PHYSICS 101

AN INTRODUCTION TO PHYSICS

This course of 45 video lectures, as well as accompanying notes, have been developed and presented by Dr. Pervez Amirali Hoodbhoy, professor of physics at Quaid-e-Azam University, Islamabad, for the Virtual University of Pakistan, Lahore.

TABLE OF CONTENTS

I. GENERAL INFORMATION

II. LECTURE SUMMARIES

		Page #
Lecture 1	Introduction to physics and this course	4
Lecture 2	Kinematics – I	6
Lecture 3	Kinematics – II	8
Lecture 4	Force and Newton's Laws	10
Lecture 5	Applications of Newton's Laws – I	12
Lecture 6	Applications of Newton's Laws – II	14
Lecture 7	Work and Energy	17
Lecture 8	Conservation of Energy	20
Lecture 9	Momentum	23
Lecture 10	Collisions	26
Lecture 11	Rotational Kinematics	28
Lecture 12	Physics of Many Particles	31
Lecture 13	Angular Momentum	36
Lecture 14	Equilibrium of Rigid Bodies	39
Lecture 15	Oscillations – I	42
Lecture 16	Oscillations – II	45
Lecture 17	Physics of Materials	48
Lecture 18	Physics of Fluids	51
Lecture 19	Physics of Sound	54
Lecture 20	Wave Motion	56
Lecture 21	Gravitation	59
Lecture 22	Electrostatics – I	62
Lecture 23	Electrostatics – II	65
Lecture 24	Electric Potential	68
Lecture 25	Capacitors and Currents	71
Lecture 26	Currents and Circuits	74
Lecture 27	The Magnetic Field	78
Lecture 28	Electromagnetic Induction	82
Lecture 29	Alternating Current	86
Lecture 30	Electromagnetic Waves	91
Lecture 31	Physics of Light	95
Lecture 32	Interaction of Light with Matter	99
Lecture 33	Interference and Diffraction	104
Lecture 34	The Particle Nature of Light	108
Lecture 35	Geometrical Optics	112
Lecture 36	Heat – I	117
Lecture 37	Heat – II	123
Lecture 38	Heat – III	127
Lecture 39	Special Relativity – I	131
Lecture 40	Special Relativity – II	137
Lecture 41	Matter as Waves	142
Lecture 42	Quantum Mechanics	149
Lecture 43	Introduction to Atomic Physics	155
Lecture 44	Introduction to Nuclear Physics	162
Lecture 45	Physics of the Sun	170

GENERAL INFORMATION

Purpose: This course aims at providing the student a good understanding of physics at the elementary level. Physics is essential for understanding the modern world, and is a definite part of its culture.

Background: It will be assumed that the student has taken physics and mathematics at the F.Sc level, i.e. the 12^{th} year of schooling. However, B.Sc students are also likely to find the course useful. Calculus is not assumed and some essential concepts will be developed as the course progresses. Algebra and trigonometry are essential. However, for physics, the more mathematics one knows the better.

Scope and Duration: The course has 45 lectures, each of somewhat less than one hour duration. All main fields of physics will be covered, together with several applications in each.

Language: For ease of communication, all lectures are in Urdu. However, English or Latin technical terms have been used where necessary. The student must remember that further study and research in science is possible only if he or she has an adequate grasp of English.

Textbook: There is no prescribed textbook. However, you are strongly recommended to read a book at the level of "College Physics" by Halliday and Resnick (any edition). There are many other such books too, such as "University Physics" by Young and Freedman. Study any book that you are comfortable with, preferably by a well-established foreign author. Avoid local authors because they usually copy. After listening to a lecture, go read the relevant chapter. Please remember that these notes cover only some things that you should know and are not meant to be complete.

Assignments: There will be total Eight Assignment in this course and its schedules will be announced from time to time. The book you choose to consult will have many more. Those students who are seriously interested in the subject are advised to work out several of the questions posed there. In physics you cannot hope to gain mastery of the subject without extensive problem solving.

Examinations: Their schedules will be announced from time to time.

Tutors: Their duty is to help you, and they will respond to all genuine questions. However, please do not overload them as they have to deal with a large number of students. Happy studying!

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Summary of Lecture 1 – INTRODUCTION TO PHYSICS

1. Physics is a science. Science works according to the *scientific method*. The *scientific method* accepts only reason, logic, and experimental evidence to tell between what is scientifically correct and what is not. Scientists do not simply believe – they test, and keep testing until satisfied. Just because some “big scientist” says something is right, that thing does not become a fact of science. Unless a discovery is repeatedly established in different laboratories at different times by different people, or the same theoretical result is derived by clear use of established rules, we do not accept it as a scientific discovery. The real strength of science lies in the fact that it continually keeps challenging itself.

