = x + c$$

Applying the condition $y(0) = 1 \Rightarrow y = 1$ when $x = 0$

$$\frac{1}{2\sqrt{0.01}}\ln\left|\frac{-\sqrt{0.01}}{\sqrt{0.01}}\right| = c \Rightarrow c = 0$$

$$\ln\left|\frac{y-1-\sqrt{0.01}}{y-1+\sqrt{0.01}}\right| = 2\sqrt{0.01}\,x$$

$$\frac{y-1-\sqrt{0.01}}{y-1+\sqrt{0.01}} = \frac{e^{2\sqrt{0.01}\,x}}{1}$$

Simplification:

By using the property $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}$

$$\frac{y-1-\sqrt{0.01}+y-1+\sqrt{0.01}}{y-1-\sqrt{0.01}-y+1-\sqrt{0.01}} = \frac{e^{2\sqrt{0.01}x}+1}{e^{2\sqrt{0.01}x}-1}$$

$$\frac{2y-2}{-2\sqrt{0.01}} = \frac{e^{2\sqrt{0.01}}+1}{e^{2\sqrt{0.01}}-1}$$

$$\frac{y-1}{-\sqrt{0.01}} = \frac{e^{2\sqrt{0.01}}+1}{e^{2\sqrt{0.01}}-1}$$

$$y-1 = -\sqrt{0.01}\left(\frac{e^{2\sqrt{0.01}}+1}{e^{2\sqrt{0.01}}-1}\right)$$

$$y = 1-\sqrt{0.01}\left(\frac{e^{2\sqrt{0.01}}+1}{e^{2\sqrt{0.01}}-1}\right)$$

Comparison:

The solutions of both the problems are

(a) $y = \quad 1+\sqrt{0.01}\ \tan\left(\sqrt{0.01}\ x\right)$

(b) $y = 1-\sqrt{0.01}\left(\frac{e^{2\sqrt{0.01}}+1}{e^{2\sqrt{0.01}}-1}\right)$

Again a radical change has occurred corresponding to a very small in the differential equation**!**

Exercise:

Solve the given differential equation by separation of variables.

1. $\frac{dy}{dx} = \left(\frac{2y+3}{4x+5}\right)^2$

2. $\sec^2 x dy + \csc y dx = 0$

3. $e^y \sin 2x dx + \cos x\left(e^{2y} - y\right)dy = 0$

4. $\frac{dy}{dx} = \frac{xy+3x-y-3}{xy-2x+4y-8}$

5. $\frac{dy}{dx} = \frac{xy+2y-x-2}{xy-3y+x-3}$

6. $y\left(4-x^2\right)^{\frac{1}{2}} dy = \left(4+y^2\right)^{\frac{1}{2}} dx$

7. $\left(x+\sqrt{x}\right)\frac{dy}{dx} = y+\sqrt{y}$

Solve the given differential equation subject to the indicated initial condition.

8. $\left(e^{-y}+1\right)\sin x dx = \left(1+\cos x\right)dy$, $y(0)=0$

9. $\left(1+x^4\right)dy + x\left(1+4y^2\right)dx = 0$, $y(1)=0$

10. $ydy = 4x\left(y^2+1\right)^{\frac{1}{2}} dx$, $y(0)=1$

Lecture 4 Homogeneous Differential Equations

A differential equation of the form

$$\frac{dy}{dx} = f(x, y)$$

Is said to be *homogeneous* if the function $f(x, y)$ is homogeneous, which means

$f(tx, ty) = t^n f(x, y)$ For some real number n, for any number t.

Example 1

Determine whether the following functions are homogeneous

$$\begin{cases} f(x, y) = \dfrac{xy}{x^2 + y^2} \\ g(x, y) = \ln\left(-3x^2 y/(x^3 + 4xy^2)\right) \end{cases}$$

Solution:

The functions $f(x, y)$ is homogeneous because

$$f(tx, ty) = \frac{t^2 xy}{t^2(x^2 + y^2)} = \frac{xy}{x^2 + y^2} = f(x, y)$$

Similarly, for the function $g(x, y)$ we see that

$$g(tx, ty) = \ln\left(\frac{-3t^3 x^2 y}{t^3(x^3 + 4xy^2)}\right) = \ln\left(\frac{-3x^2 y}{x^3 + 4xy^2}\right) = g(x, y)$$

Therefore, the second function is also homogeneous.
Hence the differential equations

$$\begin{cases} \dfrac{dy}{dx} = f(x, y) \\ \dfrac{dy}{dx} = g(x, y) \end{cases}$$

Are homogeneous differential equations

Method of Solution:

To solve the homogeneous differential equation

$$\frac{dy}{dx} = f(x, y)$$

We use the substitution

$$v = \frac{y}{x}$$

If $f(x, y)$ is homogeneous of degree zero, then we have

$$f(x, y) = f(1, v) = F(v)$$

Since $y' = xv' + v$, the differential equation becomes

$$x\frac{dv}{dx} + v = f(1, v)$$

This is a separable equation. We solve and go back to old variable y through $y = xv$.

Summary:

1. Identify the equation as homogeneous by checking $f(tx, ty) = t^n f(x, y)$;

2. Write out the substitution $v = \frac{y}{x}$;

3. Through easy differentiation, find the new equation satisfied by the new function v;

$$x\frac{dv}{dx} + v = f(1, v)$$

4. Solve the new equation (which is always separable) to find v;

5. Go back to the old function y through the substitution $y = vx$;

6. If we have an IVP, we need to use the initial condition to find the constant of integration.

Caution:

- Since we have to solve a separable equation, we must be careful about the constant solutions.
- If the substitution $y = vx$ does not reduce the equation to separable form then the equation is not homogeneous or something is wrong along the way.

Illustration:

Example 2 Solve the differential equation

$$\frac{dy}{dx} = \frac{-2x + 5y}{2x + y}$$

Solution:

Step 1. It is easy to check that the function

$$f(x, y) = \frac{-2x + 5y}{2x + y}$$

is a homogeneous function.

Step 2. To solve the differential equation we substitute

$$v = \frac{y}{x}$$

Step 3. Differentiating w.r.t x, we obtain

$$xv' + v = \frac{-2x + 5xv}{2x + xv} = \frac{-2 + 5v}{2 + v}$$

which gives

$$\frac{dv}{dx} = \frac{1}{x}\left(\frac{-2 + 5v}{2 + v} - v\right)$$

This is a separable. At this stage please refer to the **Caution!**

Step 4. Solving by separation of variables all solutions are implicitly given by

$$-4\ln(|v-2|) + 3\ln|v-1| = \ln(|x|) + C$$

Step 5. Going back to the function y through the substitution $y = vx$, we get

$$-4\ln|y-2x| + 3\ln|y-x| = C$$

$$-4\ln\left|\frac{y-2x}{x}\right| + 3\ln\left|\frac{y-x}{x}\right| = \ln|x| + c$$

$$\ln\left|\frac{y-2x}{x}\right|^{-4} + \ln\left|\frac{y-x}{x}\right|^{3} = \ln x + \ln c_1,\ c = \ln c_1$$

$$\ln\left|\frac{(y-2x)^{-4}}{x^{-4}}\right| + \ln\left|\frac{(y-x)^3}{x^3}\right| = \ln c_1 x$$

$$\ln\left|\frac{(y-2x)^{-4}}{x^{-4}} . \frac{(y-x)^3}{x^3}\right| = \ln c_1 x$$

$$\frac{(y-2x)^{-4}}{x^{-4}} . \frac{(y-x)^3}{x^3} = c_1 x$$

$$x(y-2x)^{-4}(y-x)^3 = c_1 x$$

$$(y-2x)^{-4}(y-x)^3 = c_1$$

Note that the implicit equation can be rewritten as

$$(y-x)^3 = C_1(y-2x)^4$$

Equations reducible to homogenous form

The differential equation

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$$

is not homogenous. However, it can be reduced to a homogenous form as detailed below

Case 1: $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

We use the substitution $z = a_1 x + b_1 y$ which reduces the equation to a separable equation in the variables x and z. Solving the resulting separable equation and replacing z with $a_1 x + b_1 y$, we obtain the solution of the given differential equation.

Case 2: $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

In this case we substitute

$$x = X + h, \qquad y = Y + k$$

Where h and k are constants to be determined. Then the equation becomes

$$\frac{dY}{dX} = \frac{a_1 X + b_1 Y + a_1 h + b_1 k + c_1}{a_2 X + b_2 Y + a_2 h + b_2 k + c_2}$$

We choose h and k such that

$$\left.\begin{aligned} a_1 h + b_1 k + c_1 = 0 \\ a_2 h + b_2 k + c_2 = 0 \end{aligned}\right\}$$

This reduces the equation to

$$\frac{dY}{dX} = \frac{a_1 X + b_1 Y}{a_2 X + b_2 Y}$$

Which is homogenous differential equation in X and Y, and can be solved accordingly. After having solved the last equation we come back to the old variables x and y.

Example 3

Solve the differential equation

$$\frac{dy}{dx} = -\frac{2x+3y-1}{2x+3y+2}$$

Solution:

Since $\frac{a_1}{a_2} = 1 = \frac{b_1}{b_2}$, we substitute $z = 2x+3y$, so that

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dz}{dx} - 2\right)$$

Thus the equation becomes

$$\frac{1}{3}\left(\frac{dz}{dx} - 2\right) = -\frac{z-1}{z+2}$$

i.e. $$\frac{dz}{dx} = \frac{-z+7}{z+2}$$

This is a variable separable form, and can be written as

$$\left(\frac{z+2}{-z+7}\right)dz = dx$$

Integrating both sides we get

$$-z - 9\ln(z-7) = x + A$$

Simplifying and replacing z with $2x+3y$, we obtain

$$-\ln(2x+3y-7)^9 = 3x+3y+A$$

or $$(2x+3y-7)^{-9} = ce^{3(x+y)}, \qquad c = e^A$$

Example 4

Solve the differential equation

$$\frac{dy}{dx}=\frac{(x+2y-4)}{2x+y-5}$$

Solution:

By substitution

$$x=X+h, \qquad y=Y+k$$

The given differential equation reduces to

$$\frac{dY}{dX}=\frac{(X+2Y)+(h+2k-4)}{(2X+Y)+(2h+k-5)}$$

We choose h and k such that

$$h+2k-4=0, \quad 2h+k-5=0$$

Solving these equations we have $h=2,\ k=1$. Therefore, we have

$$\frac{dY}{dX}=\frac{X+2Y}{2X+Y}$$

This is a homogenous equation. We substitute $Y=VX$ to obtain

$$X\frac{dV}{dX}=\frac{1-V^2}{2+V} \quad \text{or} \quad \left[\frac{2+V}{1-V^2}\right]dV=\frac{dX}{X}$$

Resolving into partial fractions and integrating both sides we obtain

$$\int\left[\frac{3}{2(1-V)}+\frac{1}{2(1+V)}\right]dV=\int\frac{dX}{X}$$

or

$$-\frac{3}{2}\ln(1-V)+\frac{1}{2}\ln(1+V)=\ln X+\ln A$$

Simplifying and removing ($\ln$) from both sides, we get

$$(1-V)^3/(1+V)=CX^{-2},\ C=A^{-2}$$

$$-\frac{3}{2}\ln(1-V)+\frac{1}{2}\ln(1+V)=\ln X+\ln A$$
$$\ln(1-V)^{-3/2}+\ln(1+V)^{1/2}=\ln XA$$
$$\ln(1-V)^{-3/2}(1+V)^{1/2}=\ln XA$$
$$(1-V)^{-3/2}(1+V)^{1/2}=XA$$
taking power "−2" on both sides
$$(1-V)^{3}(1+V)^{-1}=X^{-2}A^{-2}$$
$$put\ V=\frac{Y}{X}$$
$$(1-\frac{Y}{X})^{3}\left(1+\frac{Y}{X}\right)^{-1}=X^{-2}A^{-2}$$
$$\left(\frac{X-Y}{X}\right)^{3}\left(\frac{X+Y}{X}\right)^{-1}=X^{-2}A^{-2}$$
$$\frac{(X-Y)^{3}}{X+Y}X^{-3+1}=X^{-2}A^{-2}$$
$$say, c=A^{-2}$$
$$\frac{(X-Y)^{3}}{X+Y}=c$$
$$put\ X=x-2,\ Y=y-1$$
$$(x+y-1)^{3}/x+y-3=c$$

Now substituting $V=\dfrac{Y}{X}, X=x-2, Y=y-1$ and simplifying, we obtain

$$(x-y-1)^{3}/(x+y-3)=C$$

This is solution of the given differential equation, an implicit one.

Exercise

Solve the following Differential Equations

1. $(x^{4}+y^{4})dx-2x^{3}ydy=0$
2. $\dfrac{dy}{dx}=\dfrac{y}{x}+\dfrac{x^{2}}{y^{2}}+1$
3. $\left(x^{2}e^{\frac{-y}{x}}+y^{2}\right)dx=xydy$

4. $y\,dx + \left(y\cos\frac{x}{y} - x \right) dy = 0$

5. $\left(x^3 + y^2\sqrt{x^2 + y^2} \right) dx - xy\sqrt{x^2 + y^2}\, dy = 0$

Solve the initial value problems

6. $\left(3x^2 + 9xy + 5y^2 \right) dx - \left(6x^2 + 4xy \right) dy = 0, \qquad y(2) = -6$

7. $\left(x + \sqrt{y^2 - xy} \right) \frac{dy}{dx} = y, \qquad y\left(\frac{1}{2} \right) = 1$

8. $\left(x + ye^{y/x} \right) dx - xe^{y/x} dy = 0, \qquad y(1) = 0$

9. $\frac{dy}{dx} - \frac{y}{x} = \cosh\frac{y}{x}, \qquad y(1) = 0$

Lecture 5 Exact Differential Equations

Let us first rewrite the given differential equation

$$\frac{dy}{dx} = f(x, y)$$

into the alternative form

$$M(x, y)dx + N(x, y)dy = 0 \quad \text{where} \quad f(x, y) = -\frac{M(x, y)}{N(x, y)}$$

This equation is an exact differential equation if the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This condition of exactness insures the existence of a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M(x, y), \quad \frac{\partial F}{\partial y} = N(x, y)$$

Method of Solution:

If the given equation is exact then the solution procedure consists of the following steps:

Step 1. Check that the equation is exact by verifying the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Step 2. Write down the system $\frac{\partial F}{\partial x} = M(x, y)$, $\frac{\partial F}{\partial y} = N(x, y)$

Step 3. Integrate either the 1st equation *w. r. to* ***x*** or 2nd *w. r. to* ***y***. If we choose the 1st equation then

$$F(x, y) = \int M(x, y)dx + \theta(y)$$

The function $\theta(y)$ is an arbitrary function of y, integration w.r.to x; y being constant.

Step 4. Use second equation in step 2 and the equation in step 3 to find $\theta'(y)$.

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x, y)dx\right) + \theta'(y) = N(x, y)$$

$$\theta'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)dx$$

Step 5. Integrate to find $\theta(y)$ and write down the function $F(x, y)$;

Step 6. All the solutions are given by the implicit equation

$$F(x, y) = C$$

Step 7. If you are given an IVP, plug in the initial condition to find the constant C.

Caution: x should disappear from $\theta'(y)$. Otherwise something is **wrong!**

Example 1

Solve $(3x^2y+2)dx+(x^3+y)dy=0$

Solution: Here $M=3x^2y+2$ and $N=x^3+y$

$$\frac{\partial M}{\partial y}=3x^2, \frac{\partial N}{\partial x}=3x^2$$

i.e. $$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Hence the equation is exact. The LHS of the equation must be an exact differential i.e. $\exists$ a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x}=3x^2y+2=M$$

$$\frac{\partial f}{\partial y}=x^3+y=N$$

Integrating 1st of these equations w. r. t. *x*, have

$$f(x,y)=x^3y+2x+h(y),$$

where $h(y)$ is the constant of integration. Differentiating the above equation w. r. t. *y* and using 2nd, we obtain

$$\frac{\partial f}{\partial y}=x^3+h'(y)=x^3+y=N$$

Comparing $h'(y)=y$ is independent of *x*.

or.

Integrating, we have

$$h(y)=\frac{y^2}{2}$$

Thus $$f(x,y)=x^3y+2x+\frac{y^2}{2}$$

Hence the general solution of the given equation is given by

$$f(x,y)=c$$

i.e. $$x^3y + 2x + \frac{y^2}{2} = c$$

Note that we could start with the 2nd equation

$$\frac{\partial f}{\partial y} = x^3 + y = N$$

to reach on the above solution of the given equation**!**

Example 2

Solve the initial value problem

$$\left(2y\sin x\cos x + y^2\sin x\right)dx + \left(\sin^2 x - 2y\cos x\right)dy = 0.$$

$$y(0) = 3.$$

Solution: Here

$$M = 2y\sin x\cos x + y^2\sin x$$

and $$N = \sin^2 x - 2y\cos x$$

$$\frac{\partial M}{\partial y} = 2\sin x\cos x + 2y\sin x,$$

$$\frac{\partial N}{\partial x} = 2\sin x\cos x + 2y\sin x,$$

This implies $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus given equation is exact.

Hence there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 2y\sin x\cos x + y^2\sin x = M$$

$$\frac{\partial f}{\partial y} = \sin^2 x - 2y\cos x = N$$

Integrating 1st of these w. r. t. x, we have

$$f(x, y) = y\sin^2 x - y^2\cos x + h(y),$$

Differentiating this equation w. r. t. y substituting in $\frac{\partial f}{\partial y} = N$

$$\sin^2 x - 2y\cos x + h'(y) = \sin^2 x - 2y\cos x$$

$$h'(y) = 0 \quad \text{or} \quad h(y) = c_1$$

Hence the general solution of the given equation is

$$f(x, y) = c_2$$

i.e. $$y\sin^2 x - y^2\cos x = C, \text{ where } C = c_1 - c_2$$

Applying the initial condition that when $x = 0, y = 3,$ we have

$$-9 = c$$

since $$y^2\cos x - y\sin^2 x = 9$$

is the required solution.

Example 3:

Solve the DE $$\left(e^{2y} - y\cos xy\right)dx + \left(2xe^{2y} - x\cos x\,y + 2y\right)dy = 0$$

Solution:

The equation is neither separable nor homogenous.

Since, $$\left.\begin{aligned} M(x, y) &= e^{2y} - y\cos xy \\ N(x, y) &= 2xe^{2y} - x\cos xy + 2y \end{aligned}\right\}$$

and

$$\frac{\partial M}{\partial y} = 2e^{2y} + xy\sin xy - \cos xy = \frac{\partial N}{\partial x}$$

Hence the given equation is exact and a function $f(x, y)$ exist for which

$$M(x, y) = \frac{\partial f}{\partial x} \quad \text{and} \quad N(x, y) = \frac{\partial f}{\partial y}$$

which means that

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy \quad \text{and} \quad \frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

Let us start with the second equation i.e.

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

Integrating both sides w.r.to y, we obtain

$$f(x, y) = 2x\int e^{2y}dy - x\int \cos xy dy + 2\int y dy$$

Note that while integrating w.r.to y, x is treated as constant. Therefore

$$f(x,y) = xe^{2y} - \sin xy + y^2 + h(x)$$

h is an arbitrary function of x. From this equation we obtain $\frac{\partial f}{\partial x}$ and equate it to M

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy + h'(x) = e^{2y} - y\cos xy$$

So that $$h'(x) = 0 \Rightarrow h(x) = C$$

Hence a one-parameter family of solution is given by

$$xe^{2y} - \sin xy + y^2 + c = 0$$

Example 4

Solve $$2xy\, dx + (x^2 - 1)dy = 0$$

Solution:

Clearly $M(x,y) = 2xy$ and $N(x,y) = x^2 - 1$

Therefore $$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

The equation is exact and $\exists$ a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \text{ and } \frac{\partial f}{\partial y} = x^2 - 1$$

We integrate first of these equations to obtain.

$$f(x,y) = x^2 y + g(y)$$

Here $g(y)$ is an arbitrary function y. We find $\frac{\partial f}{\partial y}$ and equate it to $N(x,y)$

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1$$

$$g'(y) = -1 \Rightarrow g(y) = -y$$

Constant of integration need not to be included as the solution is given by

$$f(x,y) = c$$

Hence a one-parameter family of solutions is given by

$$x^2 y - y = c$$

Example 5

Solve the initial value problem

$$\left(\cos x \sin x - xy^2\right)dx + y\left(1 - x^2\right)dy = 0\,,\; y(0) = 2$$

Solution:

Since
$$\begin{cases} M(x,y) = \cos x.\sin x - x\,y^2 \\ N(x,y) = \quad y\left(1 - x^2\right) \end{cases}$$

and
$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}$$

Therefore the equation is exact and $\exists$ a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = \cos x\,.\,sin\,x - x\,y^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = y(1 - x^2)$$

Now integrating 2nd of these equations w.r.t. 'y' keeping 'x' constant, we obtain

$$f(x,y) = \frac{y^2}{2}\left(1 - x^2\right) + h(x)$$

Differentiate w.r.t. 'x' and equate the result to $M(x,y)$

$$\frac{\partial f}{\partial x} = -xy^2 + h'(x) = \cos x \sin x - xy^2$$

The last equation implies that.

$$h'(x) = \cos x \sin x$$

Integrating w.r.to x, we obtain

$$h(x) = -\int (\cos x)(-\sin x)dx = -\frac{1}{2}\cos^2 x$$

Thus a one parameter family solutions of the given differential equation is

$$\frac{y^2}{2}\left(1 - x^2\right) - \frac{1}{2}\cos^2 x = c_1$$

or

$$y^2\left(1-x^2\right)-\cos^2 x = c$$

where $2c_1$ has been replaced by C. The initial condition $y = 2$ when $x = 0$ demand, that $4(1)-\cos^2(0) = c$ so that $c = 3$. Thus the solution of the initial value problem is

$$y^2\left(1-x^2\right)-\cos^2 x = 3$$

Exercise

Determine whether the given equations is exact. If so, please solve.

1. $(\sin y - y\sin x)dx + (\cos x + x\cos y)dy = 0$

2. $\left(1 + \ln x + \dfrac{y}{x}\right)dx = (1 - \ln x)dy$

3. $\left(y\ln y - e^{-xy}\right)dx + \left(\dfrac{1}{y} + \ln y\right)dy = 0$

4. $\left(2y - \dfrac{1}{x} + \cos 3x\right)\dfrac{dy}{dx} + \dfrac{y}{x^2} - 4x^3 + 3y\sin 3x = 0$

5. $\left(\dfrac{1}{x} + \dfrac{1}{x^2} - \dfrac{y}{x^2 + y^2}\right)dx + \left(ye^y + \dfrac{x}{x^2 + y^2}\right)dy = 0$

Solve the given differential equations subject to indicated initial conditions.

6. $\left(e^x + y\right)dx + \left(2 + x + ye^y\right)dy = 0, \qquad y(0) = 1$

7. $\left(\dfrac{3y^2 - x^2}{y^5}\right)\dfrac{dy}{dx} + \dfrac{x}{2y^4} = 0, \qquad y(1) = 1$

8. $\left(\dfrac{1}{1 + y^2} + \cos x - 2xy\right)\dfrac{dy}{dx} = y(y + \sin x), \qquad y(0) = 1$

9. Find the value of ***k***, so that the given differential equation is exact.
   $\left(2xy^3 - y\sin xy + ky^4\right)dx - \left(20x^3 + x\sin xy\right)dy = 0$

10. $\left(6xy^3 + \cos y\right)dx - \left(kx^2y^2 - x\sin y\right)dy = 0$

Lecture 6 Integrating Factor Technique

If the equation

$$M(x,y)dx + N(x,y)dy = 0$$

is not exact, then we must have

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Therefore, we look for a function ***u (x, y)*** such that the equation

$$u(x,y)M(x,y)dx + u(x,y)N(x,y)dy = 0$$

becomes exact. The function ***u (x, y)*** (if it exists) is called the **integrating factor (IF)** and it satisfies the equation due to the condition of exactness.

$$\frac{\partial M}{\partial y}u + \frac{\partial u}{\partial y}M = \frac{\partial N}{\partial x}u + \frac{\partial u}{\partial x}N$$

This is a partial differential equation and is very difficult to solve. Consequently, the determination of the integrating factor is extremely difficult except for some special cases:

Example

Show that $1/(x^2 + y^2)$ is an integrating factor for the equation $\left(x^2 + y^2 - x\right)dx - ydy = 0,$ and then solve the equation.

Solution: Since $M = x^2 + y^2 - x, \quad \mathrm{N} = -y$

Therefore $\frac{\partial M}{\partial y} = 2y, \ \frac{\partial N}{\partial x} = 0$

So that

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

and the equation is not exact. However, if the equation is multiplied by $1/(x^2 + y^2)$ then the equation becomes

$$\left(1 - \frac{x}{x^2 + y^2}\right)dx - \frac{y}{x^2 + y^2}dy = 0$$

Now $M = 1 - \frac{x}{x^2 + y^2}$ and $N = -\frac{y}{x^2 + y^2}$

Therefore $$\frac{\partial M}{\partial y} = \frac{2xy}{\left(x^2+y^2\right)^2} = \frac{\partial N}{\partial x}$$

So that this new equation is exact. The equation can be solved. However, it is simpler to observe that the given equation can also written

$$dx - \frac{xdx + ydy}{x^2+y^2} = 0 \quad \text{or} \quad dx - \frac{1}{2}d\left[\ln(x^2+y^2)\right] = 0$$

or $$d\left[x - \frac{\ln\left(x^2+y^2\right)}{2}\right] = 0$$

Hence, by integration, we have

$$x - \ln\sqrt{x^2+y^2} = k$$

Case 1:

When $\exists$ an integrating factor $u(x)$, a function of x only. This happens if the expression

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$$

is a function of x only.

Then the integrating factor $u(x,y)$ is given by

$$u = \exp\left(\int \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}dx\right)$$

Case 2:

When $\exists$ an integrating factor $u(y)$, a function of y only. This happens if the expression

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$$

is a function of y only. Then **IF** $u(x,y)$ is given by

$$u = \exp\left(\int \frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}dy\right)$$

Case 3:

If the given equation is homogeneous and

$$xM + yN \neq 0$$

Then
$$u = \frac{1}{xM + yN}$$

Case 4:
If the given equation is of the form

$$yf(xy)dx + xg(xy)dy = 0$$

and
$$xM - yN \neq 0$$

Then
$$u = \frac{1}{xM - yN}$$

Once the **IF** is found, we multiply the old equation by *u* to get a new one, which is exact. Solve the exact equation and write the solution.

Advice: If possible, we should **check** whether or not the new equation is **exact?**

Summary:
Step 1. Write the given equation in the form

$$M(x, y)dx + N(x, y)dy = 0$$

provided the equation is not already in this form and determine M and N.
Step 2. Check for exactness of the equation by finding whether or not

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Step 3. (a) If the equation is not exact, then evaluate

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$$

If this expression is a function of x only, then

$$u(x) = \exp\left(\int \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} dx\right)$$

Otherwise, evaluate

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$$

If this expression is a function of *y* only, then

$$u(y) = \exp\left(\int \frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} dy\right)$$

In the **absence** of these **2 possibilities**, better use some other technique. However, we could also try cases 3 and 4 in step 4 and 5

Step 4. Test whether the equation is homogeneous and

$$xM + yN \neq 0$$

If yes then

$$u = \frac{1}{xM + yN}$$

Step 5. Test whether the equation is of the form

$$yf(xy)dx + xg(xy)dy = 0$$

and whether

$$xM - yN \neq 0$$

If yes then

$$u = \frac{1}{xM - yN}$$

Step 6. Multiply old equation by *u*. if possible, check whether or not the new equation is exact**?**

Step 7. Solve the new equation using steps described in the previous section.

Illustration:

Example 1

Solve the differential equation

$$\frac{dy}{dx} = -\frac{3xy + y^2}{x^2 + xy}$$

Solution:

1. The given differential equation can be written in form

$$(3xy + y^2)dx + (x^2 + xy)dy = 0$$

Therefore

$$M(x, y) = 3xy + y^2$$

$$N(x, y) = x^2 + xy$$

2. Now $\frac{\partial M}{\partial y} = 3x + 2y$, $\frac{\partial N}{\partial x} = 2x + y$.

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

3. To find an **IF** we evaluate

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{1}{x}$$

which is a function of x only.

4. Therefore, an **IF** $u(x)$ exists and is given by

$$u(x) = e^{\int \frac{1}{x} dx} = e^{\ln(x)} = x$$

5. Multiplying the given equation with the **IF,** we obtain

$$(3x^2 y + xy^2)dx + (x^3 + x^2 y)dy = 0$$

which is exact. (Please check**!**)

6. This step consists of solving this last exact differential equation.

Solution of new exact equation:

1. Since $\frac{\partial M}{\partial y} = 3x^2 + 2xy = \frac{\partial N}{\partial x}$, the equation is exact.

2. We find *F* (*x*, *y*) by solving the system

$$\begin{cases} \frac{\partial F}{\partial x} = 3x^2 y + xy^2 \\ \frac{\partial F}{\partial y} = x^3 + x^2 y. \end{cases}$$

3. We integrate the first equation to get

$$F(x, y) = x^3 y + \frac{x^2}{2} y^2 + \theta(y)$$

4. We differentiate F w. r. t. **'y'** and use the second equation of the system in step 2 to obtain

$$\frac{\partial F}{\partial y} = x^3 + x^2 y + \theta'(y) = x^3 + x^2 y$$

$\Rightarrow \theta' = 0$, **No dependence on *x***.

5. Integrating the last equation to obtain $\theta = C$. Therefore, the function $F(x, y)$ is

$$F(x, y) = x^3 y + \frac{x^2}{2} y^2$$

We don't have to keep the constant *C*, see next step.

6. All the solutions are given by the implicit equation $F(x, y) = C$ **i.e.**

$$x^3 y + \frac{x^2 y^2}{2} = C$$

Note that it can be verified that the function

$$u(x, y) = \frac{1}{2xy(2x + y)}$$

is another integrating factor for the same equation as the new equation

$$\frac{1}{2xy(2x + y)}(3xy + y^2)dx + \frac{1}{2xy(2x + y)}(x^2 + xy)dy = 0$$

is exact. This means that we may **not have uniqueness** of the integrating factor.

Example 2. Solve

$$\left(x^2 - 2x + 2y^2\right)dx + 2xydy = 0$$

Solution:

$$M = x^2 - 2x + 2y^2$$
$$N = 2xy$$

$$\frac{\partial M}{\partial y} = 4y, \frac{\partial N}{\partial x} = 2y$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The equation is not exact.

Here $$\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{1}{x}$$

Therefore, I.F. is given by

$$u = \exp\left(\int \frac{1}{x} dx\right)$$

$$u = x$$

$\therefore$ I.F is *x*.

Multiplying the equation by *x*, we have

$$\left(x^3 - 2x^2 + 2xy^2\right)dx + 2x^2 ydy = 0$$

This equation is exact. The required Solution is

$$\frac{x^4}{4} - \frac{2x^3}{3} + x^2 y^2 = c_0$$

$$3x^4 - 8x^3 + 12x^2 y^2 = c$$

Example 3

Solve $$dx + \left(\frac{x}{y} - \sin y\right)dy = 0$$

Solution: Here

$$M = 1, \quad N = \frac{x}{y} - \sin y$$
$$\frac{\partial M}{\partial y} = 0, \quad \frac{\partial N}{\partial x} = \frac{1}{y}$$
$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The equation is not exact.

Now

$$\frac{N_x - M_y}{M} = \frac{\frac{1}{y} - 0}{1} = \frac{1}{y}$$

Therefore, the IF is $$u(y) = \exp \int \frac{dy}{y} = y$$

Multiplying the equation by y, we have

$$ydx + (x - y\sin y)dy = 0$$

or $$ydx + xdy - y\sin ydy = 0$$

or $$d(xy) - y\sin ydy = 0$$

Integrating, we have

$$xy + y\cos y - \sin y = c$$

Which is the required solution.

Example 4

Solve $$\left(x^2 y - 2xy^2\right)dx - \left(x^3 - 3x^2 y\right)dy = 0$$

Solution: Comparing with

$$Mdx + Ndy = 0$$

we see that

$$M = x^2y - 2xy^2 \quad \text{and} \quad N = -(x^3 - 3x^2y)$$

Since both M and N are homogeneous. Therefore, the given equation is homogeneous.
Now

$$xM + yN = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2 \neq 0$$

Hence, the factor u is given by

$$u = \frac{1}{x^2y^2} \qquad \because u = \frac{1}{xM + yN}$$

Multiplying the given equation with the integrating factor u, we obtain.

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

Now

$$M = \frac{1}{y} - \frac{2}{x} \quad \text{and} \quad N = \frac{-x}{y^2} + \frac{3}{y}$$

and therefore

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2} = \frac{\partial N}{\partial x}$$

Therefore, the new equation is exact and solution of this new equation is given by

$$\frac{x}{y} - 2\ln|x| + 3\ln|y| = C$$

Example 5

Solve $$y\left(xy + 2x^2y^2\right)dx + x\left(xy - x^2y^2\right)dy = 0$$

Solution:
The given equation is of the form

$$yf(xy)dx + xg(xy)dy = 0$$

Now comparing with

$$Mdx + Ndy = 0$$

We see that

$$M = y\left(xy + 2x^2y^2\right) \quad \text{and} \quad N = x\left(xy - x^2y^2\right)$$

Further

$$xM - yN = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3$$
$$= \quad 3x^3y^3 \quad \neq \quad 0$$

Therefore, the integrating factor u is

$$u = \frac{1}{3x^3y^3}, \qquad \because u = \frac{1}{xM - yN}$$

Now multiplying the given equation by the integrating factor, we obtain

$$\frac{1}{3}\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \frac{1}{3}\left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$$

Therefore, solutions of the given differential equation are given by

$$-\frac{1}{xy} + 2\ln|x| - \ln|y| = C$$

where $3C_0 = C$

Exercise

Solve by finding an I.F

1. $x\,dy - y\,dx = \left(x^2 + y^2\right)dx$
2. $dy + \dfrac{y - \sin x}{x}dx = 0$
3. $\left(y^4 + 2y\right)dx + \left(xy^3 + 2y^4 - 4x\right)dy = 0$
4. $\left(x^2 + y^2\right)dx + 2xydy = 0$
5. $\left(4x + 3y^2\right)dx + 2xydy = 0$
6. $\left(3x^2y^4 + 2xy\right)dx + \left(2x^3y^3\right)dy = 0$
7. $\dfrac{dy}{dx} = e^{2x} + y - 1$
8. $\left(3xy + y^2\right)dx + \left(x^2 + xy\right)dy = 0$
9. $ydx + \left(2xy - e^{-2y}\right)dy = 0$
10. $\left(x + 2\right)\sin ydx + x\cos ydy = 0$

Lecture 7 First Order Linear Equations

The differential equation of the form:

$$a(x)\frac{dy}{dx}+b(x)y=c(x)$$

is a linear differential equation of first order. The equation can be rewritten in the following **famous form.**

$$\frac{dy}{dx}+p(x)y=q(x)$$

where $p(x)$ and $q(x)$ are continuous functions.

Method of solution:
The general solution of the first order linear differential equation is given by

$$y=\frac{\int u(x)q(x)dx+C}{u(x)}$$

Where $u(x)=\exp\left(\int p(x)dx\right)$

The function $u(x)$ is called the **integrating factor**. If it is an IVP then use it to find the constant *C*.

Summary:

1. Identify that the equation is 1st order linear equation. Rewrite it in the form

$$\frac{dy}{dx}+p(x)y=q(x)$$

if the equation is not already in this form.

2. Find the integrating factor

$$u(x)=e^{\int p(x)dx}$$

3. Write down the general solution

$$y=\frac{\int u(x)q(x)dx+C}{u(x)}$$

4. If you are given an IVP, use the initial condition to find the constant *C*.

5. Plug in the calculated value to write the particular solution of the problem.

Example 1:

Solve the initial value problem

$$y' + \tan(x)y = \cos^2(x), \qquad y(0) = 2$$

Solution:

1.The equation is already in the standard form

$$\frac{dy}{dx} + p(x)y = q(x)$$

with

$$\begin{cases} p(x) = \tan x \\ q(x) = \cos^2 x \end{cases}$$

2. Since

$$\int \tan x\, dx = -\ln\cos x = \ln\sec x$$

Therefore, the integrating factor is given by

$$u(x) = e^{\int \tan x\, dx} = \sec x$$

3. Further, because

$$\int \sec x \cos^2 x\, dx = \int \cos x\, dx = \sin x$$

So that the general solution is given by

$$y = \frac{\sin x + C}{\sec x} = (\sin x + C)\cos x$$

4. We use the initial condition $y(0) = 2$ to find the value of the constant C

$$y(0) = C = 2$$

5. Therefore the solution of the initial value problem is

$$y = (\sin x + 2)\cos x$$

Example 2: Solve the IVP $$\frac{dy}{dt} - \frac{2t}{1+t^2}y = \frac{2}{1+t^2}, \qquad y(0) = 0.4$$

Solution:

1.The given equation is a 1st order linear and is already in the requisite form

$$\frac{dy}{dx} + p(x)y = q(x)$$

with
$$\begin{cases} p(t) = -\dfrac{2t}{1+t^2} \\ q(t) = \dfrac{2}{1+t^2} \end{cases}$$

2. Since $\displaystyle\int\left(-\frac{2t}{1+t^2}\right)dt = -\ln|1+t^2|$

Therefore, the integrating factor is given by

$$u(t) = e^{\int -\frac{2t}{1+t^2}dt} = (1+t^2)^{-1}$$

3. Hence, the general solution is given by

$$y = \frac{\int u(t)q(t)dt + C}{u(t)}, \quad \int u(t)q(t)dt = \int \frac{2}{(1+t^2)^2}dt$$

Now $\displaystyle\int \frac{2}{(1+t^2)^2}dt = 2\int \frac{1+t^2-t^2}{(1+t^2)^2}dt = 2\int\left(\frac{1}{1+t^2} - \frac{t^2}{(1+t^2)^2}\right)dt$

The first integral is clearly $\tan^{-1} t$. For the 2nd we will use integration by parts with t as first function and $2t/(1+t^2)^2$ as 2nd function.

$$\int \frac{2t^2}{(1+t^2)^2}dt = t\left(-\frac{1}{1+t^2}\right) + \int \frac{1}{1+t^2}dt = -\frac{t}{1+t^2} + \tan^{-1}(t)$$

$$\int \frac{2}{(1+t^2)^2}dt = 2\tan^{-1}(t) + \frac{t}{1+t^2} - \tan^{-1}(t) = \tan^{-1}(t) + \frac{t}{1+t^2}$$

The general solution is: $\displaystyle y = (1+t^2)\left(\tan^{-1}(t) + \frac{t}{1+t^2} + C\right)$

4. The condition $y(0) = 0.4$ gives $C = 0.4$

5. Therefore, solution to the initial value problem can be written as**:**

$$y = t + (1+t^2)\tan^{-1}(t) + 0.4(1+t^2)$$

Example 3:

Find the solution to the problem

$$\cos^2 t \sin t \,.\, y' = -\cos^3 t \,.\, y + 1, \quad y\left(\frac{\pi}{4}\right) = 0$$

Solution:

1. The equation is 1st order linear and is not in the standard form

$$\frac{dy}{dx} + p(x)y = q(x)$$

Therefore we rewrite the equation as

$$y' + \frac{\cos t}{\sin t} y = \frac{1}{\cos^2 t \sin t}$$

2. Hence, the integrating factor is given by

$$u(t) = e^{\int \frac{\cos t}{\sin t} dt} = e^{\ln|\sin t|} = \sin t$$

3. Therefore, the general solution is given by

$$y = \frac{\int \sin t \frac{1}{\cos^2 t \sin t} dt + C}{\sin t}$$

Since

$$\int \sin t \frac{1}{\cos^2 t \ \sin t} dt = \int \frac{1}{\cos^2 t} dt = \tan t$$

Therefore

$$y = \frac{\tan t + C}{\sin t} = \frac{1}{\cos t} + \frac{C}{\sin t} = \sec t + C \csc t$$

(1) The initial condition $y(\pi/4) = 0$ implies

$$\sqrt{2} + C\sqrt{2} = 0$$

which gives $C = -1$.

(2) Therefore, the particular solution to the initial value problem is

$$y = \sec t - \csc t$$

Example 4

Solve $$\left(x + 2y^3\right)\frac{dy}{dx} = y$$

Solution:

We have

$$\frac{dy}{dx} = \frac{y}{x + 2y^3}$$

This equation is not linear in y. Let us regard x as dependent variable and y as independent variable. The equation may be written as

$$\frac{dx}{dy} = \frac{x + 2y^3}{y}$$

or
$$\frac{dx}{dy} - \frac{1}{y}x = 2y^2$$

Which is linear in x

$$IF = \exp\left[\int\left(-\frac{1}{y}\right)dy\right] = \exp\left[\ln\frac{1}{y}\right] = \frac{1}{y}$$

Multiplying with the $IF = \frac{1}{y}$, we get

$$\frac{1}{y}\frac{dx}{dy} - \frac{1}{y^2}x = 2y$$

$$\frac{d}{dy}\left(\frac{x}{y}\right) = 2y$$

Integrating, we have

$$\frac{x}{y} = y^2 + c$$

$$x = y\left(y^2 + c\right)$$

is the required solution.

Example 5

Solve

$$(x-1)^3 \frac{dy}{dx} + 4(x-1)^2 y = x+1$$

Solution:

The equation can be rewritten as

$$\frac{dy}{dx} + \frac{4}{x-1} y = \frac{x+1}{(x-1)^3}$$

Here $P(x) = \dfrac{4}{x-1}$.

Therefore, an integrating factor of the given equation is

$$IF = \exp\left[\int \frac{4dx}{x-1}\right] = \exp\left[\ln(x-1)^4\right] = (x-1)^4$$

Multiplying the given equation by the IF, we get

$$(x-1)^4 \frac{dy}{dx} + 4(x-1)^3 y = x^2 - 1$$

or $$\frac{d}{dx}\left[y(x-1)^4\right] = x^2 - 1$$

Integrating both sides, we obtain

$$y(x-1)^4 = \frac{x^3}{3} - x + c$$

which is the required solution.

Exercise

Solve the following differential equations

1. $\frac{dy}{dx}+\left(\frac{2x+1}{x}\right)y=e^{-2x}$

2. $\frac{dy}{dx}+3y=3x^2e^{-3x}$

3. $x\frac{dy}{dx}+(1+x\cot x)y=x$

4. $(x+1)\frac{dy}{dx}-ny=e^x(x+1)^{n+1}$

5. $(1+x^2)\frac{dy}{dx}+4xy=\frac{1}{(1+x^2)^2}$

6. $\frac{dr}{d\theta}+r\sec\theta=\cos\theta$

7. $\frac{dy}{dx}+y=\frac{1-e^{-2x}}{e^x+e^{-x}}$

8. $dx=(3e^y-2x)dy$

Solve the initial value problems

9. $\frac{dy}{dx}=2y+x(e^{3x}-e^{2x}),\qquad y(0)=2$

10. $x(2+x)\frac{dy}{dx}+2(1+x)y=1+3x^2,\qquad y(-1)=1$

Lecture 8 Bernoulli Equations

A differential equation that can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

is called Bernoulli equation.

Method of solution:

For $n = 0,1$ the equation reduces to 1st order linear DE and can be solved accordingly.

For $n \neq 0,1$ we divide the equation with y^n to write it in the form

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x)$$

and then put

$$v = y^{1-n}$$

Differentiating w.r.t. 'x', we obtain

$$v' = (1-n)y^{-n}y'$$

Therefore the equation becomes

$$\frac{dv}{dx} + (1-n)p(x)v = (1-n)q(x)$$

This is a linear equation satisfied by v. Once it is solved, you will obtain the function

$$y = v^{\frac{1}{(1-n)}}$$

If $n > 1$, then we add the solution $y = 0$ to the solutions found the above technique.

Summary:
1.Identify the equation

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

as Bernoulli equation.
Find ***n***. If $n \neq 0,1$ divide by y^n and substitute**;**

$$v = y^{1-n}$$

2. Through easy differentiation, find the new equation

$$\frac{dv}{dx} + (1-n)p(x)v = (1-n)q(x)$$

3. This is a linear equation. Solve the linear equation to find ***v*.**

4. Go back to the old function *y* through the substitution $y = v^{1/(1-n)}$.

6. If $n > 1$, then include *y* = 0 to in the solution**.**

7. If you have an IVP, use the initial condition to find the particular solution.

Example 1: Solve the equation $\frac{dy}{dx} = y + y^3$

Solution:
1. The given differential can be written as

$$\frac{dy}{dx} - y = y^3$$

which is a Bernoulli equation with

$$p(x) = -1, q(x) = 1, \mathbf{n=3.}$$

Dividing with y^3 we get

$$y^{-3}\frac{dy}{dx} - y^{-2} = 1$$

Therefore we substitute

$$v = y^{1-3} = y^{-2}$$

2. Differentiating w.r.t. 'x' we have

$$y^{-3}\frac{dy}{dx}=-\frac{1}{2}\left(\frac{dv}{dx}\right)$$

So that the equation reduces to

$$\frac{dv}{dx}+2v=-2$$

3. This is a linear equation. To solve this we find the integrating factor $u(x)$

$$u(x)=e^{\int 2dx}=e^{2x}$$

The solution of the linear equation is given by

$$v=\frac{\int u(x)q(x)dx+c}{u(x)}=\frac{\int e^{2x}(-2)dx+c}{e^{2x}}$$

Since $$\int e^{2x}(-2)dx=-e^{2x}$$

Therefore, the solution for v is given by

$$v=\frac{-e^{2x}+C}{e^{2x}}=Ce^{-2x}-1$$

4. To go back to y we substitute $v=y^{-2}$. Therefore the general solution of the given DE is

$$y=\pm\left(Ce^{-2x}-1\right)^{-\frac{1}{2}}$$

5. Since $n>1$, we include the $y=0$ in the solutions. Hence, all solutions are

$$y=0, \qquad y=\pm(Ce^{-2x}-1)^{-1/2}$$

Example 2:

Solve $$\frac{dy}{dx}+\frac{1}{x}y=xy^2$$

Solution: In the given equation we identify $P(x)=\frac{1}{x},\ q(x)=x \text{ and } n=2$.

Thus the substitution $w=y^{-1}$ gives

$$\frac{dw}{dx}-\frac{1}{x}w=-x.$$

The integrating factor for this linear equation is

$$e^{-\int\frac{dx}{x}}=e^{-\ln|x|}=e^{\ln|x|^{-1}}=x^{-1}$$

Hence $$\frac{d}{dx}\left[x^{-1}w\right]=-1.$$

Integrating this latter form, we get

$$x^{-1}w = -x + c \text{ or } w = -x^2 + cx.$$

Since $w = y^{-1}$, we obtain $y = \frac{1}{w}$ or

$$y = \frac{1}{-x^2 + cx}$$

For $n > 0$ the trivial solution $y = 0$ is a solution of the given equation. In this example, $y = 0$ is a singular solution of the given equation.

Example 3:

Solve:

$$\frac{dy}{dx} + \frac{xy}{1-x^2} = xy^{\frac{1}{2}} \qquad (1)$$

Solution: Dividing (1) by $y^{\frac{1}{2}}$, the given equation becomes

$$y^{\frac{-1}{2}}\frac{dy}{dx} + \frac{x}{1-x^2}y^{\frac{1}{2}} = x \qquad (2)$$

Put

$$y^{\frac{1}{2}} = v \text{ or. } \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = \frac{dv}{dx}$$

Then (2) reduces to

$$\frac{dv}{dx} + \frac{x}{2(1-x^2)}v = \frac{x}{2} \qquad (3)$$

This is linear in v.

$$\text{I.F} = \exp\left[\int \frac{x}{2(1-x^2)}dx\right] = \exp\left[\frac{-1}{4}\ln(1-x^2)\right] = (1-x^2)^{\frac{-1}{4}}$$

Multiplying (3) by $(1-x^2)^{\frac{-1}{4}}$, we get

$$(1-x^2)^{\frac{-1}{4}}\frac{dv}{dx} + \frac{x}{2(1-x^2)^{5/4}}v = \frac{x}{2(1-x^2)^{1/4}}$$

or

$$\frac{d}{dx}\left[(1-x^2)^{\frac{-1}{4}}v\right] = \frac{-1}{4}\left[-2x(1-x^2)^{\frac{-1}{4}}\right]$$

Integrating, we have

$$v\left(1-x^2\right)^{\frac{-1}{4}} = \frac{-1}{4}\frac{\left(1-x^2\right)^{\frac{3}{4}}}{3/4} + c$$

or $$v = c\left(1-x^2\right)^{1/4} - \frac{1-x^2}{3}$$

or $$y^{\frac{1}{2}} = c\left(1-x^2\right)^{1/4} - \frac{1-x^2}{3}$$

is the required solution.

Exercise

Solve the following differential equations

1. $x\frac{dy}{dx} + y = y^2 \ln x$

2. $\frac{dy}{dx} + y = xy^3$

3. $\frac{dy}{dx} - y = e^x y^2$

4. $\frac{dy}{dx} = y\left(xy^3 - 1\right)$

5. $x\frac{dy}{dx} - (1 + x)y = xy^2$

6. $x^2 \frac{dy}{dx} + y^2 = xy$

Solve the initial-value problems

7. $x^2 \frac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$

8. $y^{1/2} \frac{dy}{dx} + y^{3/2} = 1, \quad y(0) = 4$

9. $xy\left(1 + xy^2\right)\frac{dy}{dx} = 1, \quad y(1) = 0$

10. $2\frac{dy}{dx} = \frac{y}{x} - \frac{x}{y^2}, \quad y(1) = 1$

SUBSTITUTIONS

- Sometimes a differential equation can be transformed by means of a substitution into a form that could then be solved by one of the standard methods i.e. Methods used to solve separable, homogeneous, exact, linear, and Bernoulli's differential equation.

- An equation may look different from any of those that we have studied in the previous lectures, but through a sensible change of variables perhaps an apparently difficult problem may be readily solved.

- Although no firm rules can be given on the basis of which these substitution could be selected, a working axiom might be: Try something! It sometimes pays to be clever.

Example 1

The differential equation

$$y(1+2xy)dx + x(1-2xy)dy = 0$$

is not separable, not homogeneous, not exact, not linear, and not Bernoulli.
However, if we stare at the equation long enough, we might be prompted to try the substitution

$$u = 2xy \quad \text{or} \quad y = \frac{u}{2x}$$

Since

$$dy = \frac{xdu - udx}{2x^2}$$

The equation becomes, after we simplify

$$2u^2 dx + (1-u)xdu = 0.$$

we obtain

$$2\ln|x| - u^{-1} - \ln|u| = c$$

$$\ln\left|\frac{x}{2y}\right| = c + \frac{1}{2xy}$$

$$\frac{x}{2y} = c_1 e^{1/2xy},$$

$$x = 2c_1 y e^{1/2xy}$$

where e^c was replaced by c_1. We can also replace $2c_1$ by c_2 if desired

Note that

The differential equation in the example possesses the trivial solution $y = 0$, but then this function is not included in the one-parameter family of solution.

Example 2

Solve

$$2xy\frac{dy}{dx} + 2y^2 = 3x - 6.$$

Solution:

The presence of the term $2y\dfrac{dy}{dx}$ prompts us to try $u = y^2$

Since

$$\frac{du}{dx} = 2y\frac{dy}{dx}$$

Therefore, the equation becomes

Now

$$x\frac{du}{dx} + 2u = 3x - 6$$

or

$$\frac{du}{dx} + \frac{2}{x}u = 3 - \frac{6}{x}$$

This equation has the form of 1st order linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

with

$$P(x) = \frac{2}{x} \text{ and } Q(x) = 3 - \frac{6}{x}$$

Therefore, the integrating factor of the equation is given by

$$\text{I.F} = e^{\int \frac{2}{x}dx} = e^{\ln x^2} = x^2$$

Multiplying with the IF gives

$$\frac{d}{dx}\left[x^2 u\right] = 3x^2 - 6x$$

Integrating both sides, we obtain

$$x^2 u = x^3 - 3x^2 + c$$

or

$$x^2 y^2 = x^3 - 3x^2 + c.$$

Example 3
Solve

$$x\frac{dy}{dx} - y = \frac{x^3}{y} e^{y/x}$$

Solution:

If we let

$$u = \frac{y}{x}$$

Then the given differential equation can be simplified to

$$ue^{-u}du = dx$$

Integrating both sides, we have

$$\int ue^{-u}du = \int dx$$

Using the integration by parts on LHS, we have

$$-ue^{-u} - e^{-u} = x + c$$

or

$$u + 1 = (c_1 - x)e^u \quad \text{Where } c_1 = -c$$

We then re-substitute

$$u = \frac{y}{x}$$

and simplify to obtain

$$y + x = x(c_1 - x)e^{y/x}$$

Example 4

Solve

$$\frac{d^2y}{dx^2} = 2x\left(\frac{dy}{dx}\right)^2$$

Solution:
If we let

$$u = y'$$

Then

$$du/dx = y''$$

Then, the equation reduces to

$$\frac{du}{dx} = 2xu^2$$

Which is separable form. Separating the variables, we obtain

$$\frac{du}{u^2} = 2xdx$$

Integrating both sides yields

$$\int u^{-2}du = \int 2xdx$$

or

$$-u^{-1} = x^2 + c_1^2$$

The constant is written as c_1^2 for convenience.

Since

$$u^{-1} = 1/y'$$

Therefore

$$\frac{dy}{dx} = -\frac{1}{x^2 + c_1^2}$$

or

$$dy = -\frac{dx}{x^2 + c_1^2}$$

$$\int dy = -\int \frac{dx}{x^2 + c_1^2}$$

$$y + c_2 = -\frac{1}{c_1}\tan^{-1}\frac{x}{c_1}$$

Exercise

Solve the differential equations by using an appropriate substitution.

1. $ydx + (1 + ye^{x})dy = 0$

2. $\left(2 + e^{-x/y}\right)dx + 2\left(1 - x/y\right)dy = 0$

3. $2x\csc 2y\,\dfrac{dy}{dx} = 2x - \ln(\tan y)$

4. $\dfrac{dy}{dx} + 1 = \sin x\, e^{-(x+y)}$

5. $y\dfrac{dy}{dx} + 2x\ln x = xe^{y}$

6. $x^{2}\dfrac{dy}{dx} + 2xy = x^{4}y^{2} + 1$

7. $xe^{y}\dfrac{dy}{dx} - 2e^{y} = x^{2}$

Lecture 9 Practice Examples

Example 1: $y' = \dfrac{x^2+y^2}{xy}$

Solution: $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{xy}$

put y=wx then $\dfrac{dy}{dx} = w + x\dfrac{dw}{dx}$

$$w + x\frac{dw}{dx} = \frac{x^2+w^2x^2}{xxw} = \frac{1+w^2}{w}$$

$$w + x\frac{dw}{dx} = \frac{1}{w} + w$$

$$wdw = \frac{dx}{x}$$

Integrating

$$\frac{w^2}{2} = \ln x + \ln c$$

$$\frac{y^2}{2x^2} = \ln|xc|$$

$$y^2 = 2x^2 \ln|xc|$$

Example 2: $\frac{dy}{dx}=\frac{(2\sqrt{xy}-y)}{x}$

Solution: $\frac{dy}{dx}=\frac{(2\sqrt{xy}-y)}{x}$

put y= wx

$$w+x\frac{dw}{dx}=\frac{(2\sqrt{xwx}-xw)}{x}$$

$$w+x\frac{dw}{dx}=2\sqrt{w}-w$$

$$x\frac{dw}{dx}=2\sqrt{w}-2w$$

$$\frac{dw}{2(\sqrt{w}-w)}=\frac{dx}{x}$$

$$\int\frac{dw}{2(\sqrt{w}-w)}=\int\frac{dx}{x}$$

$$\int\frac{dw}{2\sqrt{w}(1-\sqrt{w})}=\int\frac{dx}{x}$$

put $\sqrt{w}=t$

We get $\int\frac{1}{1-t}dt=\int\frac{dx}{x}$

$$-\ln|1-t|=\ln|x|+\ln|c|$$

$$-\ln|1-t|=\ln|xc|$$

$$(1-t)^{-1}=xc$$

$$(1-\sqrt{w})^{-1}=xc$$

$$(1-\sqrt{y/x})^{-1}=xc$$

Example 3: $(2y^2x-3)dx+(2yx^2+4)dy=0$

Solution: $(2y^2x-3)dx+(2yx^2+4)dy=0$

Here $M=(2y^2x-3)$ and $N=(2yx^2+4)$

$$\frac{\partial M}{\partial y}=4xy=\frac{\partial N}{\partial x}$$

$$\frac{\partial f}{\partial x}=(2y^2x-3) \text{ and } \frac{\partial f}{\partial y}=(2yx^2+4)$$

Integrate w.r.t. 'x'

$$f(x,y)=x^2y^2-3x+h(y)$$

Differentiate w.r.t. 'y'

$$\frac{\partial f}{\partial y}=2x^2y+h'(y)=2x^2y+4=N$$

$$h'(y)=4$$

Integrate w.r.t. 'y'

$$h(y)=4y+c$$

$$x^2y^2-3x+4y=C_1$$

Example 4: $\dfrac{dy}{dx}=\dfrac{2xye^{(x/y)^2}}{y^2+y^2e^{(x/y)^2}+2x^2e^{(x/y)^2}}$

Solution: $\dfrac{dx}{dy}=\dfrac{y^2+y^2e^{(x/y)^2}+2x^2e^{(x/y)^2}}{2xye^{(x/y)^2}}$

put x/y=w

Aftersubsitution

$$y\frac{dw}{dy}=\frac{1+e^{w^2}}{2we^{w^2}}$$

$$\frac{dy}{y}=\frac{2we^{w^2}}{1+e^{w^2}}dw$$

Integrating

$$\ln|y|=\ln|1+e^{w^2}|+\ln c$$

$$\ln|y|=\ln|c(1+e^{w^2})|$$

$$y=c(1+e^{(x/y)^2})$$

Example 5: $\frac{dy}{dx}+\frac{y}{x\ln x}=\frac{3x^2}{\ln x}$

Solution: $\frac{dy}{dx}+\frac{y}{x\ln x}=\frac{3x^2}{\ln x}$

$$\frac{dy}{dx}+\frac{1}{x\ln x}y=\frac{3x^2}{\ln x}$$

$p(x)=\frac{1}{x\ln x}$ and $q(x)=\frac{3x^2}{\ln x}$

$$I.F=\exp\left(\int\frac{1}{x\ln x}dx\right)=\ln x$$

Multiply both side by lnx

$$\ln x\frac{dy}{dx}+\frac{1}{x}y=3x^2$$

$$\frac{d}{dx}(y\ln x)=3x^2$$

Integrate

$$y\ln x=\frac{3x^3}{3}+c$$

Example 6: $(y^2e^x+2xy)dx-x^2dy=0$

Solution: Here $M=y^2e^x+2xy$ $\quad N=-x^2$

$$\frac{\partial M}{\partial y}=2ye^x+2x, \qquad \frac{\partial N}{\partial x}=-2x$$

Clearly $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$

The given equation is not exact

divide the equation by y^2 to make it exact

$$\left[e^x+\frac{2x}{y}\right]dx+\left[-\frac{x^2}{y^2}\right]dy=0$$

Now $\dfrac{\partial M}{\partial y}=-\dfrac{2x}{y^2}=\dfrac{\partial N}{\partial x}$

Equation is exact

$$\frac{\partial f}{\partial x}=\left[e^x+\frac{2x}{y}\right] \qquad \frac{\partial f}{\partial y}=\left[-\frac{x^2}{y^2}\right]$$