2. It is thought that the laws of physics do not change from place to place. This is why experiments carried out in different countries by different scientists – of any religion or race – have always led to the same results if the experiments have been done honestly and correctly. We also think that the laws of physics today are the same as they were in the past. Evidence, contained in the light that left distant stars billions of years ago, strongly indicates that the laws operating at that time were no different than those today. The spectra of different elements then and now are impossible to tell apart, even though physicists have looked very carefully.

3. This course will cover the following broad categories:
    a) *Classical Mechanics*, which deals with the motion of bodies under the action of forces. This is often called Newtonian mechanics as well.
    b) *Electromagnetism*, whose objective is to study how charges behave under the influence of electric and magnetic fields as well as understand how charges can create these fields.
    c) *Thermal Physics*, in which one studies the nature of heat and the changes that the addition of heat brings about in matter.
    d) *Quantum Mechanics*, which primarily deals with the physics of small objects such as atoms, nuclei, quarks, etc. However, Quantum Mechanics will be treated only briefly for lack of time.

4. Every physical quantity can be expressed in terms of three fundamental dimensions: Mass (M), Length (L), Time (T). Some examples:

Speed	LT^{-1}
Acceleration	LT^{-2}
Force	MLT^{-2}
Energy	ML^2T^{-2}
Pressure	$ML^{-1}T^{-2}$

You cannot add quantities that have different dimensions. So force can be added to force, but force can never be added to energy, etc. A formula is definitely wrong if the dimensions on the left and right sides of the equal sign are different.

5. Remember that any function $f(x)$ takes as input a dimensionless *number x* and outputs a quantity *f* (which may, or may not have a dimension). Take, for example, the function $f(\theta) = \sin\theta$. You know the expansion: $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$ If θ had a dimension then you would be adding up quantities of different dimensions, and that is not allowed.

6. Do not confuse units and dimensions. We can use different units to measure the same physical quantity. So, for example, you can measure the mass in units of kilograms, pounds, or even in *sair* and *chatak!* In this course we shall always use the MKS or **M**etre-**K**ilogram-**S**econd system. When you want to convert from one system to another, be methodical as in the example below:

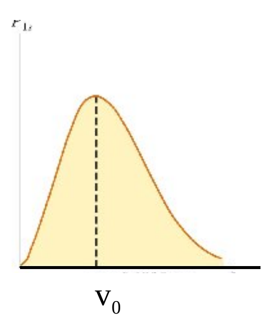
$$1\frac{mi}{hr} = 1\frac{mi}{hr}\times 5280\frac{ft}{mi}\times\frac{1}{3.28}\frac{m}{ft}\times\frac{1}{3600}\frac{hr}{s} = 0.447\frac{m}{s}$$

When you write it out in this manner, note that various quantities cancel out cleanly in the numerator and denominator. So you never make a mistake!

7. A good scientist first thinks of the larger picture and then of the finer details. So, estimating *orders of magnitude* is extremely important. Students often make the mistake of trying to get the decimal points right instead of the first digit – which obviously matters the most! So if you are asked to calculate the height of some building using some data and you come up with 0.301219 metres or 4.01219×10^6 metres, then the answer is plain nonsense even though you may have miraculously got the last six digits right. Physics is commonsense first, so use your intelligence before submitting any answer.

8. Always check your equations to see if they have the same dimensions on the left side as on the right. So, for example, from this principle we can see the equation $\mathrm{v}^2 = u^2 + 2at$ is clearly wrong, whereas $\mathrm{v}^2 = u^2 + 13a^2t^2$ could <u>possibly</u> be a correct relation. (Here v and *u* are velocities, *a* is acceleration, and *t* is time.) Note here that I use the word *possibly* because the dimensions on both sides match up in this case.

9. Whenever you derive an equation that is a little complicated, see if you can find a special limit where it becomes simple and transparent. So, sometimes it is helpful to imagine that some quantity in it is very large or very small. Where possible, make a "mental graph" so that you can picture an equation. So, for example, a formula for the distribution of molecular speeds in a gas could look like $f(\mathrm{v}) = \mathrm{v}e^{-(\mathrm{v}-\mathrm{v}_0)^2/a^2}$. Even without knowing the value of a you can immediately see that
   a) $f(\mathrm{v})$ goes to zero for large values of v, and $\mathrm{v} = 0$.
   b) The maximum value of $f(\mathrm{v})$ occurs at v_0 and the function decreases on both side of this value.



Summary of Lecture 2 – KINEMATICS I

1. *x(t)* is called displacement and it denotes the position of a body at time. If the displacement is positive then that body is to the right of the chosen origin and if negative, then it is to the left.