Integrate w.r.t. 'x'

$$f(x,y)=e^x+\frac{x^2}{y}$$

$$e^x+\frac{x^2}{y}=c$$

Example 7:

$$x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1$$

Solution: $x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1$

$$\frac{dy}{dx}+y\left[\frac{x\sin x+\cos x}{x\cos x}\right]=\frac{1}{x\cos x}$$

$$\frac{dy}{dx}+y\left[\tan x+1/x\right]=\frac{1}{x\cos x}$$

$$I.F = \exp(\int(\tan x+1/x)dx)=x\sec x$$

$$x\sec x\frac{dy}{dx}+yx\sec x\left[\tan x+1/x\right]=\frac{x\sec x}{x\cos x}$$

$$x\sec x\frac{dy}{dx}+y\left[x\sec x\tan x+\sec x\right]=\sec^2 x$$

$$\frac{d}{dx}\left[xy\sec x\right]=\sec^2 x$$

$$xy\sec x=\tan x+c$$

Example 8: $xe^{2y}\frac{dy}{dx}+e^{2y}=\frac{\ln x}{x}$

Solution: $xe^{2y}\frac{dy}{dx}+e^{2y}=\frac{\ln x}{x}$

put $e^{2y}=u$

$$2e^{2y}\frac{dy}{dx}=\frac{du}{dx}$$

$$\frac{x}{2}\frac{du}{dx}+u=\frac{\ln x}{x}$$

$$\frac{du}{dx}+\frac{2}{x}u=2\frac{\ln x}{x^2}$$

Here $p(x)=2/x$ And $Q(x)=\frac{\ln x}{x^2}$

$$I.F=\exp\left(\int\frac{2}{x}dx\right)=x^2$$

$$x^2\frac{du}{dx}+2xu=2\ln x$$

$$\frac{d}{dx}(x^2u)=2\ln x$$

Integrate

$$x^2u=2[x\ln x-x]+c$$

$$x^2e^{2y}=2[x\ln x-x]+c$$

Example 9: $\frac{dy}{dx}+y\ln y=ye^{x}$

Solution: $\frac{dy}{dx}+y\ln y=ye^{x}$

$$\frac{1}{y}\frac{dy}{dx}+\ln y=e^{x}$$

put $\ln y=u$

$$\frac{du}{dx}+u=e^{x}$$

$$I.F.=e^{\int dx}=e^{x}$$

$$\frac{d}{dx}(e^{x}u)=e^{2x}$$

Integrate

$$e^{x}.u=\frac{e^{2x}}{2}+c$$

$$e^{x}\ln y=\frac{e^{2x}}{2}+c$$

Example 10: $2x\csc 2y\dfrac{dy}{dx}=2x-\ln\tan y$

Solution: $2x\csc 2y\dfrac{dy}{dx}=2x-\ln\tan y$

put $\ln\tan y=u$

$$\frac{dy}{dx}=\sin y\cos y\frac{du}{dx}$$

$$\frac{2x\sin y\cos y}{2\sin y\cos y}\frac{du}{dx}=2x-u$$

$$x\frac{du}{dx}=2x-u$$

$$\frac{du}{dx}+\frac{1}{x}u=2$$

$$\text{I.F}\ =\exp(\int 1/x\,dx)=x$$

$$x\frac{du}{dx}+u=2x$$

$$\frac{d}{dx}(xu)=2x$$

$$xu=x^2+c$$

$$u=x+cx^{-1}$$

$$\ln\tan y=x+cx^{-1}$$

Example 11: $\frac{dy}{dx}+x+y+1=(x+y)^2e^{3x}$

Solution: $\frac{dy}{dx}+x+y+1=(x+y)^2e^{3x}$

Put $x+y=u$

$$\frac{du}{dx}+u=u^2e^{3x}$$

$$\frac{du}{dx}+u=u^2e^{3x} \text{ (Bernouli's)}$$

$$\frac{1}{u^2}\frac{du}{dx}+\frac{1}{u}=e^{3x}$$

put $1/u=w$

$$-\frac{dw}{dx}+w=e^{3x}$$

$$\frac{dw}{dx}-w=-e^{3x}$$

$$I.F=\exp(\int -dx)=e^{-x}$$

$$e^{-x}\frac{dw}{dx}-we^{-x}=-e^{2x}$$

$$\frac{d}{dx}(e^{-x}w)=-e^{2x}$$

Integrate

$$e^{-x}w=\frac{-e^{2x}}{2}+c$$

$$\frac{1}{u}=\frac{-e^{3x}}{2}+ce^{x}$$

$$\frac{1}{x+y}=\frac{-e^{3x}}{2}+ce^{x}$$

Example 12: $\frac{dy}{dx}=(4x+y+1)^2$

Solution: $\frac{dy}{dx}=(4x+y+1)^2$

put $4x+y+1=u$

we get

$$\frac{du}{dx}-4=u^2$$

$$\frac{du}{dx}=u^2+4$$

$$\frac{1}{u^2+4}du=dx$$

Integrate

$$\frac{1}{2}\tan^{-1}\frac{u}{2}=x+c$$

$$\tan^{-1}\frac{u}{2}=2x+c_1$$

$$u=2\tan(2x+c_1)$$

$$4x+y+1=2\tan(2x+c_1)$$

Example 13: $(x+y)^2 \frac{dy}{dx} = a^2$

Solution: $(x+y)^2 \frac{dy}{dx} = a^2$

put $x+y = u$

$$u^2 \left(\frac{du}{dx} - 1\right) = a^2$$

$$u^2 \frac{du}{dx} - u^2 = a^2$$

$$\frac{u^2}{u^2+a^2} du = dx$$

Integrate

$$\int \frac{u^2+a^2-a^2}{u^2+a^2} du = \int dx$$

$$\int \left(1 - \frac{a^2}{u^2+a^2}\right) du = \int dx$$

$$u - a\tan^{-1}\frac{u}{a} = x + c$$

$$(x+y) - a\tan^{-1}\frac{x+y}{a} = x + c$$

Example 14: $2y\frac{dy}{dx}+x^2+y^2+x=0$

Solution: $2y\frac{dy}{dx}+x^2+y^2+x=0$

put $x^2+y^2=u$

$$\frac{du}{dx}-2x+u+x=0$$

$$\frac{du}{dx}+u=x$$

$$\text{I.F}=\text{Exp}(\int dx)=e^x$$

$$e^x\frac{du}{dx}+ue^x=xe^x$$

$$\frac{d}{dx}(e^x u)=xe^x$$

Integrating

$$e^x u=xe^x-e^x+c$$

Example 15: $y' + 1 = e^{-(x+y)} \sin x$

Solution: $y' + 1 = e^{-(x+y)} \sin x$

put $x + y = u$

$$\frac{du}{dx} = e^{-u} \sin x$$

$$\frac{1}{e^{-u}} du = \sin x dx$$

$$e^{u} du = \sin x dx$$

Integrate

$$e^{u} = -\cos x + c$$

$$u = \ln|-\cos x + c|$$

$$x + y = \ln|-\cos x + c|$$

Example 16: $x^4y^2y'+x^3y^3=2x^3-3$

Solution: $x^4y^2y'+x^3y^3=2x^3-3$

put $x^3y^3=u$

$$3x^2y^3+3x^3y^2\frac{dy}{dx}=\frac{du}{dx}$$

$$3x^3y^2\frac{dy}{dx}=\frac{du}{dx}-3x^2y^3$$

$$x^4y^2\frac{dy}{dx}=\frac{x}{3}\frac{du}{dx}-x^3y^3$$

$$\frac{x}{3}\frac{du}{dx}=2x^3-3$$

$$\frac{du}{dx}=6x^2-9/x$$

Integrate

$$u=2x^3-9\ln x+c$$

$$x^3y^3=2x^3-9\ln x+c$$

Example 17: $\cos(x+y)dy=dx$

Solution: $\cos(x+y)dy=dx$

put $x+y=v$ or $1+\frac{dy}{dx}=\frac{dv}{dx}$, we get

$$\cos v[\frac{dv}{dx}-1]=1$$

$$dx=\frac{\cos v}{1+\cos v}dv=[1-\frac{1}{1+\cos v}]dv$$

$$dx=[1-\frac{1}{2}\sec^2\frac{v}{2}]dv$$

Integrate

$$x+c=v-\tan\frac{v}{2}$$

$$x+c=v-\tan\frac{x+y}{2}$$

Lecture 10 Applications of First Order Differential Equations

In order to translate a physical phenomenon in terms of mathematics, we strive for a set of equations that describe the system adequately. This set of equations is called a **Model** for the phenomenon. The basic steps in building such a model consist of the following steps:

Step 1: We clearly state the assumptions on which the model will be based. These assumptions should describe the relationships among the quantities to be studied.

Step 2: Completely describe the parameters and variables to be used in the model.

Step 3: Use the assumptions (from Step 1) to derive mathematical equations relating the parameters and variables (from Step 2).

The mathematical models for physical phenomenon often lead to a differential equation or a set of differential equations. The applications of the differential equations we will discuss in next two lectures include:

- Orthogonal Trajectories.
- Population dynamics.
- Radioactive decay.
- Newton's Law of cooling.
- Carbon dating.
- Chemical reactions.

etc.

Orthogonal Trajectories

- We know that that the solutions of a 1st order differential equation, e.g. separable equations, may be given by an implicit equation

$$F(x, y, C) = 0$$

with 1 parameter C, which represents a family of curves. Member curves can be obtained by fixing the parameter C. Similarly an n^{th} order DE will yields an n-parameter family of curves/solutions.

$$F(x, y, C_1, C_1, \cdots, C_n) = 0$$

- The question arises that whether or not we can turn the problem around: Starting with an n-parameter family of curves, can we find an associated n^{th} order differential equation free of parameters and representing the family. The answer in most cases is yes.

- Let us try to see, with reference to a 1-parameter family of curves, how to proceed if the answer to the question is yes.

1. Differentiate with respect to x, and get an equation-involving x, y, $\frac{dy}{dx}$ and C.
2. Using the original equation, we may be able to eliminate the parameter C from the new equation.
3. The next step is doing some algebra to rewrite this equation in an explicit form

$$\frac{dy}{dx} = f(x, y)$$

- For illustration we consider an example:

Illustration

Example

Find the differential equation satisfied by the family

$$x^2 + y^2 = C\,x$$

Solution:

1. We differentiate the equation with respect to x, to get

$$2x + 2y\frac{dy}{dx} = C$$

2. Since we have from the original equation that

$$C = \frac{x^2 + y^2}{x}$$

then we get

$$2x + 2y\frac{dy}{dx} = \frac{x^2 + y^2}{x}$$

3. The explicit form of the above differential equation is

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

This last equation is the desired DE free of parameters representing the given family.

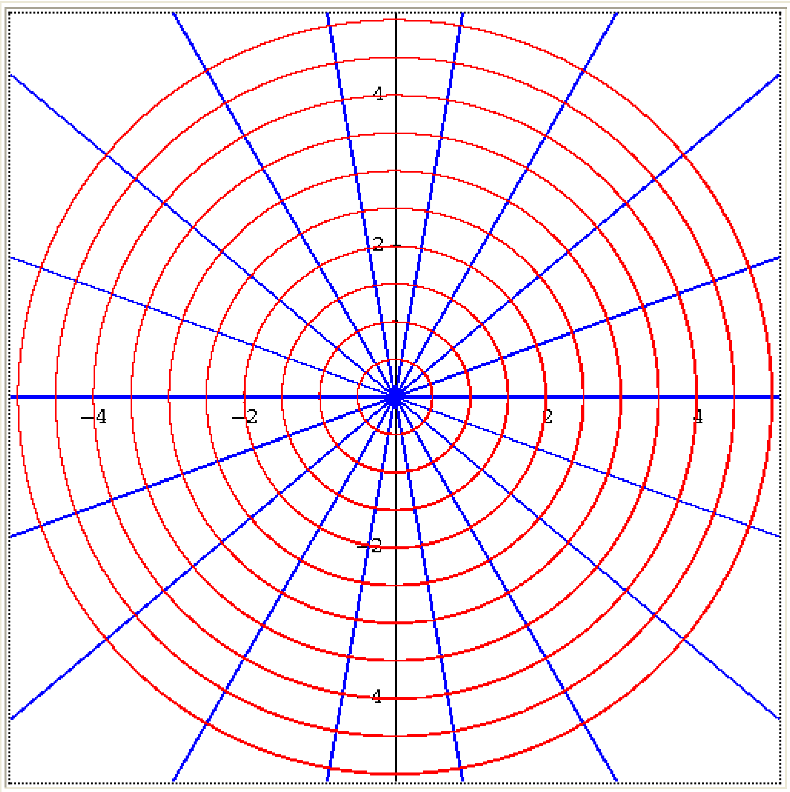
Example.

Let us consider the example of the following two families of curves

$$\begin{cases} y = mx \\ x^2 + y^2 = C^2 \end{cases}$$

The first family describes all the straight lines passing through the origin while the second family describes all the circles centered at the origin

If we draw the two families together on the same graph we get



Clearly whenever one line intersects one circle, the tangent line to the circle (at the point of intersection) and the line are perpendicular i.e. orthogonal to each other. We say that the two families of curves are orthogonal at the point of intersection.

Orthogonal curves:

Any two curves C_1 and C_2 are said to be orthogonal if their tangent lines T_1 and T_2 at their point of intersection are perpendicular. This means that slopes are negative reciprocals of each other, except when T_1 and T_2 are parallel to the coordinate axes.

Orthogonal Trajectories (OT):

When all curves of a family $\Im_1 : G(x, y, c_1) = 0$ orthogonally intersect all curves of another family $\Im_2 : H(x, y, c_2) = 0$ then each curve of the families is said to be orthogonal trajectory of the other.

Example:

As we can see from the previous figure that the family of straight lines $y = mx$ and the family of circles $x^2 + y^2 = C^2$ are orthogonal trajectories.

Orthogonal trajectories occur naturally in many areas of physics, fluid dynamics, in the study of electricity and magnetism etc. For example the lines of force are perpendicular to the equipotential curves i.e. curves of constant potential.

Method of finding Orthogonal Trajectory:

Consider a family of curves $\Im$. Assume that an associated DE may be found, which is given by**:**

$$\frac{dy}{dx} = f(x, y)$$

Since $\frac{dy}{dx}$ gives slope of the tangent to a curve of the family $\Im$ through (x, y).

Therefore, the slope of the line orthogonal to this tangent is $-\frac{1}{f(x, y)}$. So that the slope of the line that is tangent to the orthogonal curve through (x, y) is given by $-\frac{1}{f(x, y)}$. In other words, the family of orthogonal curves are solutions to the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x, y)}$$

The steps can be summarized as follows:

Summary:

In order to find Orthogonal Trajectories of a family of curves $\Im$ we perform the following steps:

Step 1. Consider a family of curves $\Im$ and find the associated differential equation.
Step 2. Rewrite this differential equation in the explicit form

$$\frac{dy}{dx} = f(x, y)$$

Step 3. Write down the differential equation associated to the orthogonal family

$$\frac{dy}{dx} = -\frac{1}{f(x, y)}$$

Step 4. Solve the new equation. The solutions are exactly the family of orthogonal curves.
Step 5. A specific curve from the orthogonal family may be required, something like an IVP.

Example 1
Find the orthogonal Trajectory to the family of circles

$$x^2 + y^2 = C^2$$

Solution:
The given equation represents a family of concentric circles centered at the origin.
Step 1. We differentiate w.r.t. 'x' to find the DE satisfied by the circles.

$$2y\frac{dy}{dx} + 2x = 0$$

Step 2. We rewrite this equation in the explicit form

$$\frac{dy}{dx} = -\frac{x}{y}$$

Step 3. Next we write down the DE for the orthogonal family

$$\frac{dy}{dx} = -\frac{1}{-(x/y)} = \frac{y}{x}$$

Step 4.This is a linear as well as a separable DE. Using the technique of linear equation, we find the integrating factor

$$u(x) = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$$

which gives the solution

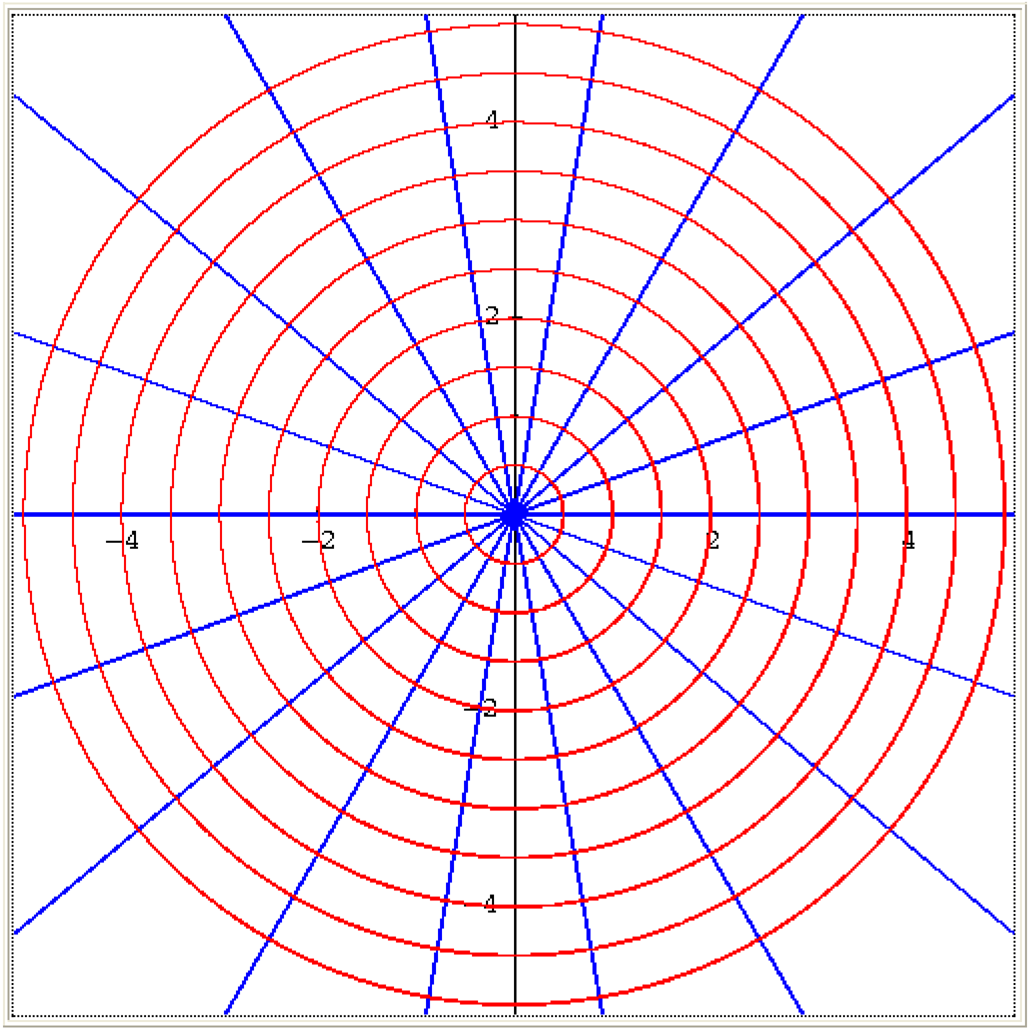
$$y\,.u(x) = m$$

or

$$y = \frac{m}{u(x)} = mx$$

Which represent a family of straight lines through origin. Hence the family of straight lines $y = mx$ and the family of circles $x^2 + y^2 = C^2$ are Orthogonal Trajectories.

Step 5. A geometrical view of these Orthogonal Trajectories is:



Example 2

Find the Orthogonal Trajectory to the family of circles

$$x^2 + y^2 = 2Cx$$

Solution:

1. We differentiate the given equation to find the DE satisfied by the circles.

$$y\frac{dy}{dx} + x = C, \qquad C = \frac{x^2 + y^2}{2x}$$

2. The explicit differential equation associated to the family of circles is

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

3. Hence the differential equation for the orthogonal family is

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

4. This DE is a homogeneous, to solve this equation we substitute $v = y/x$ or equivalently $y = vx$. Then we have

$$\frac{dy}{dx} = x\frac{dv}{dx} + v \quad \text{and} \quad \frac{2xy}{x^2 - y^2} = \frac{2v}{1-v^2}$$

Therefore the homogeneous differential equation in step 3 becomes

$$x\frac{dv}{dx} + v = \frac{2v}{1-v^2}$$

Algebraic manipulations reduce this equation to the separable form:

$$\frac{dv}{dx} = \frac{1}{x}\left\{\frac{v+v^3}{1-v^2}\right\}$$

The constant solutions are given by

$$v + v^3 = 0 \Rightarrow v(1+v^2) = 0$$

The only constant solution is $v = 0$.

To find the non-constant solutions we separate the variables

$$\frac{1-v^2}{v+v^3}dv = \frac{1}{x}dx$$

Integrate

$$\int \frac{1-v^2}{v+v^3}\, d\,v = \int \frac{1}{x} d\,x$$

Resolving into partial fractions the integrand on LHS, we obtain

$$\frac{1-v^2}{v+v^3} = \frac{1-v^2}{v(1+v^2)} = \frac{1}{v} - \frac{2v}{1+v^2}$$

Hence we have

$$\int \frac{1-v^2}{v+v^3}\, d\,v = \int \left\{\frac{1}{v} - \frac{2v}{1+v^2}\right\} d\,v = \ln|v| - \ln[v^2+1]$$

Hence the solution of the separable equation becomes

$$\ln|v| - \ln[v^2+1] = \ln|x| + \ln C$$

which is equivalent to

$$\frac{v}{v^2+1} = C\,x$$

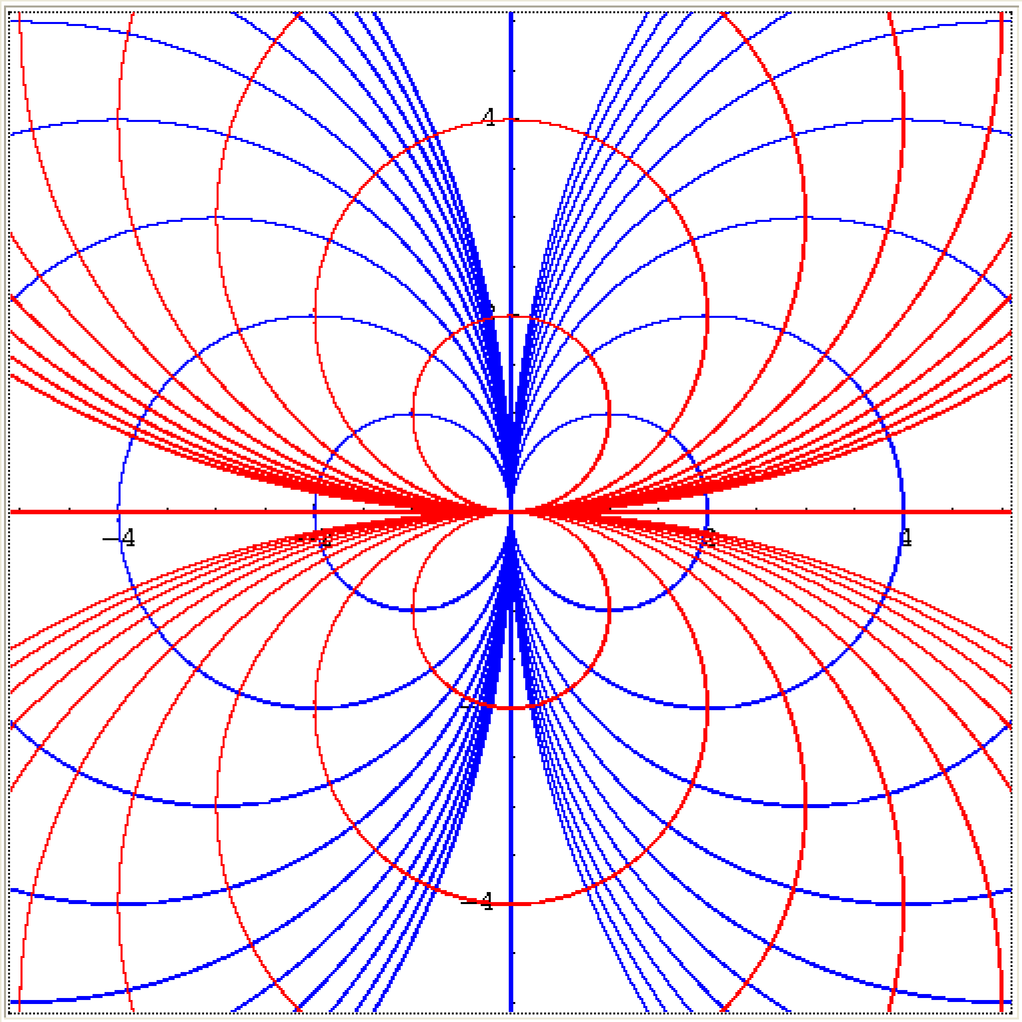
where $C \neq 0$. Hence all the solutions are

$$\begin{cases} v & = & 0 \\ \dfrac{v}{v^2+1} & = & Cx \end{cases}$$

We go back to y to get $y = 0$ and $\dfrac{y}{y^2 + x^2} = C$ which is equivalent to

$$\begin{cases} y & = & 0 \\ x^2 + y^2 & = & my \end{cases}$$

5. Which is x-axis and a family of circles centered on y-axis. A geometrical view of both the families is shown in the next slide.



Population Dynamics

Some natural questions related to population problems are the following:

- What will the population of a certain country after e.g. ten years?
- How are we protecting the resources from extinction?

The easiest population dynamics model is the **exponential model**. This model is based on the assumption:

The rate of change of the population is proportional to the existing population.

If $P(t)$ measures the population of a species at any time t then because of the above mentioned assumption we can write

$$\frac{dP}{dt} = kP$$

where the rate k is constant of proportionality. Clearly the above equation is linear as well as separable. To solve this equation we multiply the equation with the integrating factor e^{-kt} to obtain

$$\frac{d}{dt}\left[P\,e^{-kt}\right] = 0$$

Integrating both sides we obtain

$$P\,e^{-kt} = C \quad \text{or} \quad P = C\,e^{kt}$$

If P_0 is the initial population then $P(0) = P_0$. So that $C = P_0$ and obtain

$$P(t) = P_0\,e^{kt}$$

Clearly, we must have $k > 0$ for growth and $k < 0$ for the decay.

Illustration

Example:

The population of a certain community is known to increase at a rate proportional to the number of people present at any time. The population has doubled in 5 years, how long would it take to triple?. If it is known that the population of the community is 10,000 after 3 years. What was the initial population? What will be the population in 30 years?

Solution:

Suppose that P_0 is initial population of the community and $P(t)$ the population at any time t then the population growth is governed by the differential equation

$$\frac{dP}{dt} = kP$$

As we know solution of the differential equation is given by

$$P(t) = P_0\,e^{kt}$$

Since $P(5) = 2P_0$. Therefore, from the last equation we have

$$2P_0 = P_0\,e^{5k} \Rightarrow e^{5k} = 2$$

This means that

$$5k = \ln 2 = 0.69315 \quad \text{or} \quad k = \frac{0.69315}{5} = 0.13863$$

Therefore, the solution of the equation becomes

$$P(t) = P_0\, e^{0.13863t}$$

If t_1 is the time taken for the population to triple then

$$3P_0 = P_0\, e^{0.1386t_1} \Rightarrow e^{0.1386t_1} = 3$$

$$t_1 = \frac{\ln 3}{0.1386} = 7.9265 \approx 8 \text{ years}$$

Now using the information $P(3) = 10{,}000$, we obtain from the solution that

$$10{,}000 = P_0\, e^{(0.13863)(3)} \Rightarrow P_0 = \frac{10{,}000}{e^{0.41589}}$$

Therefore, the initial population of the community was

$$P_0 \approx 6598$$

Hence solution of the model is

$$P(t) = 6598e^{0.13863t}$$

So that the population in 30 years is given by

$$P(30) = 6598e^{(30)(0.13863)} = 6598e^{4.1589}$$

or $\quad P(30) = (6598)(64.0011)$

or $\quad P(30) \approx 422279$

Lecture 11 Radioactive Decay

In physics a radioactive substance disintegrates or transmutes into the atoms of another element. Many radioactive materials disintegrate at a rate proportional to the amount present. Therefore, if $A(t)$ is the amount of a radioactive substance present at time t, then the rate of change of $A(t)$ with respect to time t is given by

$$\frac{dA}{dt} = kA$$

where k is a constant of proportionality. Let the initial amount of the material be A_0 then $A(0) = A_0$. As discussed in the population growth model the solution of the differential equation is

$$A(t) = A_0 e^{kt}$$

The constant k can be determined using half-life of the radioactive material.

The half-life of a radioactive substance is the time it takes for one-half of the atoms in an initial amount A_0 to disintegrate or transmute into atoms of another element. The half-life measures stability of a radioactive substance. The longer the half-life of a substance, the more stable it is. If T denotes the half-life then

$$A(T) = \frac{A_0}{2}$$

Therefore, using this condition and the solution of the model we obtain

$$\frac{A_0}{2} = A_0 e^{kt}$$

So that $$kT = -\ln 2$$

Therefore, if we know T, we can get k and vice-versa. The half-life of some important radioactive materials is given in many textbooks of Physics and Chemistry. For example the half-life of $C-14$ is 5568 ± 30 years.

Example 1:

A radioactive isotope has a half-life of 16 days. We have 30 g at the end of 30 days. How much radioisotope was initially present?

Solution: Let $A(t)$ be the amount present at time t and A_0 the initial amount of the isotope. Then we have to solve the initial value problem.

$$\frac{dA}{dt} = kA, \qquad A(30) = 30$$

We know that the solution of the IVP is given by

$$A(t) = A_0 e^{kt}$$

If T the half-life then the constant is given k by

$$kT = -\ln 2 \quad \text{or} \quad k = -\frac{\ln 2}{T} = -\frac{\ln 2}{16}$$

Now using the condition $A(30) = 30$, we have

$$30 = A_0 e^{30k}$$

So that the initial amount is given by

$$A_0 = 30e^{-30k} = 30e^{\frac{30\ln 2}{16}} = 110.04\,\text{g}$$

Example 2:

A breeder reactor converts the relatively stable uranium 238 into the isotope plutonium 239. After 15 years it is determined that 0.043% of the initial amount A_0 of the plutonium has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.

Solution:

Let $A(t)$ denotes the amount remaining at any time t, then we need to find solution to the initial value problem

$$\frac{dA}{dt} = kA, \qquad A(0) = A_0$$

which we know is given by

$$A(t) = A_0 e^{kt}$$

If 0.043% disintegration of the atoms of A_0 means that 99.957% of the substance remains. Further 99.957% of A_0 equals $(0.99957)A_0$. So that

$$A(15) = (0.99957)A_0$$

So that

$$A_0 e^{15k} = (0.99957)A_0$$

$$15k = \ln(0.99957)$$

Or $$k = \frac{\ln(0.99957)}{15} = -0.00002867$$

Hence $$A(t) = A_0 e^{-0.00002867\,t}$$

If T denotes the half-life then $A(T) = \frac{A_0}{2}$. Thus

$$\frac{A_0}{2} = A_0 e^{-0.00002867\ T} \quad \text{or} \quad \frac{1}{2} = e^{-0.00002867\ T}$$

$$-0.00002867\ \mathrm{T} = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$T = \frac{\ln 2}{0.00002867} \approx 24{,}180\ \text{years}$$

Newton's Law of Cooling

From experimental observations it is known that the temperature ***T (t)*** of an object changes at a rate proportional to the difference between the temperature in the body and the temperature $\boldsymbol{T_m}$ of the surrounding environment. This is what is known as **Newton's law of cooling**.

If initial temperature of the cooling body is T_0 then we obtain the initial value problem

$$\frac{dT}{dt} = k(T - T_m), \quad T(0) = T_0$$

where k is constant of proportionality. The differential equation in the problem is linear as well as separable.

Separating the variables and integrating we obtain

$$\int \frac{dT}{T - T_m} = \int k\, dt$$

This means that

$$\ln|T - T_m| = kt + C$$

$$T - T_m = e^{kt + C}$$

$$T(t) = T_m + C_1 e^{kt} \quad \text{where} \quad C_1 = e^C$$

Now applying the initial condition $T(0) = T_0$, we see that $C_1 = T_0 - T_m$. Thus the solution of the initial value problem is given by

$$T(t) = T_m + (T_0 - T_m)e^{kt}$$

Hence, If temperatures at times t_1 and t_2 are known then we have

$$T(t_1) - T_m = (T_0 - T_m)e^{kt_1}\ ,\ T(t_2) - T_m = (T_0 - T_m)e^{kt_2}$$

So that we can write

$$\frac{T(t_1) - T_m}{T(t_2) - T_m} = e^{k(t_1 - t_2)}$$

This equation provides the value of k if the interval of time '$t_1 - t_2$' is known and vice-versa.

Example 3:

Suppose that a dead body was discovered at midnight in a room when its temperature was *80 ^{o}F*. The temperature of the room is kept constant at *60 ^{o}F* . Two hours later the temperature of the body dropped to $75^o\ F$. Find the time of death.

Solution:

Assume that the dead person was not sick, then

$$T(0) = 98.6^o F = T_0 \text{ and } T_m = 60^o F$$

Therefore, we have to solve the initial value problem

$$\frac{dT}{dt} = k(T - 60), \quad T(0) = 98.6$$

We know that the solution of the initial value problem is

$$T(t) = T_m + (T_0 - T_m)e^{kt}$$

So that

$$\frac{T(t_1) - T_m}{T(t_2) - T_m} = e^{k(t_1 - t_2)}$$

The observed temperatures of the cooling object, i.e. the dead body, are

$$T(t_1) = 80^o F \quad \text{and} \quad T(t_2) = 75^o F$$

Substituting these values we obtain

$$\frac{80 - 60}{75 - 60} = e^{2k} \quad \text{as} \quad t_1 - t_2 = 2 \quad \text{hours}$$

So

$$k = \frac{1}{2}\ln\frac{4}{3} = 0.1438$$

Now suppose that t_1 and t_2 denote the times of death and discovery of the dead body then

$$T(t_1) = T(0) = 98.6^o F \quad \text{and} \quad T(t_2) = 80^o F$$

For the time of death, we need to determine the interval $t_1 - t_2 = t_d$. Now

$$\frac{T(t_1) - T_m}{T(t_2) - T_m} = e^{k(t_1 - t_2)} \Rightarrow \frac{98.6 - 60}{80 - 60} = e^{kt_d}$$

or

$$t_d = \frac{1}{k}\ln\frac{38.6}{20} \approx 4.573$$

Hence the time of death is 7:42 PM.

Carbon Dating

- The isotope *C–14* is produced in the atmosphere by the action of cosmic radiation on nitrogen.
- The ratio of C-14 to ordinary carbon in the atmosphere appears to be constant.
- The proportionate amount of the isotope in all living organisms is same as that in the atmosphere.
- When an organism dies, the absorption of $C-14$ by breathing or eating ceases.
- Thus comparison of the proportionate amount of $C-14$ present, say, in a fossil with constant ratio found in the atmosphere provides a reasonable estimate of its age.
- The method has been used to date wooden furniture in Egyptian tombs.
- Since the method is based on the knowledge of half-life of the radio active $C-14$ (5600 years approximately), the initial value problem discussed in the radioactivity model governs this analysis.

Example:

A fossilized bone is found to contain $1/1000$ of the original amount of *C–14*. Determine the age of the fissile.

Solution:

Let *A(t)* be the amount present at any time ***t*** and A_0 the original amount of C–14. Therefore, the process is governed by the initial value problem.

$$\frac{dA}{dt} = kA, \quad A(0) = A_0$$

We know that the solution of the problem is

$$A(t) = A_0 e^{kt}$$

Since the half life of the carbon isotope is 5600 years. Therefore,

$$A(5600) = \frac{A_0}{2}$$

So that

$$\frac{A_0}{2} = A_0 e^{5600k} \quad \text{or} \quad 5600k = -\ln 2$$

$$k = -0.00012378$$

Hence

$$A(t) = A_0 e^{-(0.00012378)t}$$

If t denotes the time when fossilized bone was found then $A(t) = \frac{A_0}{1000}$

$$\frac{A_0}{1000} = A_0 e^{-(0.00012378)t} \Rightarrow -0.00012378t = -\ln 1000$$

Therefore

$$t = \frac{\ln 1000}{0.00012378} = 55{,}800 \text{ years}$$

Lecture 12 Application of Non Linear Equations

As we know that the solution of the exponential model for the population growth is

$$P(t) = P_0\, e^{kt}$$

P_0 being the initial population. From this solution we conclude that

(a) If $k > 0$ the population grows and expand to infinity i.e. $\lim_{t \to \infty} P(t) = +\infty$

(b) If $k < 0$ the population will shrink to approach 0, which means extinction.

Note that:

(1) The prediction in the first case ($k > 0$) differs substantially from what is actually observed, population growth is eventually limited by some factor**!**

(2) Detrimental effects on the environment such as pollution and excessive and competitive demands for food and fuel etc. can have inhibitive effects on the population growth.

Logistic equation:

Another model was proposed to remedy this flaw in the exponential model. This is called the **logistic model** (also called **Verhulst-Pearl model**).

Suppose that $a > 0$ is constant average rate of birth and that the death rate is proportional to the population $P(t)$ at any time t. Thus if $\frac{1}{P}\frac{dP}{dt}$ is the rate of growth per individual then

$$\frac{1}{P}\frac{dP}{dt} = a - bP \quad \text{or} \quad \frac{dP}{dt} = P(a - bP)$$

where b is constant of proportionality. The term $-bP^2,\ b > 0$ can be interpreted as inhibition term. When $b = 0$, the equation reduces to the one in exponential model.
Solution to the logistic equation is also very important in ecological, sociological and even in managerial sciences.

Solution of the Logistic equation:

The logistic equation

$$\frac{dP}{dt} = P(a - bP)$$

can be easily identified as a **nonlinear** equation that is separable. The constant solutions of the equation are given by

$$P(a - bP) = 0$$

$$\Rightarrow \quad \mathrm{P} = 0 \quad \text{and} \quad P = \frac{a}{b}$$

For non-constant solutions we separate the variables

$$\frac{dP}{P(a-bP)} = dt$$

Resolving into partial fractions we have

$$\left[\frac{1/a}{P} + \frac{b/a}{a-bP}\right]dP = dt$$

Integrating $$\frac{1}{a}\ln|P| - \frac{1}{a}\ln|a-bP| = t + C$$

$$\ln\left|\frac{P}{a-bP}\right| = at + aC$$

or $$\frac{P}{a-bP} = C_1 e^{at} \quad \text{where } C_1 = e^{aC}$$

Easy algebraic manipulations give

$$P(t) = \frac{aC_1 e^{at}}{1 + bC_1 e^{at}} = \frac{aC_1}{bC_1 + e^{-at}}$$

Here C_1 is an arbitrary constant. If we are given the initial condition $P(0) = P_0$, $P_0 \neq \frac{a}{b}$ we obtain $C_1 = \frac{P_0}{a - bP_0}$. Substituting this value in the last equation and simplifying, we obtain

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}$$

Clearly $$\lim_{t\to\infty} P(t) = \frac{aP_0}{bP_0} = \frac{a}{b},$$ **limited growth**

Note that $P = \frac{a}{b}$ is a **singular solution** of the logistic equation.

Special Cases of Logistic Equation:

1. Epidemic Spread

Suppose that one person infected from a contagious disease is introduced in a fixed population of n people.

The natural assumption is that the rate $\frac{dx}{dt}$ of spread of disease is proportional to the number $x(t)$ of the infected people and number $y(t)$ of people not infected people. Then

$$\frac{dx}{dt} = kxy$$

Since $$x + y = n + 1$$

Therefore, we have the following initial value problem

$$\frac{dx}{dt} = kx(n+1-x), \quad x(0) = 1$$

The last equation is a **special case of the logistic equation** and has also been used for the **spread of information** and the **impact of advertising** in centers of population.

2. A Modification of LE:

A modification of the nonlinear logistic differential equation is the following

$$\frac{dP}{dt} = P(a - b\ln P)$$

has been used in the studies of solid tumors, in actuarial predictions, and in the growth of revenue from the sale of a commercial product in addition to growth or decline of population.

Example:

Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days *x(4) =50*.

Solution

Assume that no one leaves the campus throughout the duration of the disease. We must solve the initial-value problem

$$\frac{dx}{dt} = kx(1000 - x), \quad x(0) = 1 \quad .$$

We identify

$$a = 1000k \text{ and } \mathrm{b} = \mathrm{k}$$

Since the solution of logistic equation is

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}$$

Therefore we have

$$x(t) = \frac{1000\,k}{k + 999\,ke^{-1000\,kt}} = \frac{1000}{1 + 999\,e^{-1000\,kt}}.$$

Now, using x(4)= 50, we determine *k*

$$50 = \frac{1000}{1 + 999\,e^{-4000\,k}}$$

We find

$$k = \frac{-1}{4000}\ln\frac{19}{999} = 0.0009906.$$

Thus

$$x(t) = \frac{1000}{1 + 999\,e^{-0.9906\,t}}$$

Finally

$$x(6) = \frac{1000}{1 + 999\,e^{-5.9436}} = 276 \text{ students} \quad .$$

Chemical reactions:

In a first order chemical reaction, the molecules of a substance *A* decompose into smaller molecules. This decomposition takes place at a rate proportional to the amount of the first substance that has not undergone conversion. The disintegration of a radioactive substance is an example of the first order reaction. If X is the remaining amount of the substance A at any time t then

$$\frac{dX}{dt} = k\,X$$

$k < 0$ because X is decreasing.

In a 2nd order reaction two chemicals A and B react to form another chemical C at a rate proportional to the product of the remaining concentrations of the two chemicals.

If X denotes the amount of the chemical C that has formed at time t. Then the instantaneous amounts of the first two chemicals A and B not converted to the chemical C are $\alpha - X$ and $\beta - X$, respectively. Hence the rate of formation of chemical C is given by

$$\frac{dX}{dt} = k\left(\alpha - X\right)\left(\beta - X\right)$$

where k is constant of proportionality.

Example:

A compound $\boldsymbol{C}$ is formed when two chemicals A and B are combined. The resulting reaction between the two chemicals is such that for each gram of A, 4 grams of B are used. It is observed that 30 grams of the compound C are formed in 10 minutes. Determine the amount of C at any time if the rate of reaction is proportional to the amounts of A and B remaining and if initially there are 50 grams of A and 32 grams of B. How much of the compound C is present at 15 minutes? Interpret the solution as $t \to \infty$

Solution:

If $X(t)$ denote the number of grams of chemical $\boldsymbol{C}$ present at any time t. Then

$$X(0) = 0 \text{ and } X(10) = 30$$

Suppose that there are 2 grams of the compound $\boldsymbol{C}$ and we have used a grams of A and b grams of B then

$$a + b = 2 \qquad \text{and} \qquad b = 4a$$

Solving the two equations we have

$$a = \frac{2}{5} = 2\,(1/5) \quad \text{and} \quad b = \frac{8}{5} = 2\,(4/5)$$

In general, if there were for X grams of C then we must have

$$a = \frac{X}{5} \quad \text{and} \quad b = \frac{4}{5}X$$

Therefore the amounts of A and B remaining at any time t are then

$$50 - \frac{X}{5} \text{ and } 32 - \frac{4}{5}X$$

respectively .

Therefore, the rate at which chemical C is formed satisfies the differential equation

$$\frac{dX}{dt} = \lambda\left(50 - \frac{X}{5}\right)\left(32 - \frac{4}{5}X\right)$$

or

$$\frac{dX}{dt} = k(250 - X)(40 - X), \quad k = 4\lambda/25$$

We now solve this differential equation.

By separation of variables and partial fraction, we can write

$$\frac{dX}{(250 - X)(40 - X)} = kdt$$

$$-\frac{1/210}{250 - X}dX + \frac{1/210}{40 - X}dX = kdt$$

$$\ln\left|\frac{250 - X}{40 - X}\right| = 210kt + c_1$$

$$\frac{250 - X}{40 - X} = c_2 e^{210kt} \quad Where \; c_2 = e^{c_1}$$

When $t = 0$, $X = 0$, so it follows at this point that $c_2 = 25/4$. Using $X = 30$ at $t = 10$, we find

$$210k = \frac{1}{10}\ln\frac{88}{25} = 0.1258$$

With this information we solve for X :

$$X(t) = 1000\left(\frac{1 - e^{-0.1258t}}{25 - 4e^{-0.1258t}}\right)$$

It is clear that as $e^{-0.1258t} \to 0$ as $t \to \infty$. Therefore $X \to 40$ as $t \to \infty$. This fact can also be verified from the following table that $X \to 40$ as $t \to \infty$.

t	10	15	20	25	30	35
X	30	34.78	37.25	38.54	39.22	39.59

This means that there are 40 grams of compound C formed, leaving

$$50 - \frac{1}{5}(40) = 42 \quad \text{grams of chemical } A$$

and $$32-\frac{4}{5}(40)=0 \quad \text{grams of chemical } B$$

Miscellaneous Applications

- The velocity v of a falling mass m, subjected to air resistance proportional to instantaneous velocity, is given by the differential equation

$$m\frac{dv}{dx}=mg-kv$$

Here $k>0$ is constant of proportionality.

- The rate at which a drug disseminates into bloodstream is governed by the differential equation

$$\frac{dx}{dt}=A-Bx$$

Here $A,\ B$ are positive constants and $x(t)$ describes the concentration of drug in the bloodstream at any time t.

- The rate of memorization of a subject is given by

$$\frac{dA}{dt}=k_1(M-A)-k_2A$$

Here $k_1 > 0,\ k_2 > 0$ and $A(t)$ is the amount of material memorized in time t, M is the total amount to be memorized and $M - A$ is the amount remaining to be memorized.

Lecture 13 Higher Order Linear Differential Equations

Preliminary theory

- A differential equation of the form

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

  or $\quad a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_1(x)y'+a_0(x)y=g(x)$

  where $a_0(x), a_1(x), \ldots, a_n(x), g(x)$ are functions of x and $a_n(x)\neq 0$, is called a linear differential equation with variable coefficients.
- However, we shall first study the differential equations with constant coefficients i.e. equations of the type

$$a_n\frac{d^n y}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0 y=g(x)$$

  where $a_0, a_1, \ldots, a_n$ are real constants. This equation is non-homogeneous differential equation and
- If $g(x)=0$ then the differential equation becomes

$$a_n\frac{d^n y}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0 y=0$$

  which is known as the **associated homogeneous differential equation**.

Initial -Value Problem

For a linear nth-order differential equation, the problem:

Solve: $a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$

Subject to: $\quad y(x_0)=y_0, \quad y^{/}(x_0)=y_0^{/}, \ldots y^{n-1}(x_0)=y_0^{n-1}$

$y_0, y_0^{/}, \ldots, y_0^{n-1}$ being arbitrary constants, is called an **initial-value problem** (IVP).

The specified values $y(x_0)=y_0$, $y^{/}(x_0)=y_0^{/}, \ldots, y^{n-1}(x_0)=y_0^{n-1}$ are called initial-conditions.

For $n=2$ the initial-value problem reduces to

Solve: $\quad a_2(x)\frac{d^2 y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$

Subject to: $\quad y(x_0)=y_0, \ldots, \quad y^{/}(x_0)=y_0^{/}$

Solution of IVP

A function satisfying the differential equation on I whose graph passes through (x_0, y_0) such that the slope of the curve at the point is the number $y_0^{/}$ is called solution of the initial value problem.

Theorem: *Existence and Uniqueness of Solutions*

Let $a_n(x), a_{n-1}(x),..., a_1(x), a_0(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0, \ \forall\, x \in I$. If $x = x_0 \in I$, then a solution $y(x)$ of the initial-value problem exist on I and is unique.

Example 1

Consider the function $y = 3e^{2x} + e^{-2x} - 3x$

This is a solution to the following initial value problem

$$y'' - 4y = 12x, \quad y(0) = 4, \ y'(0) = 1$$

Since $$\frac{d^2y}{dx^2} = 12e^{2x} + 4e^{-2x}$$

and $$\frac{d^2y}{dx^2} - 4y = 12e^{2x} + 4e^{-2x} - 12e^{2x} - 4e^{-2x} + 12x = 12x$$

Further $y(0) = 3 + 1 - 0 = 4$ and $y'(0) = 6 - 2 - 3 = 1$

Hence $y = 3e^{2x} + e^{-2x} - 3x$

is a solution of the initial value problem.

We observe that

- The equation is linear differential equation.
- The coefficients being constant are continuous.
- The function $g(x) = 12x$ being polynomial is continuous.
- The leading coefficient $a_2(x) = 1 \neq 0$ for all values of x.

Hence the function $y = 3e^{2x} + e^{-2x} - 3x$ is the unique solution.

Example 2

Consider the initial-value problem

$$3y''' + 5y'' - y' + 7y = 0,$$
$$y(1) = 0, \quad y'(1) = 0, \quad y''(1) = 0$$

Clearly the problem possesses the trivial solution $y = 0$.

Since

- The equation is homogeneous linear differential equation.
- The coefficients of the equation are constants.
- Being constant the coefficient are continuous.
- The leading coefficient $a_3 = 3 \neq 0$.

Hence $y = 0$ is the only solution of the initial value problem.

Note: If $a_n = 0$?

If $a_n(x) = 0$ in the differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

for some $x \in I$ then

- Solution of initial-value problem may not be unique.
- Solution of initial-value problem may not even exist.

Example 4

Consider the function

$$y = cx^2 + x + 3$$

and the initial-value problem

$$x^2 y'' - 2xy' + 2y = 6$$

$$y(0) = 3, \qquad y'(0) = 1$$

Then $\quad y' = 2cx + 1$ and $y'' = 2c$

Therefore $\quad x^2 y'' - 2xy' + 2y = x^2(2c) - 2x(2cx+1) + 2(cx^2 + x + 3)$

$$= 2cx^2 - 4cx^2 - 2x + 2cx^2 + 2x + 6$$

$$= 6.$$

Also $\quad y(0) = 3 \Rightarrow c(0) + 0 + 3 = 3$

and $\quad y'(0) = 1 \Rightarrow 2c(0) + 1 = 1$

So that for any choice of c, the function $'y'$ satisfies the differential equation and the initial conditions. Hence the solution of the initial value problem is not unique.

Note that

- The equation is linear differential equation.
- The coefficients being polynomials are continuous everywhere.
- The function $g(x)$ being constant is constant everywhere.
- The leading coefficient $a_2(x) = x^2 = 0$ at $x = 0 \in (-\infty, \infty)$.

Hence $a_2(x) = 0$ brought non-uniqueness in the solution

Boundary-value problem (BVP)

For a 2nd order linear differential equation, the problem

Solve: $a_2(x)\dfrac{d^2y}{dx^2}+a_1(x)\dfrac{dy}{dx}+a_0(x)y=g(x)$

Subject to: $y(a)=y_0,\qquad y(b)=y_1$

is called a ***boundary-value problem***. The specified values $y(a)=y_0$, and $y(b)=y_1$ are called ***boundary conditions***.

Solution of BVP

A solution of the boundary value problem is a function satisfying the differential equation on some interval I, containing a and b, whose graph passes through two points (a, y_0) and (b, y_1).

Example 5
Consider the function

$$y=3x^2-6x+3$$

We can prove that this function is a solution of the boundary-value problem

$$x^2y''-2xy'+2y=6,$$
$$y(1)=0,\qquad y(2)=3$$

Since $\dfrac{dy}{dx}=6x-6,\ \dfrac{d^2y}{dx^2}=6$

Therefore $x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=6x^2-12x^2+12x+6x^2-12x+6=6$

Also $y(1)=3-6+3=0,\quad y(2)=12-12+3=3$

Therefore, the function 'y' satisfies both the differential equation and the boundary conditions. Hence y is a solution of the boundary value problem.

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Possible Boundary Conditions
For a 2nd order linear non-homogeneous differential equation

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

all the possible pairs of boundary conditions are

$$y(a)=y_0,\qquad y(b)=y_1,$$
$$y'(a)=y'_0,\qquad y(b)=y_1,$$
$$y(a)=y_0,\qquad y'(b)=y'_1,$$
$$y'(a)=y'_0,\qquad y'(b)=y'_1$$

where y_0, y'_0, y_1 and y'_1 denote the arbitrary constants.

In General

All the four pairs of conditions mentioned above are just special cases of the general boundary conditions

$$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1$$
$$\alpha_2 y(b) + \beta_2 y'(b) = \gamma_2$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \{0,1\}$

Note that

A boundary value problem may have

- Several solutions.
- A unique solution, or
- No solution at all.

Example 1

Consider the function

$$y = c_1 \cos 4x + c_2 \sin 4x$$

and the boundary value problem

$$y'' + 16y = 0, \quad y(0) = 0, \quad y(\pi/2) = 0$$

Then

$$y' = -4c_1 \sin 4x + 4c_2 \cos 4x$$
$$y'' = -16(c_1 \cos 4x + c_2 \sin 4x)$$
$$y'' = -16y$$
$$y'' + 16y = 0$$

Therefore, the function

$$y = c_1 \cos 4x + c_2 \sin 4x$$

satisfies the differential equation

$$y'' + 16y = 0.$$

Now apply the boundary conditions

Applying $y(0) = 0$

We obtain

$$0 = c_1 \cos 0 + c_2 \sin 0$$
$$\Rightarrow c_1 = 0$$

So that

$$y = c_2 \sin 4x.$$

But when we apply the 2nd condition $y(\pi/2) = 0$, we have

$$0 = c_2 \sin 2\pi$$

Since $\sin 2\pi = 0$, the condition is satisfied for any choice of c_2, solution of the problem is the one-parameter family of functions

$$y = c_2 \sin 4x$$

Hence, there are an ***infinite number of solutions*** of the boundary value problem.

Example 2

Solve the boundary value problem

$$y^{//} + 16y = 0$$

$$y(0) = 0, \ \ y\left(\frac{\pi}{8}\right) = 0,$$

Solution:

As verified in the previous example that the function

$$y = c_1 \cos 4x + c_2 \sin 4x$$

satisfies the differential equation

$$y^{//} + 16y = 0$$

We now apply the boundary conditions

$$y(0) = 0 \Rightarrow 0 = c_1 + 0$$

and $\quad y(\pi/8) = 0 \Rightarrow 0 = 0 + c_2$

So that $\quad c_1 = 0 = c_2$

Hence

$$y = 0$$

is the only solution of the boundary-value problem.

Example 3

Solve the differential equation

$$y^{//} + 16y = 0$$

subject to the boundary conditions

$$y(0) = 0, \quad y(\pi/2) = 1$$

Solution:

As verified in an earlier example that the function

$$y = c_1 \cos 4x + c_2 \sin 4x$$

satisfies the differential equation

$$y^{//} + 16y = 0$$

We now apply the boundary conditions

$$y(0) = 0 \Rightarrow 0 = c_1 + 0$$

Therefore $\quad c_1 = 0$

So that $\quad y = c_2 \sin 4x$

However $\quad y(\pi/2) = 1 \Rightarrow c_2 \sin 2\pi = 1$

or $\quad 1 = c_2 .0 \Rightarrow 1 = 0$

This is a clear contradiction. Therefore, the boundary value problem has ***no solution***.

Definition: **Linear Dependence**

A set of functions

$$\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

is said to be ***linearly dependent*** on an interval I if $\exists$ constants $c_1, c_2, \ldots, c_n$ not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots . + c_n f_n(x) = 0, \quad \forall x \in I$$

Definition: Linear Independence

A set of functions

$$\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

is said to be linearly independent on an interval I if

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0, \quad \forall x \in I,$$

only when

$$c_1 = c_2 = \cdots = c_n = 0.$$

Case of two functions:

If $n = 2$ then the set of functions becomes

$$\{f_1(x), f_2(x)\}$$

If we suppose that

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

Also that the functions are linearly dependent on an interval I then either $c_1 \neq 0$ or $c_2 \neq 0$.

Let us assume that $c_1 \neq 0$, then

$$f_1(x) = -\frac{c_2}{c_1} f_2(x);$$

Hence $f_1(x)$ is the constant multiple of $f_2(x)$.

Conversely, if we suppose

$$f_1(x) = \mathrm{c}_2 \; f_2(x)$$

Then $$(-1) f_1(x) + c_2 f_2(x) = 0, \; \forall x \in I$$

So that the functions are linearly dependent because $c_1 = -1$.

Hence, we conclude that:

- Any two functions $f_1(x)$ and $f_2(x)$ are linearly dependent on an interval I if and only if one is the constant multiple of the other.
- Any two functions are linearly independent when neither is a constant multiple of the other on an interval I.
- In general a set of n functions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ is linearly dependent if at least one of them can be expressed as a linear combination of the remaining.

Example 1

The functions

$$f_1(x) = \sin 2x, \qquad \forall x \in (-\infty, \infty)$$

$$f_2(x) = \sin x \cos x, \quad \forall x \in (-\infty, \infty)$$

If we choose $c_1 = \frac{1}{2}$ and $c_2 = -1$ then

$$c_1 \sin 2x + c_2 \sin x \cos x = \frac{1}{2}(2\sin x \cos x) - \sin x \cos x = 0$$

Hence, the two functions $f_1(x)$ and $f_2(x)$ are linearly dependent.

Example 3

Consider the functions

$$f_1(x) = \cos^2 x, \; f_2(x) = \sin^2 x, \quad \forall x \in (-\pi/2, \pi/2),$$

$$f_3(x) = \sec^2 x, \; f_4(x) = \tan^2 x, \quad \forall x \in (-\pi/2, \pi/2)$$

If we choose $c_1 = c_2 = 1, c_3 = -1, c_4 = 1$, then

$$\begin{aligned}
&c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + c_4 f_4(x) \\
&= c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x \\
&= \cos^2 x + \sin^2 x + -1 - \tan^2 x + \tan^2 x \\
&= 1 - 1 + 0 = 0
\end{aligned}$$

Therefore, the given functions are linearly dependent.

Note that

The function $f_3(x)$ can be written as a linear combination of other three functions $f_1(x), f_2(x)$ and $f_4(x)$ because $\sec^2 x = \cos^2 x + \sin^2 x + \tan^2 x$.

Example 3

Consider the functions

$$f_1(x) = 1 + x, \ \forall\, x \in (-\infty, \infty)$$
$$f_2(x) = x, \quad \forall\, x \in (-\infty, \infty)$$
$$f_3(x) = x^2, \quad \forall\, x \in (-\infty, \infty)$$

Then

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

means that

$$c_1(1+x) + c_2 x + c_3 x^2 \qquad = 0$$

or $\quad c_1 + (c_1 + c_2)x + c_3 x^2 \qquad = 0$

Equating coefficients of x and x^2 constant terms we obtain

$$c_1 = 0 = c_3$$
$$c_1 + c_2 = 0$$

Therefore $\quad c_1 = c_2 = c_3 = 0$

Hence, the three functions $f_1(x), f_2(x)$ and $f_3(x)$ are linearly independent.

Definition: Wronskian

Suppose that the function $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n-1$ derivatives then the determinant

$$\begin{vmatrix} f_1 & f_2 & \cdots\cdots & f_n \\ f_1^{/} & f_2^{/} & \cdots\cdots & f_n^{/} \\ \vdots & \vdots & & \vdots \\ f_1^{n-1} & f_2^{n-1} & \cdots\cdots & f_n^{n-1} \end{vmatrix}$$

is called Wronskian of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ and is denoted by $W\left(f_1(x), f_1(x), \ldots, f_1(x)\right)$.

Theorem: Criterion for Linearly Independent Functions

Suppose the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possess at least *n-1* derivatives on an interval I. If

$$W(f_1(x), f_2(x), \ldots, f_n(x)) \neq 0$$

for at least one point in I, then functions $f_1(x), f_2(x), \ldots, f_n(x)$ are linearly independent on the interval I.

Note that

This is only a sufficient condition for linear independence of a set of functions.

In other words

If $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n-1$ derivatives on an interval and are linearly dependent on I, then

$$W(f_1(x),\ f_2(x),\dots,\ f_n(x)) = 0, \quad \forall x \in I$$

However, the converse is not true. i.e. a Vanishing Wronskian does not guarantee linear dependence of functions.

Example 1

The functions

$$f_1(x) = \sin^2 x$$
$$f_2(x) = 1 - \cos 2x$$

are linearly dependent because

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

We observe that for all $x \in (-\infty, \infty)$

$$W(f_1(x), f_2(x)) = \begin{vmatrix} \sin^2 x & 1 - \cos 2x \\ 2\sin x \cos x & 2\sin 2x \end{vmatrix}$$

$$= 2\sin^2 x \sin 2x - 2\sin x \cos x$$
$$+ 2\sin x \cos x \cos 2x$$
$$= \sin 2x\,[2\sin^2 x - 1 + \cos 2x]$$
$$= \sin 2x\,[2\sin^2 x - 1 + \cos^2 x - \sin^2 x]$$
$$= \sin 2x\,[\sin^2 x + \cos^2 x - 1]$$
$$= 0$$

Example 2

Consider the functions

$$f_1(x) = e^{m_1 x},\ f_2(x) = e^{m_2 x}, \quad m_1 \neq m_2$$

The functions are linearly independent because

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

if and only if $\quad c_1 = 0 = c_2 \quad$ as $\quad m_1 \neq m_2$

Now for all $x \in R$

$$W\left(e^{m_1 x}, e^{m_2 x}\right) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$$

$$= (m_2 - m_1) e^{(m_1 + m_2)x}$$

$$\neq 0$$

Thus f_1 and f_2 are linearly independent of any interval on x-axis.

Example 3

If α and β are real numbers, $\beta \neq 0$, then the functions

$$y_1 = e^{\alpha x} \cos \beta x \text{ and } y_2 = e^{\alpha x} \sin \beta x$$

are linearly independent on any interval of the x-axis because

$$W\left(e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x\right)$$

$$= \begin{vmatrix} e^{\alpha x} \cos \beta x & e^{\alpha x} \sin \beta x \\ -\beta e^{\alpha x} \sin \beta x + \alpha e^{\alpha x} \cos \beta x & \beta e^{\alpha x} \cos \beta x + \alpha e^{\alpha x} \sin \beta x \end{vmatrix}$$

$$= \beta e^{2\alpha x}\left(\cos^2 \beta x + \sin^2 \beta x\right) = \beta e^{2\alpha x} \neq 0.$$

Example 4

The functions

$$f_1(x) = e^x,\ f_2(x) = xe^x, \text{ and } f_3(x) = x^2 e^x$$

are linearly independent on any interval of the x-axis because for all $x \in R$, we have

$$W\left(e^x, xe^x, x^2 e^x\right) = \begin{vmatrix} e^x & xe^x & x^2 e^x \\ e^x & xe^x + e^x & x^2 e^x + 2xe^x \\ e^x & xe^x + 2e^x & x^2 e^x + 4xe^x + 2e^x \end{vmatrix}$$

$$= 2e^{3x} \neq 0$$

Exercise

1. Given that

$$y = c_1 e^{x} + c_2 e^{-x}$$

is a two-parameter family of solutions of the differential equation

$$y'' - y = 0$$

on $(-\infty, \infty)$, find a member of the family satisfying the boundary conditions

$$y(0) = 0, \quad y'(1) = 1.$$

2. Given that

$$y = c_1 + c_2 \cos x + c_3 \sin x$$

is a three-parameter family of solutions of the differential equation

$$y''' + y' = 0$$

on the interval $(-\infty, \infty)$, find a member of the family satisfying the initial conditions $y(\pi) = 0,\ y'(\pi) = 2, y''(\pi) = -1$.

3. Given that

$$y = c_1 x + c_2 x \ln x$$

is a two-parameter family of solutions of the differential equation $x^2 y'' - xy' + y = 0$ on $(-\infty, \infty)$. Find a member of the family satisfying the initial conditions

$$y(1) = 3,\ y'(1) = -1.$$

Determine whether the functions in problems 4-7 are linearly independent or dependent on $(-\infty, \infty)$.

4. $f_1(x) = x, \quad f_2(x) = x^2, \quad f_3(x) = 4x - 3x^2$
5. $f_1(x) = 0, \quad f_2(x) = x, \ f_3(x) = e^{x}$
6. $f_1(x) = \cos 2x, \quad f_2(x) = 1, \quad f_3(x) = \cos^2 x$
7. $f_1(x) = e^{x}, \quad f_2(x) = e^{-x}, \quad f_3(x) = \sinh x$

Show by computing the Wronskian that the given functions are linearly independent on the indicated interval.

8. $\tan x,\ \cot x; \quad (-\infty, \infty)$
9. $e^{x},\ e^{-x},\ e^{4x}; \quad (-\infty, \infty)$
10. $x, x \ln x, x^2 \ln x; \quad (0, \infty)$

Lecture 14 Solutions of Higher Order Linear Equations

Preliminary Theory

- In order to solve an *n*th order non-homogeneous linear differential equation
$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$
we first solve the associated homogeneous differential equation
$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

- Therefore, we first concentrate upon the preliminary theory and the methods of solving the homogeneous linear differential equation.

- We recall that a function $y=f(x)$ that satisfies the associated homogeneous equation
$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$
is called solution of the differential equation.

Superposition Principle

Suppose that $y_1, y_2, \ldots, y_n$ are solutions on an interval I of the homogeneous linear differential equation

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

Then

$$y=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x),$$

$c_1, c_2, \ldots, c_n$ being arbitrary constants is also a solution of the differential equation.

Note that

- A constant multiple $y=c_1y_1(x)$ of a solution $y_1(x)$ of the homogeneous linear differential equation is also a solution of the equation.
- The homogeneous linear differential equations always possess the trivial solution $y=0$.
- The superposition principle is a property of linear differential equations and it does not hold in case of non-linear differential equations.

Example 1

The functions

$$y_1 = e^x, y_2 = c^{2x}, \text{ and } y_3 = e^{3x}$$

all satisfy the homogeneous differential equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$$

on $(-\infty, \infty)$. Thus y_1, y_2 and y_3 are all solutions of the differential equation

Now suppose that

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}.$$

Then

$$\frac{dy}{dx} = c_1e^x + 2c_2e^{2x} + 3c_3e^{3x}.$$

$$\frac{d^2y}{dx^2} = c_1e^x + 4c_2e^{2x} + 9c_3e^{3x}.$$

$$\frac{d^3y}{dx^3} = c_1e^x + 8c_2e^{2x} + 27c_3e^{3x}.$$

Therefore

$$\begin{aligned}
&\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y \\
&= c_1\left(e^x - 6e^x + 11e^x - 6e^x\right) + c_2\left(8e^{2x} - 24e^{2x} + 22e^{2x} - 6e^{2x}\right) \\
&\quad + c_3\left(27e^{3x} - 54e^{3x} + 33e^{3x} - 6e^{3x}\right) \\
&= c_1(12-12)e^x + c_2(30-30)e^{2x} + c_3(60-60)e^{3x} \\
&= 0
\end{aligned}$$

Thus $$y = c_1e^x + c_2e^{2x} + c_3e^{3x}.$$

is also a solution of the differential equation.

Example 2

The function

$$y = x^2$$

is a solution of the homogeneous linear equation

$$x^2 y'' - 3xy' + 4y = 0$$

on $(0, \infty)$.

Now consider

$$y = cx^2$$

Then $\quad y' = 2cx \quad$ and $\quad y'' = 2c$

So that $\quad x^2 y'' - 3xy' + 4y = 2cx^2 - 6cx^2 + 4cx^2 = 0$

Hence the function

$$y = cx^2$$

is also a solution of the given differential equation.

The Wronskian

Suppose that y_1, y_2 are 2 solutions, on an interval I, of the second order homogeneous linear differential equation

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

Then either $\quad W(y_1, y_2) = 0, \quad \forall\, x \in I$

or $\quad W(y_1, y_2) \neq 0, \quad \forall\, x \in I$

To verify this we write the equation as

$$\frac{d^2y}{dx^2} + \frac{Pdy}{dx} + Qy = 0$$

Now $\quad W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$

Differentiating *w.r.to* x, we have

$$\frac{dW}{dx} = y_1 y_2'' - y_1'' y_2$$

Since y_1 and y_2 are solutions of the differential equation

$$\frac{d^2y}{dx^2} + \frac{Pdy}{dx} + Qy = 0$$

Therefore

$$y_1'' + Py_1' + Qy_1 = 0$$

$$y_2'' + Py_2' + Qy_2 = 0$$

Multiplying 1st equation by y_2 and 2nd by y_1 the have

$$y_1''y_2 + Py_1'y_2 + Qy_1y_2 = 0$$

$$y_1y_2'' + Py_1y_2' + Qy_1y_2 = 0$$

Subtracting the two equations we have:

$$\left(y_1y_2'' - y_2y_1''\right) + P\left(y_1y_2' - y_1'y_2\right) = 0$$

or
$$\frac{dW}{dx} + PW = 0$$

This is a linear 1st order differential equation in W, whose solution is

$$W = ce^{-\int P dx}$$

Therefore

- If $c \neq 0$ then $W\left(y_1, y_2\right) \neq 0, \quad \forall\, x \in I$
- If $c = 0$ then $W\left(y_1, y_2\right) = 0, \quad \forall\, x \in I$

Hence Wronskian of y_1 and y_2 is either identically zero or is never zero on I.

In general

If $y_1, y_2, \ldots, y_n$ are n solutions, on an interval I, of the homogeneous *n*th order linear differential equation with constants coefficients

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = 0$$

Then

Either
$$W\left(y_1, y_2, \ldots, y_n\right) = 0, \quad \forall x \in I$$

or
$$W\left(y_1, y_2, \ldots, y_n\right) \neq 0, \quad \forall x \in I$$

Linear Independence of Solutions:

Suppose that

$$y_1, y_2, \ldots, y_n$$

are n solutions, on an interval I, of the homogeneous linear nth-order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Then the set of solutions is linearly independent on I if and only if

$$W(y_{1,} y_2, \ldots, y_n) \neq 0$$

In other words

The solutions

$$y_1, y_2, \ldots, y_n$$

are linearly dependent if and only if

$$W(y_{1,} y_2, \ldots, y_n) = 0, \quad \forall x \in I$$

Fundamental Set of Solutions

A set

$$\{y_1, y_2, \ldots, y_n\}$$

of n linearly independent solutions, on interval I, of the homogeneous linear nth-order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

is said to be a fundamental set of solutions on the interval I.

Existence of a Fundamental Set

There always exists a fundamental set of solutions for a linear nth-order homogeneous differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on an interval I.

General Solution-Homogeneous Equations

Suppose that

$$\{y_1, y_2, \ldots, y_n\}$$

is a fundamental set of solutions, on an interval I, of the homogeneous linear nth-order differential equation

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

Then the general solution of the equation on the interval I is defined to be

$$y=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)$$

Here $c_1,c_2,\ldots,c_n$ are arbitrary constants.

Example 1

The functions

$$y_1=e^{3x} \text{ and } y_2=e^{-3x}$$

are solutions of the differential equation

$$y''-9y=0$$

Since $W\left(e^{3x},e^{-3x}\right)=\begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x}\end{vmatrix}=-6\neq 0,\quad \forall x\in I$

Therefore y_1 and y_2 from a fundamental set of solutions on $(-\infty,\infty)$. **Hence** general solution of the differential equation on the $(-\infty,\infty)$ is

$$y=c_1e^{3x}+c_2e^{-3x}$$

Example 2

Consider the function $y=4\sinh 3x-5e^{-3x}$

Then $y'=12\cosh 3x+15e^{-3x}$, $\quad y''=36\sinh 3x-45e^{-3x}$

$\Rightarrow \quad y''=9\left(4\sinh 3x-5e^{-3x}\right)$ or $y''=9y$,

Therefore $y''-9y=0$

Hence $y=4\sinh 3x-5e^{-3x}$

is a particular solution of differential equation.

$$y''-9y=0$$

The general solution of the differential equation is

$$y=c_1e^{3x}+c_2e^{-3x}$$

Choosing $c_1=2, c_2=-7$

We obtain $y = 2e^{3x} - 7e^{-3x}$

$$y = 2e^{3x} - 2e^{-3x} - 5e^{-3x}$$

$$y = 4\left(\frac{e^{3x} - e^{-3x}}{2}\right) - 5e^{-3x}$$

$$y = 4\sinh 3x - 5e^{-3x}$$

Hence, the particular solution has been obtained from the general solution.

Example 3

Consider the differential equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$$

and suppose that $y_1 = e^x,\ y_2 = e^{2x}$ and $y_3 = e^{3x}$

Then $\dfrac{dy_1}{dx} = e^x = \dfrac{d^2y_1}{dx^2} = \dfrac{d^3y_1}{dx^3}$

Therefore $\dfrac{d^3y_1}{dx^3} - 6\dfrac{d^2y_1}{dx^2} + 11\dfrac{dy_1}{dx} - 6y_1 = e^x - 6e^x + 11e^x - 6e^x$

or $\dfrac{d^3y_1}{dx^3} - 6\dfrac{d^2y_1}{dx^2} + 11\dfrac{dy_1}{dx} - 6y_1 = 12e^x - 12e^x = 0$

Thus the function y_1 is a solution of the differential equation. Similarly, we can verify that the other two functions i.e. y_2 and y_3 also satisfy the differential equation.

Now for all $x \in R$

$$W\left(e^x, e^{2x}, e^{3x}\right) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0 \quad \forall\, x \in I$$

Therefore $y_1, y_{2,}$ and y_3 form a fundamental solution of the differential equation on $(-\infty, \infty)$. We conclude that

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}$$

is the general solution of the differential equation on the interval $(-\infty, \infty)$.

Non-Homogeneous Equations

A function y_p that satisfies the non-homogeneous differential equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

and is free of parameters is called the particular solution of the differential equation

Example 1

Suppose that

$$y_p = 3$$

Then $$y_p'' = 0$$

So that

$$y_p'' + 9y_p = 0 + 9(3)$$
$$= 27$$

Therefore

$$y_p = 3$$

is a particular solution of the differential equation

$$y_p'' + 9y_p = 27$$

Example 2

Suppose that

$$y_p = x^3 - x$$

Then $$y_p' = 3x^2 - 1, \quad y_p'' = 6x$$

Therefore $$x^2 y_p'' + 2xy_p' - 8y_p = x^2(6x) + 2x\left(3x^2 - 1\right) - 8\left(x^3 - x\right)$$
$$= 4x^3 + 6x$$

Therefore

$$y_p = x^3 - x$$

is a particular solution of the differential equation

$$x^2 y'' + 2xy' - 8y = 4x^3 + 6x$$

Complementary Function

The general solution

$$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

of the homogeneous linear differential equation

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

is known as the complementary function for the non-homogeneous linear differential equation.

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

General Solution of Non-Homogeneous Equations

Suppose that

- The particular solution of the non-homogeneous equation

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

  is y_p.

- The complementary function of the non-homogeneous differential equation

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

  is

$$y_c=c_1y_1+c_2y_2+\cdots+c_ny_n.$$

- Then general solution of the non-homogeneous equation on the interval I is given by

$$y=y_c+y_p$$

  or

$$y=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)+y_p(x)=y_c(x)+y_p(x)$$

Hence

General Solution = Complementary solution + any particular solution.

Example

Suppose that

$$y_p=-\frac{11}{12}-\frac{1}{2}x$$

Then $\quad y'_p=-\frac{1}{2},\ y''_p=0=y'''_p$

$\therefore \qquad \frac{d^3y_p}{dx^3}-6\frac{d^2y_p}{dx^2}+11\frac{dy_p}{dx}-6y_p=0-0-\frac{11}{2}+\frac{11}{2}+3x=3x$

Hence

$$y_p=-\frac{11}{12}-\frac{1}{2}x$$

is a particular solution of the non-homogeneous equation

$$\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y=3x$$

Now consider

$$y_c=c_1e^x+c_2e^{2x}+c_3e^{3x}$$

Then

$$\frac{dy_c}{dx}=c_1e^x+2c_2e^{2x}+3c_3e^{3x}$$
$$\frac{d^2y_c}{dx^2}=c_1e^x+4c_2e^{2x}+9c_3e^{3x}$$
$$\frac{d^3y_c}{dx^3}=c_1e^x+8c_2e^{2x}+27c_3e^{3x}$$

Since,

$$\frac{d^3y_c}{dx^3}-6\frac{d^2y_c}{dx^2}+11\frac{dy_c}{dx}-6y_c$$
$$=c_1e^x+8c_2e^{2x}+27c_3e^{3x}-6\left(c_1e^x+4c_2e^{2x}+9c_3e^{3x}\right)$$
$$+11\left(c_1e^x+2c_2e^{2x}+3c_3e^{3x}\right)-6\left(c_1e^x+c_2e^{2x}+c_3e^{3x}\right)$$
$$=12c_1e^x-12c_1e^x+30c_2e^{2x}-30c_2e^{2x}+60c_3e^{3x}-60c_3e^{3x}$$
$$=0$$

Thus y_c is general solution of associated homogeneous differential equation

$$\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y=0$$

Hence general solution of the non-homogeneous equation is

$$y=y_c+y_p=c_1e^x+c_2e^{2x}+c_3e^{3x}-\frac{11}{12}-\frac{1}{2}x$$

Superposition Principle for Non-homogeneous Equations

Suppose that

$$y_{p_1}, y_{p_2}, \ldots, y_{p_k}$$

denote the particular solutions of the k differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_i(x),$$

$i = 1,2,\ldots k$, on an interval I. Then

$$y_p = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x)$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x)$$

Example

Consider the differential equation

$$y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^x - e^x$$

Suppose that

$$y_{p_1} = -4x^2, \quad y_{p_2} = e^{2x}, \quad y_{p_3} = xe^x$$

Then $\quad y''_{p_1} - 3y'_{p_1} + 4y_{p_1} = -8 + 24x - 16x^2$

Therefore $\quad y_{p_1} = -4x^2$

is a particular solution of the non-homogenous differential equation

$$y'' - 3y' + 4y = -16x^2 + 24x - 8$$

Similarly, it can be verified that

$$y_{p_2} = e^{2x} \text{ and } y_{p_3} = xe^x$$

are particular solutions of the equations:

$$y'' - 3y' + 4y = 2e^{2x}$$

and $\quad y''\text{-}3y' + 4y = 2xe^x - e^x$

respectively.

Hence $\quad y_p = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^x$

is a particular solution of the differential equation

$$y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^x - e^x$$

Exercise

Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Form the general solution.

11. $y'' - y' - 12y = 0; \quad e^{-3x}, e^{4x}, \quad (-\infty, \infty)$

12. $y''-2y'+5y=0;\quad e^{x}\cos 2x, e^{x}\sin 2x,\quad (-\infty,\infty)$

13. $x^2y''+xy'+y=0;\quad \cos(\ln x), \sin(\ln x),\quad (0,\infty)$

14. $4y''-4y'+y=0;\quad e^{x/2}, xe^{x/2},\quad (-\infty,\infty)$

15. $x^2y''-6xy'+12y=0;\quad x^3,\ x^4\quad (0,\infty)$

16. $y''-4y=0;\quad \cosh 2x,\ \sinh 2x,\quad (-\infty,\infty)$

Verify that the given two-parameter family of functions is the general solution of the non-homogeneous differential equation on the indicated interval.

17. $y''+y=\sec x,\quad y=c_1\cos x+c_2\sin x+x\sin x+(\cos x)\ln(\cos x),\quad (-\pi/2,\pi/2)$.

18. $y''-4y'+4y=2e^{2x}+4x-12,\quad y=c_1e^{2x}+c_2xe^{2x}+x^2e^{2x}+x-2$

19. $y''-7y'+10y=24e^{x},\quad y=c_1e^{2x}+c_2e^{5x}+6e^{x},\quad (-\infty,\infty)$

20. $x^2y''+5xy'+y=x^2-x,\quad y=c_1x^{-1/2}+c_2x^{-1}+\frac{1}{15}x^2-\frac{1}{6}x,\quad (0,\infty)$

Lecture 15 Construction of a Second Solution

General Case

Consider the differential equation

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

We divide by $a_2(x)$ to put the above equation in the form

$$y''+P(x)y'+Q(x)y=0$$

Where $P(x)$ and $Q(x)$ are continuous on some interval I .

Suppose that $y_1(x)\neq 0,\ \forall\, x\in \mathrm{I}$ is a solution of the differential equation

Then $$y_1''+P\,y_1'+Q\,y_1=0$$

We define $y=u(x)y_1\ (x)$ then

$$y'=uy_1'+y_1u',\qquad y''=uy_1''+2y_1'u_1'+y_1u''$$

$$y''+Py'+Qy=u[\underbrace{y_1''+Py_1'+Qy_1}_{zero}]+y_1u''+(2y_1'+Py_1)u'=0$$

This implies that we must have

$$y_1u''+(2y_1'+Py_1)u'=0$$

If we suppose $w=u'$, then

$$y_1w'+(2y_1'+Py_1)w=0$$

The equation is separable. Separating variables we have from the last equation

$$.\frac{dw}{w}+(2\frac{y_1'}{y_1}+P)dx=0$$

Integrating

$$\ln|w|+2\ln|y_1|=-\int Pdx+c$$

$$\ln\left|wy_1^2\right|=-\int Pdx+c$$

$$wy_1^2=c_1e^{-\int Pdx}$$

$$w=\frac{c_1e^{-\int Pdx}dx}{y_1^2}$$

or $$u' = \frac{c_1 e^{-\int P dx}}{{y_1}^2}$$

Integrating again, we obtain

$$u = c_1 \int \frac{e^{-\int P dx}}{{y_1}^2} dx + c_2$$

Hence $$y = u(x) y_1(x) = c_1 y_1(x) \int \frac{e^{-\int P dx}}{{y_1}^2} dx + c_2 y_1(x).$$

Choosing $c_1 = 1$ and $c_2 = 0$, we obtain a second solution of the differential equation

$$y_2 = y_1(x) \int \frac{e^{-\int P dx}}{{y_1}^2} dx$$

The Woolskin

$$W(y_1(x), y_2(x)) = \begin{vmatrix} y_1 & y_1 \int \frac{e^{-\int P dx}}{{y_1}^2} dx \\ y_1' & \frac{e^{-\int P dx}}{y_1} + y_1' \int \frac{e^{-\int P dx}}{{y_1}^2} dx \end{vmatrix}$$

$$= e^{-\int P dx} \neq 0, \forall x$$

Therefore $y_1(x)$ and $y_2(x)$ are linear independent set of solutions. So that they form a fundamental set of solutions of the differential equation

$$y'' + P(x) y' + Q(x) y = 0$$

Hence the general solution of the differential equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Example 1

Given that

$$y_1 = x^2$$

is a solution of

$$x^2 y'' - 3xy' + 4y = 0$$

Find general solution of the differential equation on the interval $(0,\infty)$.

Solution:

The equation can be written as

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0,$$

The 2nd solution y_2 is given by

$$y_2 = y_1(x)\int \frac{e^{-\int P dx}}{y_1^2}dx$$

or $$y_2 = x^2\int \frac{e^{3\int dx/x}}{x^4}dx = x^2\int \frac{e^{\ln x^3}}{x^4}dx$$

$$y_2 = x^2\int \frac{1}{x}dx = x^2 \ln x$$

Hence the general solution of the differential equation on $(0,\infty)$ is given by

$$y = c_1\, y_1 + c_2 y_2$$

or $$y = c_1\, x^2 + c_2 x^2 \ln x$$

Example 2

Verify that

$$y_1 = \frac{\sin x}{\sqrt{x}}$$

is a solution of

$$x^2 y'' + xy' + (x^2 - 1/4)y = 0$$

on $(0, \pi)$. Find a second solution of the equation.

Solution:

The differential equation can be written as

$$y'' + \frac{1}{x}y' + (1 - \frac{1}{4x^2})y = 0$$

The 2nd solution is given by

$$y_2 = y_1 \int \frac{e^{-\int P dx}}{{y_1}^2} dx$$

Therefore
$$y_2 = \frac{\sin x}{\sqrt{x}} \int \frac{e^{-\int \frac{dx}{x}}}{(\frac{\sin x}{\sqrt{x}})^2} dx$$

$$= \frac{-\sin x}{\sqrt{x}} \int \frac{x}{x \sin^2 x} dx$$

$$= \frac{-\sin x}{\sqrt{x}} \int \csc^2 x dx$$

$$= \frac{-\sin x}{\sqrt{x}}(-\cot x) = \frac{\cos x}{\sqrt{x}}$$

Thus the second solution is

$$y_2 = \frac{\cos x}{\sqrt{x}}$$

Hence, general solution of the differential equation is

$$y = c_1\left(\frac{\sin x}{\sqrt{x}}\right) + c_2\left(\frac{\cos x}{\sqrt{x}}\right)$$

Order Reduction

Example 3

Given that

$$y_1 = x^3$$

is a solution of the differential equation

$$x^2 y'' - 6y = 0,$$

Find second solution of the equation

Solution

We write the given equation as:

$$y'' - \frac{6}{x^2} y = 0$$

So that $$P(x) = -\frac{6}{x^2}$$

Therefore

$$y_2 = y_1 \int \frac{e^{-\int P dx}}{y_1^{\,2}} dx$$

$$y_2 = x^3 \int \frac{e^{-\int \frac{6}{x^2}}}{x^6} dx$$

$$y_2 = x^3 \int \frac{e^{\frac{6}{x}}}{x^6} dx$$

Therefore, using the formula

$$y_2 = y_1 \int \frac{e^{-\int P dx}}{y_1^{\,2}} dx$$

We encounter an integral that is difficult or impossible to evaluate.

Hence, we conclude sometimes use of the formula to find a second solution is not suitable. We need to try something else.

Alternatively, we can try the reduction of order to find y_2 . For this purpose, we again define

$$y(x) = u(x) y_1(x) \quad \text{or} \quad y = u(x).x^3$$

then

$$y' = 3x^2 u + x^3 u'$$
$$y'' = x^3 u'' + 6x^2 u' + 6xu$$

Substituting the values of y, y'' in the given differential equation

$$x^2 y'' - 6y = 0$$

we have

$$x^2(x^3u'' + 6x^2u' + 6xu) - 6ux^3 = 0$$

or $$x^5u'' + 6x^4u' = 0$$

or $$u'' + \frac{6}{x}u' = 0,$$

If we take $w = u'$ then

$$w' + \frac{6}{x}w = 0$$

This is separable as well as linear first order differential equation in w. For using the latter, we find the integrating factor

$$I.F = e^{6\int \frac{1}{x}dx} = e^{6\ln x} = x^6$$

Multiplying with the $IF = x^6$, we obtain

$$x^6w' + 6x^5w = 0$$

or $$\frac{d}{dx}(x^6w) = 0$$

Integrating w.r.t. ' x ', we have

$$x^6w = c_1$$

or $$u' = \frac{c_1}{x^6}$$

Integrating once again, gives

$$u = -\frac{c_1}{5x^5} + c_2$$

Therefore $$y = ux^3 = \frac{-c_1}{5x^2} + c_2x^3$$

Choosing $c_2 = 0$ and $c_1 = -5$, we obtain

$$y_2 = \frac{1}{x^2}$$

Thus the second solution is given by

$$y_2 = \frac{1}{x^2}$$

Hence, general solution of the given differential equation is

$$y = c_1 y_1 + c_2 y_2$$

i.e.
$$y = c_1 x^3 + c_2 \left(1/x^2\right)$$

Where c_1 and c_2 are constants.

Exercise

Find the 2nd solution of each of Differential equations by reducing order or by using the formula.

1. $y'' - y' = 0; \qquad y_1 = 1$

2. $y'' + 2y' + y = 0; \qquad y_1 = xe^{-x}$

3. $y'' + 9y = 0; \qquad y_1 = \sin x$

4. $y'' - 25y = 0; \qquad y_1 = e^{5x}$

5. $6y'' + y' - y = 0; \qquad y_1 = e^{x/2}$

6. $x^2 y'' + 2xy' - 6y = 0; \qquad y_1 = x^2$

7. $4x^2 y'' + y = 0; \qquad y_1 = x^{1/2} \ln x$

8. $(1 - x^2) y'' - 2xy' = 0; \qquad y_1 = 1$

9. $x^2 y'' - 3xy' + 5y = 0; \qquad y_1 = x^2 \cos(\ln x)$

10. $(1 + x) y'' + xy' - y = 0; \qquad y_1 = x$

Lecture 16 Homogeneous Linear Equations with Constant Coefficients

We know that the linear first order differential equation

$$\frac{dy}{dx} + my = 0$$

m being a constant, has the exponential solution on $(-\infty, \infty)$

$$y = c_1 e^{-mx}$$

The question?

- The question is whether or not the exponential solutions of the higher-order differential equations
$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y^{//} + a_1 y^{/} + a_0 y = 0,$$
exist on $(-\infty, \infty)$.
- In fact all the solutions of this equation are exponential functions or constructed out of exponential functions.

Recall

That the linear differential of order n is an equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Method of Solution

Taking $n = 2$, the nth-order differential equation becomes

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

This equation can be written as

$$a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

We now try a solution of the exponential form

$$y = e^{mx}$$

Then

$$y' = me^{mx} \text{ and } y'' = m^2 e^{mx}$$

Substituting in the differential equation, we have

$$e^{mx}(am^2 + bm + c) = 0$$

Since $e^{mx} \neq 0, \quad \forall x \in (-\infty, \infty)$

Therefore $am^2 + bm + c = 0$

This algebraic equation is known as the Auxiliary equation (AE).The solution of the auxiliary equation determines the solutions of the differential equation.

Case 1: *Distinct Real Roots*

If the auxiliary equation has distinct real roots m_1 and m_2 then we have the following two solutions of the differential equation.

$$y_1 = e^{m_1 x} \text{ and } y_2 = e^{m_2 x}$$

These solutions are linearly independent because

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = (m_2 - m_1)e^{(m_1 + m_2)x}$$

Since $m_1 \neq m_2$ and $e^{(m_1+m_2)x} \neq 0$

Therefore $W(y_1, y_2) \neq 0 \;\; \forall x \in (-\infty, \infty)$

Hence

- y_1 and y_2 form a fundamental set of solutions of the differential equation.
- The general solution of the differential equation on $(-\infty, \infty)$ is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Case 2. *Repeated Roots*

If the auxiliary equation has real and equal roots i.e

$$m = m_1, m_2 \quad \text{with} \quad m_1 = m_2$$

Then we obtain only one exponential solution

$$y = c_1 e^{mx}$$

To construct a second solution we rewrite the equation in the form

$$y'' + \frac{b}{a} y' + \frac{c}{a} y = 0$$

Comparing with $y'' + Py' + Qy = 0$

We make the identification

$$P = \frac{b}{a}$$

Thus a second solution is given by

$$y_2 = y_1 \int \frac{e^{-\int P dx}}{{y_1}^2} dx = e^{mx} \int \frac{e^{-\frac{b}{a}x}}{e^{2mx}} dx$$

Since the auxiliary equation is a quadratic algebraic equation and has equal roots

Therefore, $Disc. = b^2 - 4ac = 0$

We know from the quadratic formula

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we have $2m = -\frac{b}{a}$

Therefore

$$y_2 = e^{mx} \int \frac{e^{2mx}}{e^{2mx}} dx = xe^{mx}$$

Hence the general solution is

$$y = c_1 e^{mx} + c_2 x e^{mx} = (c_1 + c_2 x) e^{mx}$$

Case 3: *Complex Roots*

If the auxiliary equation has complex roots $\alpha \pm i\beta$ then, with

$$m_1 = \alpha + i\beta \text{ and } m_2 = \alpha - i\beta$$

Where α >0 and β >0 are real, the general solution of the differential equation is

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

First we choose the following two pairs of values of c_1 and c_2

$$c_1 = c_2 = 1$$

$$c_1 = 1, c_2 = -1$$

Then we have

$$y_1 = e^{(\alpha + i\beta)x} + e^{(\alpha - i\beta)x}$$
$$y_2 = e^{(\alpha + i\beta)x} - e^{(\alpha - i\beta)x}$$

We know by the Euler's Formula that

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathbb{R}$$

Using this formula, we can simplify the solutions y_1 and y_2 as

$$y_1 = e^{\alpha x}(e^{i\beta x} + e^{-i\beta x}) = 2e^{\alpha x}\cos\beta x$$
$$y_2 = e^{\alpha x}(e^{i\beta x} - e^{-i\beta x}) = 2ie^{\alpha x}\sin\beta x$$

We can drop constant to write

$$y_1 = e^{\alpha x}\cos\beta x\,,\quad y_2 = e^{\alpha x}\sin\beta x$$

The Wronskian

$$W\left(e^{\alpha x}\cos\beta x\,,\ e^{\alpha x}\sin\beta x\right) = \beta e^{2\alpha x} \neq 0 \quad \forall\ x$$

Therefore, $e^{\alpha x}\cos(\beta\, x),\ e^{\alpha x}\sin(\beta\, x)$

form a fundamental set of solutions of the differential equation on $(-\infty,\infty)$.

Hence general solution of the differential equation is

$$y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$$

or $$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

Example:

Solve

$$2y'' - 5y' - 3y = 0$$

Solution:

The given differential equation is

$$2y'' - 5y' - 3y = 0$$

Put $y = e^{mx}$

Then $y' = me^{mx},\ \ y'' = m^2 e^{mx}$

Substituting in the give differential equation, we have

$$\left(2m^2 - 5m - 3\right)e^{mx} = 0$$

Since $e^{mx} \neq 0\ \ \forall\ x$, the auxiliary equation is

$$2m^2 - 5m - 3 = 0 \qquad \text{as} \quad e^{mx} \neq 0$$

$$(2m+1)(m-3) = 0 \Rightarrow m = -\frac{1}{2},\ 3$$

Therefore, the auxiliary equation has distinct real roots

$$m_1 = -\frac{1}{2} \text{ and } m_2 = 3$$

Hence the general solution of the differential equation is

$$y = c_1 e^{(-1/2)x} + c_2 e^{3x}$$

Example 2

Solve $y'' - 10y' + 25y = 0$

Solution:

We put $y = e^{mx}$

Then $y' = me^{mx}, y'' = m^2e^{mx}$

Substituting in the given differential equation, we have

$$(m^2 - 10m + 25)e^{mx} = 0$$

Since $e^{mx} \neq 0 \ \forall \ x$, the auxiliary equation is

$$m^2 - 10m + 25 = 0$$

$$(m-5)^2 = 0 \Rightarrow m = 5, 5$$

Thus the auxiliary equation has repeated real roots i.e

$$m_1 = 5 = m_2$$

Hence general solution of the differential equation is

$$y = c_1e^{5x} + c_2xe^{5x}$$

or $$y = (c_1 + c_2x)e^{5x}$$

Example 3

Solve the initial value problem

$$y'' - 4y' + 13y = 0$$
$$y(0) = -1, \ y'(0) = 2$$

Solution:

Given that the differential equation

$$y'' - 4y' + 13y = 0$$

Put $y = e^{mx}$

Then $y' = me^{mx}, \ y'' = m^2e^{mx}$

Substituting in the given differential equation, we have:

$$(m^2 - 4m + 13)e^{mx} = 0$$

Since $e^{mx} \neq 0 \forall x$, the auxiliary equation is

$$m^2 - 4m + 13 = 0$$

By quadratic formula, the solution of the auxiliary equation is

$$m = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i$$

Thus the auxiliary equation has complex roots

$$m_1 = 2 + 3i, \qquad m_2 = 2 - 3i$$

Hence general solution of the differential equation is

$$y = e^{2x}\left(c_1 \cos 3x + c_2 \sin 3x\right)$$

Example 4

Solve the differential equations

(a) $y'' + k^2 y = 0$

(b) $y'' - k^2 y = 0$

Solution

First consider the differential equation

$$y'' + k^2 y = 0,$$

Put $y = e^{mx}$

Then $y' = me^{mx}$ and $y'' = m^2 e^{mx}$

Substituting in the given differential equation, we have:

$$\left(m^2 + k^2\right) e^{mx} = 0$$

Since $e^{mx} \neq 0 \forall x$, the auxiliary equation is

$$m^2 + k^2 = 0$$

or $m = \pm ki,$

Therefore, the auxiliary equation has complex roots

$$m_1 = 0 + ki, \quad m_2 = 0 - ki$$

Hence general solution of the differential equation is

$$y = c_1 \cos kx + c_2 \sin kx$$

Next consider the differential equation

$$\frac{d^2 y}{dx^2} - k^2 y = 0$$

Substituting values y and y'', we have.

$$\left(m^2 - k^2\right)e^{mx} = 0$$

Since $e^{mx} \neq 0$, the auxiliary equation is

$$m^2 - k^2 = 0$$

$$\Rightarrow \quad m = \pm k$$

Thus the auxiliary equation has distinct real roots

$$m_1 = +k, \; m_2 = -k$$

Hence the general solution is

$$y = c_1 e^{kx} + c_2 e^{-kx}.$$

Higher Order Equations

If we consider nth order homogeneous linear differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = 0$$

Then, the auxiliary equation is an nth degree polynomial equation

$$a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0 = 0$$

Case 1: ***Real distinct roots***

If the roots $m_1, m_2, \ldots, m_n$ of the auxiliary equation are all real and distinct, then the general solution of the equation is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$$

Case 2: *Real & repeated roots*

We suppose that m_1 is a root of multiplicity n of the auxiliary equation, then it can be shown that

$$e^{m_1 x}, xe^{m_1 x}, \ldots, x^{n-1} e^{m_1 x}$$

are n linearly independent solutions of the differential equation. Hence general solution of the differential equation is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + \ldots + c_n x^{n-1} e^{m_1 x}$$

Case 3: *Complex roots*

Suppose that coefficients of the auxiliary equation are real.

- We fix n at 6, all roots of the auxiliary are complex, namely

  $$\alpha_1 \pm i\beta_1, \quad \alpha_2 \pm i\beta_2, \ \alpha_3 \pm i\beta_3$$

  - Then the general solution of the differential equation

$$y = e^{\alpha_1 x}(c_1 \cos\beta_1 x + c_2 \sin\beta_1 x) + e^{\alpha_2 x}(c_3 \cos\beta_2 x + c_4 \sin\beta_2 x)$$
$$+ e^{\alpha_3 x}(c_5 \cos\beta_3 x + c_6 \sin\beta_3 x)$$

- If $n = 6$, two roots of the auxiliary equation are real and equal and the remaining 4 are complex, namely $\alpha_1 \pm i\beta_1, \quad \alpha_2 \pm i\beta_2$

  Then the general solution is

  $$y = e^{\alpha_1 x}(c_1 \cos\beta_1 x + c_2 \sin\beta_1 x) + e^{\alpha_2 x}(c_3 \cos\beta_2 x + c_4 \sin\beta_2 x) + c_5 e^{m_1 x} + c_6 x e^{m_1 x}$$

- If $m_1 = \alpha + i\beta$ is a complex root of multiplicity k of the auxiliary equation. Then its conjugate $m_2 = \alpha - i\beta$ is also a root of multiplicity k. Thus from Case 2 **,** the differential equation has $2k$ solutions

  $$e^{(\alpha+i\beta)x}, \ xe^{(\alpha+i\beta)x}, \ x^2 e^{(\alpha+i\beta)x}, \ldots, x^{k-1} e^{(\alpha+i\beta)x}$$
  $$e^{(\alpha-i\beta)x}, \ xe^{(\alpha-i\beta)x}, \ x^2 e^{(\alpha-i\beta)x}, \ldots, x^{k-1} e^{(\alpha-i\beta)x}$$

- By using the Euler's formula, we conclude that the general solution of the differential equation is a linear combination of the linearly independent solutions

  $$e^{\alpha x}\cos\beta x, \ xe^{\alpha x}\cos\beta x, \ x^2 e^{\alpha x}\cos\beta x, \ldots, x^{k-1} e^{\alpha x}\cos\beta x$$
  $$e^{\alpha x}\sin\beta x, \ xe^{\alpha x}\sin\beta x, \ x^2 e^{\alpha x}\sin\beta x, \ldots, x^{k-1} e^{\alpha x}\sin\beta x$$

- Thus if $k = 3$ then

  $$y = e^{\alpha x}\left[\left(c_1 + c_2 x + c_3 x^2\right)\cos\beta x + \left(d_1 + d_2 x + d_3 x^2\right)\sin\beta x\right]$$

Solving the Auxiliary Equation

Recall that the auxiliary equation of nth degree differential equation is nth degree polynomial equation

- Solving the auxiliary equation could be difficult

  $$P_n(m) = 0, \quad \text{n} > 2$$

- One way to solve this polynomial equation is to guess a root m_1. Then $m - m_1$ is a factor of the polynomial $P_n(m)$.
- Dividing with $m - m_1$ synthetically or otherwise, we find the factorization
$$P_n(m) = (m - m_1)\,Q(m)$$
- We then try to find roots of the quotient i.e. roots of the polynomial equation
$$Q(m) = 0$$
- Note that if $m_1 = \dfrac{p}{q}$ is a rational real root of the equation
$$P_n(m) = 0, \quad \text{n} > 2$$
then p is a factor of a_0 and q of a_n.
- By using this fact we can construct a list of all possible rational roots of the auxiliary equation and test each of them by synthetic division.

Example 1

Solve the differential equation

$$y''' + 3y'' - 4y = 0$$

Solution:

Given the differential equation

$$y''' + 3y'' - 4y = 0$$

Put $y = e^{mx}$

Then $\text{y}^{/} = me^{mx}, y^{//} = m^2 e^{mx}$ and $y^{///} = m^3 e^{mx}$

Substituting this in the given differential equation, we have

$$(m^3 + 3m^2 - 4)e^{mx} = 0$$

Since $e^{mx} \neq 0$

Therefore $m^3 + 3m^2 - 4 = 0$

So that the auxiliary equation is

$$m^3 + 3m^2 - 4 = 0$$

Solution of the AE

If we take $m = 1$ then we see that

$$m^3 + 3m^2 - 4 = 1 + 3 - 4 = 0$$

Therefore $m = 1$ satisfies the auxiliary equations so that m-1 is a factor of the polynomial

$$m^3 + 3m^2 - 4$$

By synthetic division, we can write

$$m^3 + 3m^2 - 4 = (m-1)(m^2 + 4m + 4)$$

or $m^3+3m^2-4=(m-1)(m+2)^2$

Therefore $m^3+3m^2-4=0$

$$\Rightarrow (m-1)(m+2)^2=0$$

or $m=1,-2,-2$

Hence solution of the differential equation is

$$y=c_1e^x+c_2e^{-2x}+c_3xe^{-2x}$$

Example 2

Solve

$$3y^{///}+5y^{//}+10y^{/}-4y=0$$

Solution:

Given the differential equation

$$3y^{///}+5y^{//}+10y^{/}-4y=0$$

Put $y=e^{mx}$

Then $y^{/}=me^{mx}, y^{//}=m^2e^{mx}$ and $y^{///}=m^3e^{mx}$

Therefore the auxiliary equation is

$$3m^3+5m^2+10m-4=0$$

Solution of the auxiliary equation:

a) $a_\circ=-4$ and all its factors are:

$p:\quad \pm1,\pm2,\pm4$

b) $a_n=3$ and all its factors are:

$q:\quad \pm1,\ \pm3$

c) List of possible rational roots of the auxiliary equation is

$\frac{p}{q}:\quad -1, 1, -2, 2, -4, 4, \frac{-1}{3},\frac{1}{3},\frac{-2}{3},\frac{2}{3},\frac{-4}{3},\frac{4}{3}$

d) Testing each of these successively by synthetic division we find

$\frac{1}{3}$	3	5	10	−4
		1	2	4
	3	6	12	0

Consequently a root of the auxiliary equation is

$$m=1/3$$

The coefficients of the quotient are

3 6 12

Thus we can write the auxiliary equation as:

$$(m-1/3)(3m^2+6m+12)=0$$

$$m-\frac{1}{3}=0 \quad \text{or} \quad 3m^2+6m+12=0$$

Therefore $m = 1/3$ or $m = -1 \pm i\sqrt{3}$

Hence solution of the given differential equation is

$$y = c_1 e^{(1/3)x} + e^{-x}\left(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right)$$

Example 3

Solve the differential equation

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$$

Solution:

Given the differential equation

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$$

Put $y = e^{mx}$

Then $y' = me^{mx},\ y'' = m^2 e^{mx}$

Substituting in the differential equation, we obtain

$$\left(m^4 + 2m^2 + 1\right) e^{mx} = 0$$

Since $e^{mx} \neq 0$, the auxiliary equation is

$$m^4 + 2m^2 + 1 = 0$$

$$(m^2 + 1)^2 = 0$$

$$\Rightarrow m = \pm i,\ \pm i$$

$$m_1 = m_3 = i \quad \text{and} \quad m_2 = m_4 = -i$$

Thus i is a root of the auxiliary equation of multiplicity 2 and so is $-i$.

Now $\alpha = 0$ and $\beta = 1$

Hence the general solution of the differential equation is

$$y = e^{0x}\left[(c_1 + c_2 x)\cos x + (d_1 + d_2 x)\sin x\right]$$

or $y = c_1 \cos x + d_1 \sin x + c_2 x \cos x + d_2 x \sin x$

Exercise

Find the general solution of the given differential equations.

1. $y'' - 8y = 0$
2. $y'' - 3y' + 2y = 0$
3. $y'' + 4y' - y = 0$
4. $2y'' - 3y' + 4y = 0$
5. $4y''' + 4y'' + y' = 0$
6. $y''' + 5y'' = 0$
7. $y''' + 3y'' - 4y' - 12y = 0$

Solve the given differential equations subject to the indicated initial conditions.

8. $y''' + 2y'' - 5y' - 6y = 0, \qquad y(0) = y'(0) = 0,\ y''(0) = 1$
9. $\dfrac{d^4y}{dx^4} = 0,\ y(0) = 2,\ y'(0) = 3,\ y''(0) = 4,\ y'''(0) = 5$
10. $\dfrac{d^4y}{dx^4} - y = 0,\ y(0) = y'(0) = y''(0) = 0,\ y'''(0) = 1$

Lecture 17 Method of Undetermined Coefficients Superposition Approach

Recall

1. That a non-homogeneous linear differential equation of order n is an equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

   The coefficients $a_0, a_1, \ldots, a_n$ can be functions of x. However, we will discuss equations with constant coefficients.

2. That to obtain the general solution of a non-homogeneous linear differential equation we must find**:**

   - The complementary function y_c, which is general solution of the associated homogeneous differential equation.
   - Any particular solution y_p of the non-homogeneous differential equation.

3. That the general solution of the non-homogeneous linear differential equation is given by

   General solution = Complementary function + Particular Integral

Finding

Complementary function has been discussed in the previous lecture. In the next three lectures we will discuss methods for finding a particular integral for the non-homogeneous equation, namely

- The method of undetermined coefficients-*superposition approach*
- The method undetermined coefficients-*annihilator operator approach*.
- The method of variation of parameters.

The Method of Undetermined Coefficient

The method of undetermined coefficients developed here is limited to non-homogeneous linear differential equations

- That have constant coefficients, and
- Where the function $g(x)$ has a specific form.

The form of $g(x)$

The input function $g(x)$ can have one of the following forms:

- A constant function k.
- A polynomial function
- An exponential function e^x
- The trigonometric functions $\sin(\beta x)$, $\cos(\beta x)$
- Finite sums and products of these functions.

Otherwise, we cannot apply the method of undetermined coefficients.

The method

Consist of performing the following steps.

Step 1 Determine the form of the input function $g(x)$.

Step 2 Assume the general form of y_p according to the form of $g(x)$

Step 3 Substitute in the given non-homogeneous differential equation.

Step 4 Simplify and equate coefficients of like terms from both sides.

Step 5 Solve the resulting equations to find the unknown coefficients.

Step 6 Substitute the calculated values of coefficients in assumed y_p

Restriction on g ?

The input function g is restricted to have one of the above stated forms because of the reason**:**

- The derivatives of sums and products of polynomials, exponentials etc are again sums and products of similar kind of functions.
- The expression $ay_p^{//} + by_p^{/} + cy_p$ has to be identically equal to the input function $g(x)$.

Therefore, to make an educated guess, y_p is assured to have the same form as g.

Caution!

- In addition to the form of the input function $g(x)$, the educated guess for y_p must take into consideration the functions that make up the complementary function y_c.
- No function in the assumed y_p must be a solution of the associated homogeneous differential equation. This means that the assumed y_p should not contain terms that duplicate terms in y_c.

Taking for granted that no function in the assumed y_p is duplicated by a function in y_c, some forms of g and the corresponding forms of y_p are given in the following table.

Trial particular solutions

Number	The input function $g(x)$	The assumed particular solution y_p
1	Any constant e.g. 1	A
2	$5x+7$	$Ax+B$
3	$3x^2-2$	Ax^2+Bx+c
4	x^3-x+1	Ax^3+Bx^2+Cx+D
5	$\sin 4x$	$A\cos 4x+B\sin 4x$
6	$\cos 4x$	$A\cos 4x+B\sin 4x$
7	e^{5x}	Ae^{5x}
8	$(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
9	x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
10	$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x+Be^{3x}\sin 4x$
11	$5x^2\sin 4x$	$(A_1x^2+B_1x+C_1)\cos 4x+(A_2x^2+B_2x+C_2)\sin 4x$
12	$xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x+(Cx+D)e^{3x}\sin 4x$

If $g(x)$ equals a sum?

Suppose that

- The input function $g(x)$ consists of a sum of m terms of the kind listed in the above table i.e.
$$g(x)=g_1(x)+g_2(x)+\cdots+g_m(x).$$
- The trial forms corresponding to $g_1(x), g_2(x), \ldots, g_m(x)$ be $y_{p_1}, y_{p_2}, \ldots, y_{p_m}$.

Then the particular solution of the given non-homogeneous differential equation is
$$y_p=y_{p_1}+y_{p_2}+\cdots+y_{p_m}$$
In other words, *the form of y_p is a linear combination of all the linearly independent functions generated by repeated differentiation of the input function $g(x)$.*

Example 1

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$

Solution:

Complementary function

To find y_c, we first solve the associated homogeneous equation

$$y'' + 4y' - 2y = 0$$

We put $y = e^{mx}, \quad y' = me^{mx}, \; y'' = m^2 e^{mx}$

Then the associated homogeneous equation gives

$$(m^2 + 4m - 2)e^{mx} = 0$$

Therefore, the auxiliary equation is

$$m^2 + 4m - 2 = 0 \quad \text{as} \quad e^{mx} \neq 0, \quad \forall\, x$$

Using the quadratic formula, roots of the auxiliary equation are

$$m = -2 \pm \sqrt{6}$$

Thus we have real and distinct roots of the auxiliary equation

$$m_1 = -2 - \sqrt{6} \quad \text{and} \quad m_2 = -2 + \sqrt{6}$$

Hence the complementary function is

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}$$

Next we find a particular solution of the non-homogeneous differential equation.

Particular Integral

Since the input function

$$g(x) = 2x^2 - 3x + 6$$

is a quadratic polynomial. Therefore, we assume that

$$y_p = Ax^2 + Bx + C$$

Then $y_p' = 2Ax + B$ and $y_p'' = 2A$

Therefore $y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C$

Substituting in the given equation, we have

$$2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6$$

or $-2Ax^2 + (8A - 2B)x + (2A + 4B - 2C) = 2x^2 - 3x + 6$

Equating the coefficients of the like powers of x, we have

$$-2A=2,\ 8A-2B=-3,\quad 2A+4B-2C=6$$

Solving this system of equations leads to the values

$$A=-1,\quad B=-5/2,\quad C=-9.$$

Thus a particular solution of the given equation is

$$y_p=-x^2-\frac{5}{2}x-9.$$

Hence, the general solution of the given non-homogeneous differential equation is given by

$$y=y_c+y_p$$

or $$y=-x^2-\frac{5}{2}x-9+c_1e^{-(2+\sqrt{6})x}+c_2e^{(-2+\sqrt{6})x}$$

Example 2

Solve the differential equation

$$y''-y'+y=2\sin 3x$$

Solution:

Complementary function

To find y_c, we solve the associated homogeneous differential equation

$$y''-y'+y=0$$

Put $y=e^{mx},\ \ y'=me^{mx},\ \ y''=m^2e^{mx}$

Substitute in the given differential equation to obtain the auxiliary equation

$$m^2-m+1=0 \qquad \text{or} \qquad m=\frac{1\pm i\sqrt{3}}{2}$$

Hence, the auxiliary equation has complex roots. Hence the complementary function is

$$y_c=e^{(1/2)x}\left(c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right)$$

Particular Integral

Since successive differentiation of

$$g(x)=\sin 3x$$

produce $\sin 3x$ and $\cos 3x$

Therefore, we include both of these terms in the assumed particular solution, see table

$$y_p = A\cos 3x + B\sin 3x.$$

Then $\quad y_p' = -3A\sin 3x + 3B\cos 3x.$

$$y_p'' = -9A\cos 3x - 9B\sin 3x.$$

Therefore $\quad y_p'' - y_p' + y_p = (-8A - 3B)\cos 3x + (3A - 8B)\sin 3x.$

Substituting in the given differential equation

$$(-8A - 3B)\cos 3x + (3A - 8B)\sin 3x = 0\cos 3x + 2\sin 3x.$$

From the resulting equations

$$-8A - 3B = 0,\ 3A - 8B = 2$$

Solving these equations, we obtain

$$A = 6/73, B = -16/73$$

A particular solution of the equation is

$$y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x$$

Hence the general solution of the given non-homogeneous differential equation is

$$y = e^{(1/2)x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x$$

Example 3

Solve $\quad y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$

Solution:

Complementary function

To find y_c, we solve the associated homogeneous equation

$$y'' - 2y' - 3y = 0$$

Put $\quad y = e^{mx},\ \ y' = me^{mx},\ y'' = m^2e^{mx}$

Substitute in the given differential equation to obtain the auxiliary equation

$$m^2 - 2m - 3 = 0$$

$$\Rightarrow (m+1)(m-3) = 0$$

$$m = -1, 3$$

Therefore, the auxiliary equation has real distinct root

$$m_1 = -1, m_2 = 3$$

Thus the complementary function is

$$y_c = c_1e^{-x} + c_2e^{3x}.$$

Particular integral

Since $\quad g(x) = (4x - 5) + 6xe^{2x} = g_1(x) + g_2(x)$

Corresponding to $g_1(x)$ $\quad y_{p_1} = Ax + B$

Corresponding to $g_2(x)$ $\quad y_{p_2} = (Cx + D)e^{2x}$

The superposition principle suggests that we assume a particular solution

$$y_p = y_{p_1} + y_{p_2}$$

i.e. $$y_p = Ax + B + (Cx + D)\, e^{2x}$$

Then $$y'_p = A + 2(Cx + D)\, e^{2x} + Ce^{2x}$$

$$y''_p = 4(Cx + D)\, e^{2x} + 4Ce^{2x}$$

Substituting in the given

$$y_p{}^{//} - 2y_p{}^{/} - 3y_p = 4Cxe^{2x} + 4De^{2x} + 4Ce^{2x} - 2A - 4Cxe^{2x}$$
$$- 4De^{2x} - 2Ce^{2x} - 3Ax - 3B - 3Cxe^{2x} - 3De^{2x}$$

Simplifying and grouping like terms

$$y_p{}^{//} - 2y_p{}^{/} - 3y_p = -3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3D)e^{2x} = 4x - 5 + 6xe^{2x}.$$

Substituting in the non-homogeneous differential equation, we have

$$-3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3D)e^{2x} = 4x - 5 + 6xe^{2x} + 0e^{2x}$$

Now equating constant terms and coefficients of x, xe^{2x} and e^{2x}, we obtain

$$-2A - 3B = -5, \qquad -3A = 4$$
$$-3C = 6, \qquad 2C - 3D = 0$$

Solving these algebraic equations, we find

$$A = -4/3, \quad B = 23/9$$
$$C = -2, \quad D = -4/3$$

Thus, a particular solution of the non-homogeneous equation is

$$y_p = -(4/3)x + (23/9) - 2\, xe^{2x} - (4/3)e^{2x}$$

The general solution of the equation is

$$y = y_c + y_p = c_1e^{-x} + c_2e^{3x} - (4/3)x + (23/9) - 2\,x\,e^{2x} - (4/3)e^{2x}$$

Duplication between y_p and y_c?

- If a function in the assumed y_p is also present in y_c then this function is a solution of the associated homogeneous differential equation. In this case the obvious assumption for the form of y_p is not correct**.**
- In this case we suppose that the input function is made up of terms of n kinds i.e.

  $$g(x) = g_1(x) + g_2(x) + \cdots + g_n(x)$$

  and corresponding to this input function the assumed particular solution y_p is

  $$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_n}$$
- If a y_{p_i} contain terms that duplicate terms in y_c, then that y_{p_i} must be multiplied with x^n, n being the least positive integer that eliminates the duplication.

Example 4

Find a particular solution of the following non-homogeneous differential equation

$$y^{//} - 5y^{/} + 4y = 8e^x$$

Solution:

To find y_c, we solve the associated homogeneous differential equation

$$y'' - 5y' + 4y = 0$$

We put $y = e^{mx}$ in the given equation, so that the auxiliary equation is

$$m^2 - 5m + 4 = 0 \Rightarrow m = 1, 4$$

Thus $\quad y_c = c_1 e^x + c_2 e^{4x}$

Since $\quad g(x) = 8e^x$

Therefore, $\quad y_p = Ae^x$

Substituting in the given non-homogeneous differential equation, we obtain

$$Ae^x - 5Ae^x + 4Ae^x = 8e^x$$

So $\quad 0 = 8e^x$

Clearly we have made a wrong assumption for y_p, as we did not remove the duplication.

Since Ae^x is present in y_c. Therefore, it is a solution of the associated homogeneous differential equation

$$y'' - 5y' + 4y = 0$$

To avoid this we find a particular solution of the form

$$y_p = Axe^x$$

We notice that there is no duplication between y_c and this new assumption for y_p

Now $\quad y_p' = Axe^x + Ae^x, \quad y_p'' = Axe^x + 2Ae^x$

Substituting in the given differential equation, we obtain

$$Axe^x + 2Ae^x - 5Axe^x - 5Ae^x + 4Axe^x = 8e^x.$$

or $\quad -3Ae^x = 8e^x \Rightarrow A = -8/3.$

So that a particular solution of the given equation is given by

$$y_p = -(8/3)e^x$$

Hence, the general solution of the given equation is

$$y = c_1 e^x + c_2 e^{4x} - (8/3)\, x\, e^x$$

Example 5

Determine the form of the particular solution

(a) $\quad y'' - 8y' + 25y = 5x^3 e^{-x} - 7e^{-x}$

(b) $\quad y'' + 4y = x\cos x.$

Solution:

(a) To find y_c we solve the associated homogeneous differential equation

$$y'' - 8y' + 25y = 0$$

Put $y = e^{mx}$

Then the auxiliary equation is

$$m^2 - 8m + 25 = 0 \Rightarrow m = 4 \pm 3i$$

Roots of the auxiliary equation are complex

$$y_c = e^{4x}(c_1 \cos 3x + c_2 \sin 3x)$$

The input function is

$$g(x) = 5x^3 e^{-x} - 7e^{-x} = (5x^3 - 7)e^{-x}$$

Therefore, we assume a particular solution of the form

$$y_p = (Ax^3 + Bx^2 + Cx + D)e^{-x}$$

Notice that there is ***no duplication*** between the terms in y_p and the terms in y_c. Therefore, while proceeding further we can easily calculate the value A, B, C and D.

(b) Consider the associated homogeneous differential equation

$$y'' + 4y = 0$$

Since $g(x) = x \cos x$

Therefore, we assume a particular solution of the form

$$y_p = (Ax + B)\cos x + (Cx + D)\sin x$$

Again observe that there is ***no duplication*** of terms between y_c and y_p

Example 6

Determine the form of a particular solution of

$$y'' - y' + y = 3x^2 - 5\sin 2x + 7xe^{6x}$$

Solution:

To find y_c, we solve the associated homogeneous differential equation

$$y'' - y' + y = 0$$

Put $y = e^{mx}$

Then the auxiliary equation is

$$m^2 - m + 1 = 0 \Rightarrow m = \frac{1 \pm i\sqrt{3}}{2}$$

Therefore $$y_c = e^{(1/2)x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

Since $$g(x) = 3x^2 - 5\sin 2x + 7xe^{6x} = g_1(x) + g_2(x) + g_3(x)$$

Corresponding to $g_1(x) = 3x^2$: $\quad y_{p_1} = Ax^2 + Bx + C$

Corresponding to $g_2(x) = -5\sin 2x$: $\quad y_{p_2} = D\cos 2x + E\sin 2x$

Corresponding to $g_3(x) = 7xe^{6x}$: $\quad y_{p_3} = (Fx + G)\,e^{6x}$

Hence, the assumption for the particular solution is

$$y_p = y_{p_1} + y_{p_2} + y_{p_3}$$

or $$y_p = Ax^2 + Bx + C + D\cos 2x + E\sin 2x + (Fx + G)e^{6x}$$

No term in this assumption duplicate any term in the complementary function

$$y_c = c_1 e^{2x} + c_2 e^{7x}$$

Example 7

Find a particular solution of

$$y'' - 2y' + y = e^x$$

Solution:

Consider the associated homogeneous equation

$$y'' - 2y' + y = 0$$

Put $\quad y = e^{mx}$

Then the auxiliary equation is

$$m^2 - 2m + 1 = (m-1)^2 = 0$$

$$\Rightarrow m = 1,\ 1$$

Roots of the auxiliary equation are real and equal. Therefore,

$$y_c = c_1 e^x + c_2 xe^x$$

Since $\quad g(x) = e^x$

Therefore, we assume that

$$y_p = Ae^x$$

This assumption fails because of duplication between y_c and y_p. We multiply with x

Therefore, we now assume

$$y_p = Axe^x$$

However, the duplication is still there. Therefore, we again multiply with x and assume

$$y_p = Ax^2e^x$$

Since there is no duplication, this is acceptable form of the trial y_p

$$y_p = \frac{1}{2}x^2e^x$$

Example 8

Solve the initial value problem

$$y'' + y = 4x + 10\sin x,$$

$$y(\pi) = 0,\ y'(\pi) = 2$$

Solution

Consider the associated homogeneous differential equation

$$y'' + y = 0$$

Put $y = e^{mx}$

Then the auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

The roots of the auxiliary equation are complex. Therefore, the complementary function is

$$y_c = c_1 \cos x + c_2 \sin x$$

Since $g(x) = 4x + 10\sin x = g_1(x) + g_2(x)$

Therefore, we assume that

$$y_{p_1} = Ax + B, \quad y_{p_2} = C\cos x + D\sin x$$

So that $y_p = Ax + B + C\cos x + D\sin x$

Clearly, there is duplication of the functions $\cos x$ and $\sin x$. To remove this duplication we multiply y_{p_2} with x. Therefore, we assume that

$$y_p = Ax + B + C\,x\cos x + Dx\sin x.$$

$$y_p'' = -2C\sin x - Cx\cos x + 2D\cos x - Dx\sin x$$

So that $y_p'' + y_p = Ax + B - 2C\sin x + 2Dx\cos x$

Substituting into the given non-homogeneous differential equation, we have

$$Ax + B - 2C\sin x + 2Dx\cos x = 4x + 10\sin x$$

Equating constant terms and coefficients of x, $\sin x$, $x\cos x$, we obtain

$$B = 0,\ A = 4,\ -2C = 10,\ 2D = 0$$

So that $A = 4, B = 0, C = -5,\ D = 0$

Thus $y_p = 4x - 5x\cos x$

Hence the general solution of the differential equation is

$$y = y_c + y_p = c_1\cos x + c_2\sin x + 4x - 5x\cos x$$

We now apply the initial conditions to find c_1 and c_2.

$$y(\pi) = 0 \Rightarrow c_1\cos\pi + c_2\sin\pi + 4\pi - 5\pi\cos\pi = 0$$

Since $\sin\pi = 0, \cos\pi = -1$

Therefore $c_1 = 9\pi$

Now $y' = -9\pi\sin x + c_2\cos x + 4 + 5x\sin x - 5\cos x$

Therefore $y'(\pi) = 2 \Rightarrow -9\pi\sin\pi + c_2\cos\pi + 4 + 5\pi\sin\pi - 5\cos\pi = 2$

$\therefore$ $c_2 = 7.$

Hence the solution of the initial value problem is

$$y = 9\pi\cos x + 7\sin x + 4x - 5x\cos x.$$

Example 9

Solve the differential equation

$$y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$$

Solution:

The associated homogeneous differential equation is

$$y'' - 6y' + 9y = 0$$

Put $y = e^{mx}$

Then the auxiliary equation is

$$m^2 - 6m + 9 = 0 \Rightarrow m = 3, 3$$

Thus the complementary function is

$$y_c = c_1 e^{3x} + c_2 x e^{3x}$$

Since $\qquad g(x) = (x^2 + 2) - 12e^{3x} = g_1(x) + g_2(x)$

We assume that

Corresponding to $g_1(x) = x^2 + 2$: $\qquad y_{p_1} = Ax^2 + Bx + C$

Corresponding to $g_2(x) = -12e^{3x}$: $\qquad y_{p_2} = De^{3x}$

Thus the assumed form of the particular solution is

$$y_p = Ax^2 + Bx + C + De^{3x}$$

The function e^{3x} in y_{p_2} is duplicated between y_c and y_p. Multiplication with x does not remove this duplication. However, if we multiply y_{p_2} with x^2, this duplication is removed.

Thus the operative from of a particular solution is

$$y_p = Ax^2 + Bx + C + Dx^2 e^{3x}$$

Then $\qquad y'_p = 2Ax + B + 2Dxe^{3x} + 3Dx^2 e^{3x}$

and $\qquad y''_p = 2A + 2De^{3x} + 6Dxe^{3x} + 9Dx^2 e^{3x}$

Substituting into the given differential equation and collecting like term, we obtain

$$y_p'' - 6y_p' + y_p = 9Ax^2 + (-12A + 9B)x + 2A - 6B + 9C + 2De^{3x} = 6x^2 + 2 - 12e^{3x}$$

Equating constant terms and coefficients of x, x^2 and e^{3x} yields

$$2A - 6B + 9C = 2, \quad -12A + 9B = 0$$

$$9A = 6, \qquad 2D = -12$$

Solving these equations, we have the values of the unknown coefficients

$$A = 2/3, B = 8/9, C = 2/3 \text{ and } D = -6$$

Thus $\qquad y_p = \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2 e^{3x}$

Hence the general solution

$$y = y_c + y_p = c_1 e^{3x} + c_2 x e^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2 e^{3x}.$$

Higher –Order Equation

The method of undetermined coefficients can also be used for higher order equations of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

with constant coefficients. The only requirement is that $g(x)$ consists of the proper kinds of functions as discussed earlier.

Example 10

Solve $\quad y''' + y'' = e^x \cos x$

Solution:

To find the complementary function we solve the associated homogeneous differential equation

$$y''' + y'' = 0$$

Put $\quad y = e^{mx}, y' = me^{mx}, y'' = m^2 e^{mx}$

Then the auxiliary equation is

$$m^3 + m^2 = 0$$

or $\quad m^2(m+1) = 0 \Rightarrow m = 0,0,-1$

The auxiliary equation has equal and distinct real roots. Therefore, the complementary function is

$$y_c = c_1 + c_2 x + c_3 e^{-x}$$

Since $\quad g(x) = e^x \cos x$

Therefore, we assume that

$$y_p = Ae^x \cos x + Be^x \sin x$$

Clearly, there is no duplication of terms between y_c and y_p.

Substituting the derivatives of y_p in the given differential equation and grouping the like terms, we have

$$y_p''' + y_p'' = (-2A + 4B)e^x \cos x + (-4A - 2B)e^x \sin x = e^x \cos x.$$

Equating the coefficients, of $e^x \cos x$ and $e^x \sin x$, yields

$$-2A + 4B = 1, -4A - 2B = 0$$

Solving these equations, we obtain

$$A = -1/10, B = 1/5$$

So that a particular solution is

$$y_p = c_1 + c_2 x + c_3 e^{-x} - (1/10)e^x \cos x + (1/5)e^x \sin x$$

Hence the general solution of the given differential equation is

$$y_p = c_1 + c_2 x + c_3 e^{-x} - (1/10)e^x \cos x + (1/5)e^x \sin x$$

Example 12

Determine the form of a particular solution of the equation

$$y'''' + y''' = 1 - e^{-x}$$

Solution:

Consider the associated homogeneous differential equation

$$y'''' + y''' = 0$$

The auxiliary equation is

$$m^4 + m^3 = 0 \Rightarrow m = 0, 0, 0, -1$$

Therefore, the complementary function is

$$y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x}$$

Since $\qquad g(x) = 1 - e^{-x} = g_1(x) + g_2(x)$

Corresponding to $g_1(x) = 1$: $\qquad y_{p_1} = A$

Corresponding to $g_2(x) = -e^{-x}$: $\qquad y_{p_2} = Be^{-x}$

Therefore, the normal assumption for the particular solution is

$$y_p = A + Be^{-x}$$

Clearly there is duplication of

(i) The constant function between y_c and y_{p_1}.

(ii) The exponential function e^{-x} between y_c and y_{p_2}.

To remove this duplication, we multiply y_{p_1} with x^3 and y_{p_2} with x. This duplication can't be removed by multiplying with x and x^2. Hence, the correct assumption for the particular solution y_p is

$$y_p = Ax^3 + Bxe^{-x}$$

Exercise

Solve the following differential equations using the undetermined coefficients.

1. $\frac{1}{4}y'' + y' + y = x^2 + 2x$

2. $y'' - 8y' + 20y = 100x^2 - 26xe^x$

3. $y'' + 3y = -48x^2 e^{3x}$

4. $4y'' - 4y' - 3y = \cos 2x$

5. $y'' + 4y = (x^2 - 3)\sin 2x$

6. $y'' - 5y' = 2x^3 - 4x^2 - x + 6$

7. $y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x)$

Solve the following initial value problems.

8. $y'' + 4y' + 4y = (3 + x)e^{-2x}, \quad y(0) = 2, y'(0) = 5$

9. $\dfrac{d^2x}{dt^2} + \omega^2 x = F_0 \cos\gamma t, \qquad x(0) = 0, x^{/}(0) = 0$

10. $y^{///} + 8y = 2x - 5 + 8e^{-2x}, \quad y(0) = -5, \; y^{/}(0) = 3, y^{//}(0) = -4$

Lecture 18 Undetermined Coefficient: Annihilator Operator Approach

Recall

1. That a non-homogeneous linear differential equation of order n is an equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

The following differential equation is called the associated homogeneous equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$$

The coefficients $a_0, a_1, \ldots, a_n$ can be functions of x. However, we will discuss equations with constant coefficients.

2. That to obtain the general solution of a non-homogeneous linear differential equation we must find**:**

   - The complementary function y_c, which is general solution of the associated homogeneous differential equation.
   - Any particular solution y_p of the non-homogeneous differential equation.

3. That the general solution of the non-homogeneous linear differential equation is given by

General Solution = Complementary Function + Particular Integral

   - Finding the complementary function has been completely discussed in an earlier lecture
   - In the previous lecture, we studied a method for finding particular integral of the non-homogeneous equations. This was the *method of undetermined coefficients developed from the viewpoint of superposition* principle.
   - In the present lecture, we will learn to find particular integral of the non-homogeneous equations by the same method utilizing the concept of differential annihilator operators.

Differential Operators

- In calculus, the differential coefficient d/dx is often denoted by the capital letter D. So that

$$\frac{dy}{dx} = Dy$$

The symbol D is known as differential operator.

- This operator transforms a differentiable function into another function, e.g.

$$D(e^{4x}) = 4e^{4x},\ D(5x^3 - 6x^2) = 15x^2 - 12x,\ D(\cos 2x) = -2\sin 2x$$

- The differential operator D possesses the property of linearity. This means that if f, g are two differentiable functions, then

$$D\{af(x) + bg(x)\} = aDf(x) + bDg(x)$$

Where a and b are constants. Because of this property, we say that D is a linear differential operator.

- Higher order derivatives can be expressed in terms of the operator D in a natural manner:

$$\frac{d^2y}{d^2x} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = D(Dy) = D^2y$$

Similarly

$$\frac{d^3y}{dx^3} = D^3y, \ldots, \frac{d^ny}{d^nx} = D^ny$$

- The following polynomial expression of degree n involving the operator D

$$a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0$$

is also a linear differential operator.

For example, the following expressions are all linear differential operators

$$D+3,\ D^2 + 3D - 4,\ 5D^3 - 6D^2 + 4D$$

Differential Equation in Terms of *D*

Any linear differential equation can be expressed in terms of the notation D. Consider a 2nd order equation with constant coefficients

$$ay'' + by' + cy = g(x)$$

Since $$\frac{dy}{dx} = Dy, \frac{d^2y}{dx^2} = D^2y$$

Therefore the equation can be written as

$$aD^2y + bDy + cy = g(x)$$

or $$(aD^2 + bD + c)y = g(x)$$

Now, we define another differential operator L as

$$L = aD^2 + bD + c$$

Then the equation can be compactly written as

$$L(y) = g(x)$$

The operator L is a second-order linear differential operator with constant coefficients.

Example 1

Consider the differential equation

$$y'' + y' + 2y = 5x - 3$$

Since $\frac{dy}{dx} = Dy, \frac{d^2y}{dx^2} = D^2y$

Therefore, the equation can be written as

$$(D^2 + D + 2)y = 5x - 3$$

Now, we define the operator L as

$$L = D^2 + D + 2$$

Then the given differential can be compactly written as

$$L(y) = 5x - 3$$

Factorization of a differential operator

- An nth-order linear differential operator

  $$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

  with constant coefficients can be factorized, whenever the characteristics polynomial equation

  $$L = a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0$$

  can be factorized.

- The factors of a linear differential operator with constant coefficients commute.

Example 2

(a) Consider the following 2nd order linear differential operator

$$D^2 + 5D + 6$$

If we treat D as an algebraic quantity, then the operator can be factorized as

$$D^2 + 5D + 6 = (D+2)(D+3)$$

(b) To illustrate the commutative property of the factors, we consider a twice-differentiable function $y = f(x)$. Then we can write

$$(D^2 + 5D + 6)y = (D+2)(D+3)y = (D+3)(D+2)y$$

To verify this we let

$$w = (D+3)y = y' + 3y$$

Then

$$(D+2)w = Dw + 2w$$

or $$(D+2)w = (y'' + 3y') + (2y' + 6y)$$

or $$(D+2)w = y'' + 5y' + 6y$$

or $(D+2)(D+3)y = y'' + 5y' + 6y$

Similarly if we let

$$w = (D+2)y = (y' + 2y)$$

Then $(D+3)w = Dw + 3w = (y'' + 2y') + (3y' + 6y)$

or $(D+3)w = y'' + 5y' + 6y$

or $(D+3)(D+2)y = y'' + 5y' + 6y$

Therefore, we can write from the two expressions that

$$(D+3)(D+2)y = (D+2)(D+3)y$$

Hence $(D+3)(D+2)y = (D+2)(D+3)y$

Example 3

(a) The operator $D^2 - 1$ can be factorized as

$$D^2 - 1 = (D+1)(D-1) .$$

or $D^2 - 1 = (\mathrm{D} - 1)(D+1)$

(b) The operator $D^2 + D + 2$ does not factor with real numbers.

Example 4

The differential equation

$$y'' + 4y' + 4y = 0$$

can be written as

$$\left(D^2 + 4D + 4\right)y = 0$$

or $(D+2)(D+2)y = 0$

or $(D+2)^2 y = 0.$

Annihilator Operator

Suppose that

- ***L*** is a linear differential operator with constant coefficients.
- *y = f(x)* defines a sufficiently differentiable function.
- The function *f* is such that *L(y)=0*

Then the differential operator *L* is said to be an **annihilator operator** of the function *f*.

Example 5
Since

$$Dx=0,\ \ D^2x=0,\ \ D^3x^2=0,\ \ D^4x^3=0,\ \ldots$$

Therefore, the differential operators

$$D,\ \ D^2,\ D^3,\ \ D^4,\ \ldots$$

are annihilator operators of the following functions

$$k(\text{a constant}),\ \ x,\ \ x^2,\ \ x^3,\ \ldots$$

In general, the differential operator D^n annihilates each of the functions

$$1, x, x^2, \ldots, x^{n-1}$$

Hence, we conclude that the polynomial function

$$c_0+c_1x+\cdots+c_{n-1}x^{n-1}$$

can be annihilated by finding an operator that annihilates the highest power of x.

Example 6

Find a differential operator that annihilates the polynomial function

$$y=1-5x^2+8x^3.$$

Solution

Since $D^4x^3=0,$

Therefore $D^4y=D^4\left(1-5x^2+8x^3\right)=0.$

Hence, D^4 is the differential operator that annihilates the function y.

Note that the functions that are annihilated by an nth-order linear differential operator L are simply those functions that can be obtained from the general solution of the homogeneous differential equation

$$L(y)=0.$$

Example 7
Consider the homogeneous linear differential equation of order n

$$(D-\alpha)^n y=0$$

The auxiliary equation of the differential equation is

$$(m-\alpha)^n=0$$

$$\Rightarrow \qquad m = \alpha, \alpha, \ldots, \alpha \ (n \ \text{times})$$

Therefore, the auxiliary equation has a real root α of multiplicity n. So that the differential equation has the following linearly independent solutions:

$$e^{\alpha x}, xe^{\alpha x}, x^2 e^{\alpha x}, \ldots, x^{n-1} e^{\alpha x}.$$

Therefore, the general solution of the differential equation is

$$y = c_1 e^{\alpha x} + c_2 x e^{\alpha x} + c_3 x^2 e^{\alpha x} + \cdots + c_n x^{n-1} e^{\alpha x}$$

So that the differential operator

$$(D-\alpha)^n$$

annihilates each of the functions

$$e^{\alpha x}, xe^{\alpha x}, x^2 e^{\alpha x}, \ldots, x^{n-1} e^{\alpha x}$$

Hence, as a consequence of the fact that the differentiation can be performed term by term, the differential operator

$$(D-\alpha)^n$$

annihilates the function

$$y = c_1 e^{\alpha x} + c_2 x e^{\alpha x} + c_3 x^2 e^{\alpha x} + \cdots + c_n x^{n-1} e^{\alpha x}$$

Example 8

Find an annihilator operator for the functions

(a) $\qquad f(x) = e^{5x}$

(b) $\qquad g(x) = 4e^{2x} - 6xe^{2x}$

Solution

(a) Since

$$(D-5)e^{5x} = 5e^{5x} - 5e^{5x} = 0.$$

Therefore, the annihilator operator of function f is given by

$$L = D - 5$$

We notice that in this case $\alpha = 5,\ n = 1$.

(b) Similarly

$$(D-2)^2\left(4e^{2x} - 6xe^{2x}\right) = (D^2 - 4D + 4)(4e^{2x}) - (D^2 - 4D + 4)(6xe^{2x})$$

or $\quad (D-2)^2\left(4e^{2x} - 6xe^{2x}\right) = 32e^{2x} - 32e^{2x} + 48xe^{2x} - 48xe^{2x} + 24e^{2x} - 24e^{2x}$

or $\quad (D-2)^2\left(4e^{2x} - 6xe^{2x}\right) = 0$

Therefore, the annihilator operator of the function g is given by

$$L = (D-2)^2$$

We notice that in this case $\alpha = 2 = n$.

Example 9

Consider the differential equation

$$\left(D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right)^n y = 0$$

The auxiliary equation is

$$\left(m^2 - 2\alpha m + \left(\alpha^2 + \beta^2\right)\right)^n = 0$$

$$\Rightarrow \qquad m^2 - 2\alpha m + \left(\alpha^2 + \beta^2\right) = 0$$

Therefore, when $\alpha,\ \beta$ are real numbers, we have from the quadratic formula

$$m = \frac{2\alpha \pm \sqrt{4\alpha^2 - 4\left(\alpha^2 + \beta^2\right)}}{2} = \alpha \pm i\beta$$

Therefore, the auxiliary equation has the following two complex roots of multiplicity n.

$$m_1 = \alpha + i\beta, \quad m_2 = \alpha - i\beta$$

Thus, the general solution of the differential equation is a linear combination of the following linearly independent solutions

$$e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x,\ x^2e^{\alpha x}\cos\beta x,\ \cdots,\ x^{n-1}e^{\alpha x}\cos\beta x$$

$$e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x,\ x^2e^{\alpha x}\sin\beta x,\ \cdots,\ x^{n-1}e^{\alpha x}\sin\beta x$$

Hence, the differential operator

$$\left(D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right)^n$$

is the annihilator operator of the functions

$$e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x,\ x^2e^{\alpha x}\cos\beta x,\ \cdots,\ x^{n-1}e^{\alpha x}\cos\beta x$$

$$e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x,\ x^2e^{\alpha x}\sin\beta x,\ \cdots,\ x^{n-1}e^{\alpha x}\sin\beta x$$

Example 10

If we take

$$\alpha = -1,\ \beta = 2,\ n = 1$$

Then the differential operator

$$\left(D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right)^n$$

becomes $\qquad D^2 + 2D + 5.$

Also, it can be verified that

$$\left(D^2 + 2D + 5\right)e^{-x}\cos 2x = 0$$

$$\left(D^2 + 2D + 5\right)e^{-x}\sin 2x = 0$$

Therefore, the linear differential operator

$$D^2 + 2D + 5$$

annihilates the functions

$$y_1(x) = e^{-x}\cos 2x$$
$$y_2(x) = e^{-x}\sin 2x$$

Now, consider the differential equation

$$\left(D^2 + 2D + 5\right)y = 0$$

The auxiliary equation is

$$m^2 + 2m + 5 = 0$$
$$\Rightarrow m = -1 \pm 2i$$

Therefore, the functions

$$y_1(x) = e^{-x}\cos 2x$$
$$y_2(x) = e^{-x}\sin 2x$$

are the two linearly independent solutions of the differential equation

$$\left(D^2 + 2D + 5\right)y = 0,$$

Therefore, the operator also annihilates a linear combination of y_1 and y_2, e.g.

$$5y_1 - 9y_2 = 5e^{-x}\cos 2x - 9e^{-x}\sin 2x.$$

Example 11
If we take

$$\alpha = 0,\ \beta = 1,\ n = 2$$

Then the differential operator

$$\left(D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right)^n$$

becomes

$$(D^2 + 1)^2 = D^4 + 2D^2 + 1$$

Also, it can be verified that

$$\left(D^4 + 2D^2 + 1\right)\cos x = 0$$
$$\left(D^4 + 2D^2 + 1\right)\sin x = 0$$

and

$$\left(D^4 + 2D^2 + 1\right)x\cos x = 0$$
$$\left(D^4 + 2D^2 + 1\right)x\sin x = 0$$

Therefore, the linear differential operator

$$D^4 + 2D^2 + 1$$

annihilates the functions

$$\begin{array}{ll} \cos x, & \sin x \\ x\cos x, & x\sin x \end{array}$$

Example 12

Taking $\alpha = 0,\ \mathrm{n} = 1$, the operator

$$\left(D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right)^n$$

becomes

$$D^2 + \beta^2$$

Since
$$\left(D^2 + \beta^2\right)\cos\beta x = -\beta^2\cos\beta\, x + \beta^2\cos\beta\, x = 0$$
$$\left(D^2 + \beta^2\right)\sin\beta\, x = -\beta^2\sin\beta\, x + \beta^2\sin\beta\, x = 0$$

Therefore, the differential operator annihilates the functions

$$f(x) = \cos\beta\, x, \quad g(x) = \sin\beta\, x$$

Note that

- If a linear differential operator with constant coefficients is such that
  $$L(y_1) = 0\,, \qquad L(y_2) = 0$$
  i.e. the operator L annihilates the functions y_1 and y_2. Then the operator L annihilates their linear combination.
  $$L[c_1 y_1(x) + c_2 y_2(x)] = 0\,.$$
  This result follows from the linearity property of the differential operator L.

- Suppose that L_1 and L_2 are linear operators with constant coefficients such that
  $$L_1(y_1) = 0, \quad L_2(y_2) = 0$$
  and
  $$L_1(y_2) \neq 0, \quad L_2(y_1) \neq 0$$
  then the product of these differential operators $L_1 L_2$ annihilates the linear sum
  $$y_1(x) + y_2(x)$$
  So that
  $$L_1 L_2[y_1(x) + y_2(x)] = 0$$

  To demonstrate this fact we use the linearity property for writing
  $$L_1 L_2(y_1 + y_2) = L_1 L_2(y_1) + L_1 L_2(y_2)$$

  Since $\quad L_1 L_2 = L_2 L_1$

  therefore $\quad L_1 L_2(y_1 + y_2) = L_2 L_1(y_1) + L_1 L_2(y_2)$

  or $\quad L_1 L_2(y_1 + y_2) = L_2[L_1(y_1)] + L_1[L_2(y_2)]$

  But we know that $\quad L_1(y_1) = 0, \quad L_2(y_2) = 0$

  Therefore $\quad L_1 L_2(y_1 + y_2) = L_2[0] + L_1[0] = 0$

Example 13

Find a differential operator that annihilates the function

$$f(x) = 7 - x + 6\sin 3x$$

Solution

Suppose that

$$y_1(x) = 7 - x, \quad y_2(x) = 6\sin 3x$$

Then

$$D^2 y_1(x) = D^2(7 - x) = 0$$

$$(D^2 + 9)y_2(x) = (D^2 + 9)\sin 3x = 0$$

Therefore, $D^2(D^2 + 9)$ annihilates the function $f(x)$.

Example 14

Find a differential operator that annihilates the function

$$f(x) = e^{-3x} + xe^x$$

Solution

Suppose that

$$y_1(x) = e^{-3x}, \quad y_2(x) = xe^x$$

Then

$$(D+3)y_1 = (D+3)e^{-3x} = 0,$$

$$(D-1)^2 y_2 = (D-1)^2 xe^x = 0.$$

Therefore, the product of two operators

$$(D+3)(D-1)^2$$

annihilates the given function $f(x) = e^{-3x} + xe^x$

Note that

- The differential operator that annihilates a function is not unique. For example,

  $$(D-5)\,e^{5x} = 0,$$

  $$(D-5)(D+1)e^{5x} = 0,$$

  $$(D-5)D^2 e^{5x} = 0$$

  Therefore, there are 3 annihilator operators of the functions, namely

  $$(D-5),\ (D-5)(D+1),\ (D-5)D^2$$

- When we seek a differential annihilator for a function, we want the operator of lowest possible order that does the job.

Exercises

Write the given differential equation in the form $L(y) = g(x)$, where L is a differential operator with constant coefficients.

1. $\dfrac{dy}{dx} + 5y = 9\sin x$
2. $4\dfrac{dy}{dx} + 8y = x + 3$

3. $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 4x$
4. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 6y = 1 - \sin x$

Factor the given differentiable operator, if possible.

5. $9D^2 - 4$
6. $D^2 - 5$
7. $D^3 + 2D^2 - 13D + 10$
8. $D^4 - 8D^2 + 16$

Verify that the given differential operator annihilates the indicated functions

9. $2D - 1; \quad y = 4e^{x/2}$
10. $D^4 + 64; \quad y = 2\cos 8x - 5\sin 8x$

Find a differential operator that annihilates the given function.

11. $x + 3xe^{6x}$
12. $1 + \sin x$

Lecture 19 Undetermined Coefficients: Annihilator Operator Approach

The method of undetermined coefficients that utilizes the concept of annihilator operator approach is also limited to non-homogeneous linear differential equations

- That have constant coefficients, and
- Where the function $g(x)$ has a specific form.

The form of $g(x)$: The input function $g(x)$ has to have one of the following forms:

- A constant function k.
- A polynomial function
- An exponential function e^{x}
- The trigonometric functions $\sin(\beta x)$, $\cos(\beta x)$
- Finite sums and products of these functions.

Otherwise, we cannot apply the method of undetermined coefficients.

The Method

Consider the following non-homogeneous linear differential equation with constant coefficients of order n

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

If L denotes the following differential operator

$$L = a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$$

Then the non-homogeneous linear differential equation of order n can be written as

$$L(y) = g(x)$$

The function $g(x)$ should consist of finite sums and products of the proper kind of functions as already explained.

The method of undetermined coefficients, annihilator operator approach, for finding a particular integral of the non-homogeneous equation consists of the following steps**:**

Step 1 Write the given non-homogeneous linear differential equation in the form

$$L(y) = g(x)$$

Step 2 Find the complementary solution y_c by finding the general solution of the associated homogeneous differential equation**:**

$$L(y) = 0$$

Step 3 Operate on both sides of the non-homogeneous equation with a differential operator L_1 that annihilates the function g(x).

Step 4 Find the general solution of the higher-order homogeneous differential equation

$$L_1 L(y) = 0$$

Step 5 Delete all those terms from the solution in step 4 that are duplicated in the complementary solution y_c, found in step 2.

Step 6 Form a linear combination y_p of the terms that remain. This is the form of a particular solution of the non-homogeneous differential equation

$$L(\mathrm{y}) = g(x)$$

Step 7 Substitute y_p found in step 6 into the given non-homogeneous linear differential equation

$$L(y) = g(x)$$

Match coefficients of various functions on each side of the equality and solve the resulting system of equations for the unknown coefficients in y_p.

Step 8 With the particular integral found in step 7, form the general solution of the given differential equation as:

$$y = y_c + y_p$$

Example 1

Solve $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x^2 .$$

Solution:

Step 1 Since $$\frac{dy}{dx} = Dy,\ \frac{d^2y}{dx^2} = D^2y$$

Therefore, the given differential equation can be written as

$$\left(D^2 + 3D + 2\right)y = 4x^2$$

Step 2 To find the complementary function y_c, we consider the associated homogeneous differential equation

$$\left(D^2 + 3D + 2\right)y = 0$$

The auxiliary equation is

$$m^2 + 3m + 2 = (m+1)(m+2) = 0$$

$$\Rightarrow \quad m \quad = -1, -2$$

Therefore, the auxiliary equation has two distinct real roots.

$$m_1 = -1,\ m_2 = -2,$$

Thus, the complementary function is given by

$$y_c = c_1e^{-x} + c_2e^{-2x}$$

Step 3 In this case the input function is

$$g(x) = 4x^2$$

Further $$D^3g(x) = 4D^3x^2 = 0$$

Therefore, the differential operator D^3 annihilates the function g. Operating on both sides of the equation in step 1, we have

$$D^3(D^2 + 3D + 2)y = 4D^3x^2$$

$$D^3(D^2 + 3D + 2)y = 0$$

This is the homogeneous equation of order 5. Next we solve this higher order equation.

Step 4 The auxiliary equation of the differential equation in step 3 is

$$m^3(m^2+3m+2)=0$$

$$m^3(m+1)(m+2)=0$$

$$m=0,0,0,-1,-2$$

Thus its general solution of the differential equation must be

$$y=c_1+c_2x+c_3x^2+c_4e^{-x}+c_5e^{-2x}$$

Step 5 The following terms constitute y_c

$$c_4e^{-x}+c_5e^{-2x}$$

Therefore, we remove these terms and the remaining terms are

$$c_1+c_2x+c_3x^2$$

Step 6 This means that the basic structure of the particular solution y_p is

$$y_p=A+Bx+Cx^2,$$

Where the constants c_1, c_2 and c_3 have been replaced, with A, *B*, and *C*, respectively.

Step 7 Since $y_p=A+Bx+Cx^2$

$$y_p'=B+2Cx,$$

$$y_p''=2C$$

Therefore $y_p''+3y_p'+2y_p=2C+3B+6Cx+2A+2Bx+2Cx^2$

or $y_p''+3y_p'+2y_p=(2C)x^2+(2B+6C)x+(2A+3B+2C)$

Substituting into the given differential equation, we have

$$(2C)x^2+(2B+6C)x+(2A+3B+2C)=4x^2+0x+0$$

Equating the coefficients of x^2, x and the constant terms, we have

$$2C = 4$$

$$2B + 6C = 0$$

$$2A+3B+2C = 0$$

Solving these equations, we obtain

$$A=7,\quad B=-6,\quad C=2$$

Hence $y_p=7-6x+2x^2$

Step 8 The general solution of the given non-homogeneous differential equation is

$$y=y_c+y_p$$

$$y=c_1e^{-x}+c_2e^{-2x}+7-6x+2x^2.$$

Example 2

Solve $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 8e^{3x} + 4\sin x$$

Solution:

Step 1 Since $$\frac{dy}{dx} = Dy,\ \frac{d^2y}{dx^2} = D^2y$$

Therefore, the given differential equation can be written as
$$\left(D^2 - 3D\right)y = 8e^{3x} + 4\sin x$$

Step 2 We first consider the associated homogeneous differential equation to find y_c
The auxiliary equation is
$$m(m-3) = 0 \Rightarrow m = 0, 3$$
Thus the auxiliary equation has real and distinct roots. So that we have
$$y_c = c_1 + c_2e^{3x}$$

Step 3 In this case the input function is given by
$$g(x) = 8e^{3x} + 4\sin x$$
Since $$(D-3)(8e^{3x}) = 0,\ (D^2+1)(4\sin x) = 0$$
Therefore, the operators $D-3$ and D^2+1 annihilate $8e^{3x}$ and $4\sin x$, respectively. So the operator $(D-3)(D^2+1)$ annihilates the input function $g(x)$. This means that
$$(D-3)(D^2+1)g(x) = (D-3)(D^2+1)(8e^{3x} + \sin x) = 0$$
We apply $(D-3)(D^2+1)$ to both sides of the differential equation in step 1 to obtain
$$(D-3)(D^2+1)(D^2-3D)y = 0\,.$$
This is homogeneous differential equation of order 5.

Step 4 The auxiliary equation of the higher order equation found in step 3 is
$$(m-3)(m^2+1)(m^2-3m) = 0$$
$$m(m-3)^2(m^2+1) = 0$$
$$\Rightarrow m = 0,\ 3,\ 3,\ \pm i$$
Thus, the general solution of the differential equation
$$y = c_1 + c_2e^{3x} + c_3xe^{3x} + c_4\cos x + c_5\sin x$$

Step 5 First two terms in this solution are already present in y_c
$$c_1 + c_2e^{3x}$$
Therefore, we eliminate these terms. The remaining terms are
$$c_3xe^{3x} + c_4\cos x + c_5\sin x$$

Step 6 Therefore, the basic structure of the particular solution y_p must be
$$y_p = Axe^{3x} + B\cos x + C\sin x$$

The constants c_3, c_4 and c_5 have been replaced with the constants A, B and C, respectively.

Step 7 Since $y_p = Axe^{3x} + B\cos x + C\sin x$

Therefore $y_p'' - 3y_p' = 3Ae^{3x} + (-B-3C)\cos x + (3B-C)\sin x$

Substituting into the given differential equation, we have

$$3Ae^{3x} + (-B-3C)\cos x + (3B-C)\sin x = 8e^{3x} + 4\sin x.$$

Equating coefficients of e^{3x}, $\cos x$ and $\sin x$, we obtain

$$3A = 8,\ -B-3C = 0,\ 3B-C = 4$$

Solving these equations we obtain

$$A = 8/3,\quad B = 6/5,\quad C = -2/5$$

$$y_p = \frac{8}{3}xe^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x.$$

Step 8 The general solution of the differential equation is then

$$y = c_1 + c_2e^{3x} + \frac{8}{3}xe^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x.$$

Example 3

Solve $$\frac{d^2y}{dx^2} + 8y = 5x + 2e^{-x} .$$

Solution:

Step 1 The given differential equation can be written as

$$(D^2 + 8)y = 5x + 2e^{-x}$$

Step 2 The associated homogeneous differential equation is

$$(D^2 + 8)y = 0$$

Roots of the auxiliary equation are complex

$$m = \pm 2\sqrt{2}\, i$$

Therefore, the complementary function is

$$y_c = c_1\cos 2\sqrt{2}\,x + c_2\sin 2\sqrt{2}\,x$$

Step 3 Since $D^2x = 0,\ (D+1)e^{-x} = 0$

Therefore the operators D^2 and $D+1$ annihilate the functions $5x$ and $2e^{-x}$. We apply $D^2(D+1)$ to the non-homogeneous differential equation

$$D^2(D+1)(D^2+8)y = 0.$$

This is a homogeneous differential equation of order 5.

Step 4 The auxiliary equation of this differential equation is

$$m^2(m+1)(m^2+8)=0$$
$$\Rightarrow m=0,0,-1,\pm 2\sqrt{2}\,i$$

Therefore, the general solution of this equation must be

$$y=c_1\cos 2\sqrt{2}x+c_2\sin 2\sqrt{2}x+c_3+c_4x+c_5e^{-x}$$

Step 5 Since the following terms are already present in y_c

$$c_1\cos 2\sqrt{2}x+c_2\sin 2\sqrt{2}x$$

Thus we remove these terms. The remaining ones are

$$c_3+c_4x+c_5e^{-x}$$

Step 6 The basic form of the particular solution of the equation is

$$y_p=A+Bx+Ce^{-x}$$

The constants c_3, c_4 and c_5 have been replaced with A, B and C.

Step 7 Since $\quad y_p=A+Bx+Ce^{-x}$

Therefore $\quad y_p''+8y_p=8A+8Bx+9Ce^{-x}$

Substituting in the given differential equation, we have

$$8A+8Bx+9Ce^{-x}=5x+2e^{-x}$$

Equating coefficients of x, e^{-x} and the constant terms, we have

$$A=0, B=5/8,\ C=2/9$$

Thus $\quad y_p=\frac{5}{8}x+\frac{2}{9}e^{-x}$

Step 8 Hence, the general solution of the given differential equation is

$$y=y_c+y_p$$

or $\quad y=c_1\cos 2\sqrt{2}x+c_2\sin 2\sqrt{2}x+\frac{5}{8}x+\frac{2}{9}e^{-x}.$

Example 4

Solve $\quad \frac{d^2y}{dx^2}+y=x\cos x-\cos x$

Solution:

Step 1 The given differential equation can be written as

$$(D^2+1)y=x\cos x-\cos x$$

Step 2 Consider the associated differential equation

$$(D^2+1)y=0$$

The auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

Therefore $y_c = c_1 \cos x + c_2 \sin x$

Step 3 Since $(D^2+1)^2(x\cos x) = 0$

$$(D^2+1)^2 \cos x = 0 \;; \quad \because x \neq 0$$

Therefore, the operator $(D^2+1)^2$ annihilates the input function

$$x\cos x - \cos x$$

Thus operating on both sides of the non-homogeneous equation with $(D^2+1)^2$, we have

$$(D^2+1)^2(D^2+1)y = 0$$

or $(D^2+1)^3 y = 0$

This is a homogeneous equation of order 6.

Step 4 The auxiliary equation of this higher order differential equation is

$$(m^2+1)^3 = 0 \Rightarrow m = i, i, i, -i, -i, -i$$

Therefore, the auxiliary equation has complex roots i, and $-i$ both of multiplicity 3. We conclude that

$$y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x + c_5 x^2 \cos x + c_6 x^2 \sin x$$

Step 5 Since first two terms in the above solution are already present in y_c

$$c_1 \cos x + c_2 \sin x$$

Therefore, we remove these terms.

Step 6 The basic form of the particular solution is

$$y_p = Ax\cos x + Bx\sin x + Cx^2 \cos x + Ex^2 \sin x$$

Step 7 Since $y_p = Ax\cos x + Bx\sin x + Cx^2 \cos x + Ex^2 \sin x$

Therefore

$$y_p'' + y_p = 4Ex\cos x - 4Cx\sin x + (2B + 2C)\cos x + (-2A + 2E)\sin x$$

Substituting in the given differential equation, we obtain

$$4Ex\cos x - 4Cx\sin x + (2B+2C)\cos x + (-2A+2E)\sin x = x\cos x - \cos x$$

Equating coefficients of $x\cos x, x\sin x, \cos x$ and $\sin x$, we obtain

$$4E = 1, \quad -4C = 0$$

$$2B + 2C = -1, \quad -2A + 2E = 0$$

Solving these equations we obtain

$$A = 1/4,\ B = -1/2,\ C = 0,\ E = 1/4$$

Thus $y_p = \frac{1}{4}x\cos x - \frac{1}{2}x\sin x + \frac{1}{4}x^2 \sin x$

Step 8 Hence the general solution of the differential equation is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{4}x\cos x - \frac{1}{2}x\sin x + \frac{1}{4}x^2 \sin x.$$

Example 5
Determine the form of a particular solution for

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=10e^{-2x}\cos x$$

Solution

Step 1 The given differential equation can be written as

$$(D^2-2D+1)y=10e^{-2x}\cos x$$

Step 2 To find the complementary function, we consider

$$y''-2y'+y=0$$

The auxiliary equation is

$$m^2-2m+1=0 \Rightarrow (m-1)^2=0 \Rightarrow m=1,1$$

The complementary function for the given equation is

$$y_c=c_1e^x+c_2xe^x$$

Step 3 Since $(D^2+4D+5)e^{-2x}\cos x=0$
Applying the operator (D^2+4D+5) to both sides of the equation, we have

$$(D^2+4D+5)(D^2-2D+1)y=0$$

This is homogeneous differential equation of order 4.

Step 4 The auxiliary equation is

$$(m^2+4m+5)(m^2-2m+1)=0$$

$$\Rightarrow m=-2\pm i,1,1$$

Therefore, general solution of the 4th order homogeneous equation is

$$y=c_1e^x+c_2xe^x+c_3e^{-2x}\cos x+c_4e^{-2x}\sin x$$

Step 5 Since the terms $c_1e^x+c_2xe^x$ are already present in y_c, therefore, we remove these and the remaining terms are $c_3e^{-2x}\cos x+c_4e^{-2x}\sin x$

Step 6 Therefore, the form of the particular solution of the non-homogeneous equation is

$$\therefore \qquad y_p=Ae^{-2x}\cos x+Be^{-2x}\sin x$$

Note that the steps 7 and 8 are not needed, as we don't have to solve the given differential equation.

Example 6
Determine the form of a particular solution for

$$\frac{d^3y}{dx^3}-4\frac{d^2y}{dx^2}+4\frac{dy}{dx}=5x^2-6x+4x^2e^{2x}+3e^{5x}.$$

Solution:

Step 1 The given differential can be rewritten as

$$\left(D^3 - 4D^2 + 4D\right)y = 5x^2 - 6x + 4x^2e^{2x} + 3e^{5x}$$

Step 2 To find the complementary function, we consider the equation

$$\left(D^3 - 4D^2 + 4D\right)y = 0$$

The auxiliary equation is

$$m^3 - 4m^2 + 4m = 0$$
$$m(m^2 - 4m + 4) = 0$$
$$m(m-2)^2 = 0 \Rightarrow m = 0, 2, 2$$

Thus the complementary function is

$$y_c = c_1 + c_2e^{2x} + c_3xe^{2x}$$

Step 3 Since $\quad g(x) = 5x^2 - 6x + 4x^2e^{2x} + 3e^{5x}$

Further $\quad D^3(5x^2 - 6x) = 0$

$$(D-2)^3 x^2e^{2x} = 0$$
$$(D-5)e^{5x} = 0$$

Therefore the following operator must annihilate the input function $g(x)$. Therefore, applying the operator $D^3(D-2)^3(D-5)$ to both sides of the non-homogeneous equation, we have

$$D^3(D-2)^3(D-5)(D^3 - D^2 + 4D)y = 0$$

or $\quad D^4(D-2)^5(D-5)y = 0$

This is homogeneous differential equation of order 10.

Step 4 The auxiliary equation for the 10th order differential equation is

$$m^4(m-2)^5(m-5) = 0$$
$$\Rightarrow m = 0, 0, 0, 0, 2, 2, 2, 2, 2, 5$$

Hence the general solution of the 10th order equation is

$$y = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5e^{2x} + c_6xe^{2x} + c_7x^2e^{2x} + c_8x^3e^{2x} + c_9x^4e^{2x} + c_{10}e^{5x}$$

Step 5 Since the following terms constitute the complementary function y_c, we remove these $\quad c_1 + c_5e^{2x} + c_6xe^{2x}$

Thus the remaining terms are

$$c_2x + c_3x^2 + c_4x^3 + c_7x^2e^{2x} + c_8x^3e^{2x} + c_9x^4e^{2x} + c_{10}e^{5x}$$

Hence, the form of the particular solution of the given equation is

$$y_p = Ax + Bx^2 + Cx^3 + Ex^2e^{2x} + Fx^3e^{2x} + Gx^4e^{2x} + He^{5x}$$

Exercise

Solve the given differential equation by the undetermined coefficients.

1. $2y'' - 7y' + 5y = -29$
2. $y'' + 3y' = 4x - 5$
3. $y'' + 2y' + 2y = 5e^{6x}$
4. $y'' + 4y = 4\cos x + 3\sin x - 8$
5. $y'' + 2y' + y = x^2 e^{-x}$
6. $y'' + y = 4\cos x - \sin x$
7. $y''' - y'' + y' - y = xe^x - e^{-x} + 7$
8. $y'' + y = 8\cos 2x - 4\sin x$, $y(\pi/2) = -1$, $y'(\pi/2) = 0$
9. $y''' - 2y'' + y' = xe^x + 5$, y(0)=2, $y'(0) = 2$, $y''(0) = -1$
10. $y^{(4)} - y''' = x + e^x$, y(0)=0, $y'(0) = 0$, $y''(0) = 0$, $y'''(0) = 0$

Lecture 20 Variation of Parameters

Recall

- That a non-homogeneous linear differential equation with constant coefficients is an equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

- The general solution of such an equation is given by

General Solution = Complementary Function + Particular Integral

- Finding the complementary function has already been completely discussed.

- In the last two lectures, we learnt how to find the particular integral of the non-homogeneous equations by using the undetermined coefficients.

- That the general solution of a linear first order differential equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)$$

is given by $y = e^{-\int P dx} . \int e^{\int P dx} f(x) dx + c_1 e^{-\int P dx}$

Note that

- In this last equation, the 2nd term

$$y_c = c_1 e^{-\int P dx}$$

is solution of the associated homogeneous equation:

$$\frac{dy}{dx} + P(x)y = 0$$

- Similarly, the 1st term

$$y_p = e^{-\int P dx} . \int e^{\int P dx} . f(x) dx$$

is a particular solution of the first order non-homogeneous linear differential equation.

- Therefore, the solution of the first order linear differential equation can be written in the form

$$y = y_c + y_p$$

In this lecture, we will use the variation of parameters to find the particular integral of the non-homogeneous equation.

The Variation of Parameters

First order equation

The particular solution y_p of the first order linear differential equation is given by

$$y_p = e^{-\int P dx}.\int e^{\int P dx}.f(x)dx$$

This formula can also be derived by another method, known as the variation of parameters. The basic procedure is same as discussed in the lecture on construction of a second solution

Since $y_1 = e^{-\int P dx}$

is the solution of the homogeneous differential equation

$$\frac{dy}{dx} + P(x)y = 0,$$

and the equation is linear. Therefore, the general solution of the equation is

$$y = c_1 y_1(x)$$

The variation of parameters consists of finding a function $u_1(x)$ such that

$$y_p = u_1(x)\, y_1(x)$$

is a particular solution of the non-homogeneous differential equation

$$\frac{dy}{dx} + P(x)\, y = f(x)$$

Notice that the parameter c_1 has been replaced by the variable u_1. We substitute y_p in the given equation to obtain

$$u_1\left[\frac{dy_1}{dx} + P(x)y_1\right] + y_1\frac{du_1}{dx} = f(x)$$

Since y_1 is a solution of the non-homogeneous differential equation. Therefore we must have

$$\frac{dy_1}{dx} + P(x)y_1 = 0$$

So that we obtain

$$\therefore \qquad y_1\frac{du_1}{dx} = f(x)$$

This is a variable separable equation. By separating the variables, we have

$$du_1 = \frac{f(x)}{y_1(x)}dx$$

Integrating the last expression *w.r.to* x, we obtain

$$u_1(x) = \int \frac{f(x)}{y_1} dx = \int e^{\int Pdx} \cdot f(x) dx$$

Therefore, the particular solution y_p of the given first-order differential equation is .

$$y = u_1(x) y_1$$

or $$y_p = e^{-\int Pdx} . \int e^{\int Pdx} . f(x) dx$$

$$u_1 = \int \frac{f(x)}{y_1(x)} dx$$

Second Order Equation

Consider the 2nd order linear non-homogeneous differential equation

$$a_2(x) y'' + a_1(x) y' + a_0(x) y = g(x)$$

By dividing with $a_2(x)$, we can write this equation in the standard form

$$y'' + P(x) y' + Q(x) y = f(x)$$

The functions $P(x)$, $Q(x)$ and $f(x)$ are continuous on some interval I . For the complementary function we consider the associated homogeneous differential equation

$$y'' + P(x) y' + Q(x) y = 0$$

Complementary function

Suppose that y_1 and y_2 are two linearly independent solutions of the homogeneous equation. Then y_1 and y_2 form a fundamental set of solutions of the homogeneous equation on the interval I . Thus the complementary function is

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

Since y_1 and y_2 are solutions of the homogeneous equation. Therefore, we have

$$y_1'' + P(x) y_1' + Q(x) y_1 = 0$$

$$y_2'' + P(x) y_2' + Q(x) y_2 = 0$$

Particular Integral

For finding a particular solution y_p, we replace the parameters c_1 and c_2 in the complementary function with the unknown variables $u_1(x)$ and $u_2(x)$. So that the assumed particular integral is

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

Since we seek to determine two unknown functions u_1 and u_2, we need two equations involving these unknowns. One of these two equations results from substituting the assumed y_p in the given differential equation. We impose the other equation to simplify the first derivative and thereby the 2nd derivative of y_p .

$$y'_p = u_1 y'_1 + y_1 u'_1 + u_2 y'_2 + u'_2 y_2 = u_1 y'_1 + u_2 y'_2 + u'_1 y_1 + u'_2 y_2$$

To avoid 2nd derivatives of u_1 and u_2, we impose the condition

$$u'_1 y_1 + u'_2 y_2 = 0$$

Then $y'_p = u_1 y'_1 + u_2 y'_2$

So that

$$y''_p = u_1 y''_1 + u'_1 y'_1 + u_2 y''_2 + u'_2 y'_2$$

Therefore

$$\begin{aligned} y''_p + P\,y'_p + Q\,y_p = {} & u_1 y''_1 + u'_1 y'_1 + u_2 y''_2 + u'_2 y'_2 \\ & + P u_1 y'_1 + P u_2 y'_2 + Q u_1 y_1 + Q u_2 y_2 \end{aligned}$$

Substituting in the given non-homogeneous differential equation yields

$$u_1 y''_1 + u'_1 y'_1 + u_2 y''_2 + u'_2 y'_2 + P u_1 y'_1 + P u_2 y'_2 + Q u_1 y_1 + Q u_2 y_2 = f(x)$$

or $$u_1[y''_1 + P\,y'_1 + Q\,y_1] + u_2[y''_2 + P y'_2 + Q y_2] + u'_1 y'_1 + u'_2 y'_2 = f(x)$$

Now making use of the relations

$$y''_1 + P(x) y'_1 + Q(x) y_1 = 0$$

$$y''_2 + P(x) y'_2 + Q(x) y_2 = 0$$

we obtain

$$u'_1 y'_1 + u'_2 y'_2 = f(x)$$

Hence u_1 and u_2 must be functions that satisfy the equations

$$u'_1 y_1 + u'_2 y_2 = 0$$

$$u'_1 y'_1 + u'_2 y'_2 = f(x)$$

By using the Cramer's rule, the solution of this set of equations is given by

$$u'_1 = \frac{W_1}{W}, \qquad u'_2 = \frac{W_2}{W}$$

Where W, W_1 and W_2 denote the following determinants

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}, \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y'_2 \end{vmatrix}, \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y'_1 & f(x) \end{vmatrix}$$

The determinant W can be identified as the Wronskian of the solutions y_1 and y_2. Since the solutions y_1 and y_2 are linearly independent on I. Therefore

$$W(y_1(x), y_2(x)) \neq 0, \ \forall \ x \in I.$$

Now integrating the expressions for u_1' and u_2', we obtain the values of u_1 and u_2, hence the particular solution of the non-homogeneous linear differential equation.

Summary of the Method

To solve the 2nd order non-homogeneous linear differential equation

$$a_2 y'' + a_1 y' + a_0 y = g(x),$$

using the variation of parameters, we need to perform the following steps:

Step 1 We find the complementary function by solving the associated homogeneous differential equation

$$a_2 y'' + a_1 y' + a_0 y = 0$$

Step 2 If the complementary function of the equation is given by

$$y_c = c_1 y_1 + c_2 y_2$$

then y_1 and y_2 are two linearly independent solutions of the homogeneous differential equation. Then compute the Wronskian of these solutions.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Step 3 By dividing with a_2, we transform the given non-homogeneous equation into the standard form

$$y'' + P(x)y' + Q(x)y = f(x)$$

and we identify the function $f(x)$.

Step 4 We now construct the determinants W_1 and W_2 given by

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Step 5 Next we determine the derivatives of the unknown variables u_1 and u_2 through the relations

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}$$

Step 6 Integrate the derivatives u_1' and u_2' to find the unknown variables u_1 and u_2. So that

$$u_1 = \int \frac{W_1}{W}\, dx, \quad u_2 = \int \frac{W_2}{W}\, dx$$

Step 7 Write a particular solution of the given non-homogeneous equation as

$$y_p = u_1 y_1 + u_2 y_2$$

Step 8 The general solution of the differential equation is then given by

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2.$$

Constants of Integration

We don't need to introduce the constants of integration, when computing the indefinite integrals in step 6 to find the unknown functions of u_1 and u_2. For, if we do introduce these constants, then

$$y_p = (u_1 + a_1)y_1 + \quad (u_2 + b_1)y_2$$

So that the general solution of the given non-homogeneous differential equation is

$$y = y_c + y_p = c_1y_1 + c_2y_2 + (u_1 + a_1)y_1 + (u_2 + b_1)y_2$$

or $$y = (c_1 + a_1)y_1 + (c_2 + b_1)y_2 + u_1y_1 + u_2y_2$$

If we replace $c_1 + a_1$ with C_1 and $c_2 + b_1$ with C_2, we obtain

$$y = C_1y_1 + C_2y_2 + u_1y_1 + u_2y_2$$

This does not provide anything new and is similar to the general solution found in step 8, namely

$$y = c_1y_1 + c_2y_2 + u_1y_1 + u_2y_2$$

Example 1

Solve $$y'' - 4y' + 4y = (x+1)e^{2x}.$$

Solution:

Step 1 To find the complementary function

$$y'' - 4y' + 4y = 0$$

Put $$y = e^{mx}, y' = me^{mx}, y'' = m^2e^{mx}$$

Then the auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0 \Rightarrow m = 2, 2$$

Repeated real roots of the auxiliary equation

$$y_c = c_1e^{2x} \quad + \quad c_2xe^{2x}$$

Step 2 By the inspection of the complementary function y_c, we make the identification

$$y_1 = e^{2x} \text{ and } y_2 = xe^{2x}$$

Therefore $$W(y_1, y_2) = W\left(e^{2x}, xe^{2x}\right) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x} \neq 0, \forall x$$

Step 3 The given differential equation is

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Since this equation is already in the standard form

$$y'' + P(x)y' + Q(x)y = f(x)$$

Therefore, we identify the function $f(x)$ as

$$f(x) = (x+1)e^{2x}$$

Step 4 We now construct the determinants

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x+1)xe^{4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

Step 5 We determine the derivatives of the functions u_1 and u_2 in this step

$$u_1' = \frac{W_1}{W} = -\frac{(x+1)xe^{4x}}{e^{4x}} = -x^2 - x$$

$$u_2' = \frac{W_2}{W} = \frac{(x+1)e^{4x}}{e^{4x}} = x+1$$

Step 6 Integrating the last two expressions, we obtain

$$u_1 = \int (-x^2 - x)dx = -\frac{x^3}{3} - \frac{x^2}{2}$$

$$u_2 = \int (x+1)\, dx \quad = \frac{x^2}{2} + x.$$

Remember! We don't have to add the constants of integration.
Step 7 Therefore, a particular solution of then given differential equation is

$$y_p = \left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}$$

or $$y_p = \left(\frac{x^3}{6} + \frac{x^2}{2}\right)e^{2x}$$

Step 8 Hence**,** the general solution of the given differential equation is

$$y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \left(\frac{x^3}{6} + \frac{x^2}{2}\right)e^{2x}$$

Example 2

Solve $$4y'' + 36y = \csc 3x.$$

Solution:

Step 1 To find the complementary function we solve the associated homogeneous differential equation

$$4y''+36y=0\Rightarrow y''+9y=0$$

The auxiliary equation is

$$m^2+9=0\Rightarrow m=\pm 3i$$

Roots of the auxiliary equation are complex. Therefore, the complementary function is

$$y_c=c_1\cos 3x+c_2\sin 3x$$

Step 2 From the complementary function, we identify

$$y_1=\cos 3x,\ \ y_2=\sin 3x$$

as two linearly independent solutions of the associated homogeneous equation. Therefore

$$W(\cos 3x,\sin 3x)=\begin{vmatrix}\cos 3x & \sin 3x\\ -3\sin 3x & 3\cos 3x\end{vmatrix}=3$$

Step 3 By dividing with 4, we put the given equation in the following standard form

$$y''+9y=\frac{1}{4}\csc 3x.$$

So that we identify the function $f(x)$ as

$$f(x)=\frac{1}{4}\csc 3x$$

Step 4 We now construct the determinants W_1 and W_2

$$W_1=\begin{vmatrix}0 & \sin 3x\\ \frac{1}{4}\csc 3x & 3\cos 3x\end{vmatrix}=-\frac{1}{4}\csc 3x\cdot\sin 3x=-\frac{1}{4}$$

$$W_2=\begin{vmatrix}\cos 3x & 0\\ -3\sin 3x & \frac{1}{4}\csc 3x\end{vmatrix}=\frac{1}{4}\frac{\cos 3x}{\sin 3x}$$

Step 5 Therefore, the derivatives u_1' and u_2' are given by

$$u_1'=\frac{W_1}{W}=-\frac{1}{12},\quad u_2'=\frac{W_2}{W}=\frac{1}{12}\frac{\cos 3x}{\sin 3x}$$

Step 6 Integrating the last two equations $w.r.to\ x$, we obtain

$$u_1 = -\frac{1}{12}x \quad \text{and} \quad u_2 = \frac{1}{36}\ln|\sin 3x|$$

Note that no constants of integration have been added.

Step 7 The particular solution of the non-homogeneous equation is

$$y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|$$

Step 8 Hence, the general solution of the given differential equation is

$$y = y_c + y_p = c_1\cos 3x + c_2\sin 3x - \frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|$$

Example 3

Solve $\qquad y'' - y = \frac{1}{x}.$

Solution:

Step 1 For the complementary function consider the associated homogeneous equation

$$y'' - y = 0$$

To solve this equation we put

$$y = e^{mx}, y' = m\,e^{mx}, y'' = m^2 e^{mx}$$

Then the auxiliary equation is:

$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

The roots of the auxiliary equation are real and distinct. Therefore, the complementary function is

$$y_c = c_1 e^x + c_2 e^{-x}$$

Step 2 From the complementary function we find

$$y_1 = e^x, \quad y_2 = e^{-x}$$

The functions y_1 and y_2 are two linearly independent solutions of the homogeneous equation. The Wronskian of these solutions is

$$W\left(e^x,\ e^{-x}\right) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Step 3 The given equation is already in the standard form

$$y'' + p(x)y' + Q(x)y = f(x)$$

Here $\qquad f(x) = \frac{1}{x}$

Step 4 We now form the determinants

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ 1/x & -e^{-x} \end{vmatrix} = -e^{-x}(1/x)$$

$$W_2 = \begin{vmatrix} e^{x} & 0 \\ e^{x} & 1/x \end{vmatrix} = e^{x}(1/x)$$

Step 5 Therefore, the derivatives of the unknown functions u_1 and u_2 are given by

$$u_1' = \frac{W_1}{W} = -\frac{e^{-x}(1/x)}{-2} = \frac{e^{-x}}{2x}$$

$$u_2' = \frac{W_2}{W} = \frac{e^{x}(1/x)}{-2} = -\frac{e^{x}}{2x}$$

Step 6 We integrate these two equations to find the unknown functions u_1 and u_2.

$$u_1 = \frac{1}{2}\int \frac{e^{-x}}{x}dx, \quad u_2 = -\frac{1}{2}\int \frac{e^{x}}{x}dx$$

The integrals defining u_1 and u_2 cannot be expressed in terms of the elementary functions and it is customary to write such integral as:

$$u_1 = \frac{1}{2}\int_{x_\circ}^{x} \frac{e^{-t}}{t}dt, \quad u_2 = -\frac{1}{2}\int_{x_\circ}^{x} \frac{e^{t}}{t}dt$$

Step 7 A particular solution of the non-homogeneous equations is

$$y_p = \frac{1}{2}e^{x}\int_{x_\circ}^{x} \frac{e^{-t}}{t}dt - \frac{1}{2}e^{-x}\int_{x_\circ}^{x} \frac{e^{t}}{t}dt$$

Step 8 Hence, the general solution of the given differential equation is

$$y = y_c + y_p = c_1e^{x} + c_2e^{-x} + \frac{1}{2}e^{x}\int_{x_\circ}^{x} \frac{e^{-t}}{t}dt - \frac{1}{2}e^{-x}\int_{x_\circ}^{x} \frac{e^{t}}{t}dt$$

Exercise

Solve the differential equations by variations of parameters.

1. $y'' + y = \tan x$
2. $y'' + y = \sec x \tan x$
3. $y'' + y = \sec^2 x$
4. $y'' - y = 9x/e^{3x}$
5. $y'' - 2y' + y = e^x/(1 + x^2)$
6. $4y'' - 4y' + y = e^{x/2}\sqrt{1 - x^2}$
7. $y''' + 4y' = \sec 2x$
8. $2y''' - 6y'' = x^2$

Solve the initial value problems.

9. $2y'' + y' - y = x + 1$
10. $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$