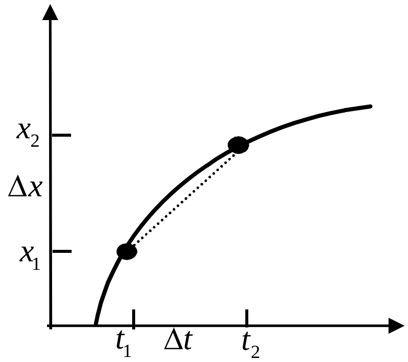
2. If a body is moving with average speed v then in time t it will cover a distance d=vt. But, in fact, the speed of a car changes from time to time and so one should limit the use of this formula to small time differences only. So, more accurately, one defines an average speed over the small time interval Δt as:

$$\text{average speed } = \frac{\text{distance travelled in time } \Delta t}{\Delta t}$$

3. We define *instantaneous velocity* at any time t as:

$$\mathrm{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} \equiv \frac{\Delta x}{\Delta t} \quad .$$

Here Δx and Δt are both very small quantities that tend to zero but their ratio v does not.



4. Just as we have defined velocity as the rate of change of distance, similarly we can define *instantaneous acceleration* at any time t as:

$$a = \frac{\mathrm{v}(t_2) - \mathrm{v}(t_1)}{t_2 - t_1} \equiv \frac{\Delta \mathrm{v}}{\Delta t} \quad .$$

Here Δv and Δt are both very small quantities that tend to zero but their ratio a is not zero, in general. Negative acceleration is called deceleration. The speed of a decelerating body decreases with time.

5. Some students are puzzled by the fact that a body can have a very large acceleration but can be standing still at a given time. In fact, it can be moving in the opposite direction to its acceleration. There is actually nothing strange here because position,

6. For constant acceleration and a body that starts from rest at $t = 0$, v increases linearly with time, $\mathrm{v} \propto t$ (or $\mathrm{v} = at$). If the body has speed v_0 at $t = 0$, then at time t, $\mathrm{v} = at + \mathrm{v}_0$.
7. We know in (6) above how far a body moving at constant speed moves in time *t*. But what if the body is changing its speed? If the speed is increasing linearly (i.e. constant acceleration), then the answer is particularly simple: just use the same formula as in (6) but use the average speed: $(\mathrm{v}_0 + \mathrm{v}_0 + at)/2$. So we get $x = x_0 + (\mathrm{v}_0 + \mathrm{v}_0 + at)t/2 = x_0 + \mathrm{v}_0 t + \frac{1}{2}at^2$. This formula tells you how far a body moves in time *t* if it moves with constant acceleration *a*, and if started at position x_0 at $t=0$ with speed v_0 .

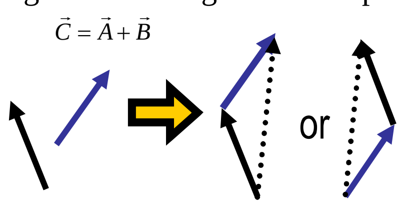
8. We can eliminate the time using (7), and arrive at another useful formula that tells us what the final speed will be after the body has traveled a distance equal to $x - x_0$ after time t, $\mathrm{v}^2 = \mathrm{v}_0^2 + 2a(x - x_0)$.

9. Vectors: a quantity that has a size as well as direction is called a *vector*. So, for example, the wind blows with some speed and in some direction. So the *wind velocity* is a vector.

10. If we choose axes, then a vector is fixed by its components along those axes. In one dimension, a vector has only one component (call it the x-component). In two dimensions, a vector has both x and y components. In three dimensions, the components are along the x,y,z axes.

11. If we denote a vector $\vec{r} = (x, y)$ then, $r_x = x = r\cos\theta$, and $r_y = y = r\sin\theta$. Note that $x^2 + y^2 = r^2$. Also, that $\tan\theta = y/x$.

13. Two vectors can be added together geometrically. We take any one vector, move it without changing its direction so that both vectors start from the same point, and then make a parallelogram. The diagonal of the parallelogram is the resultant.



14. Two vectors can also be added algebraically. In this case, we simply add the components of the two vectors along each axis separately. So, for example, Two vectors can be put together as $(1.5, 2.4) + (1, -1) = (2.5, 1.4)$.

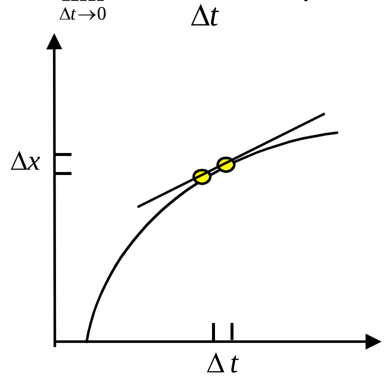
Summary of Lecture 3 – KINEMATICS II

1. The concept of the derivative of a function is exceedingly important. The derivative shows how fast a function changes when its argument is changed. (Remember that for $f(x)$ we say that f is a function that depends upon the argument x. You should think of f as a machine that gives you the value f when you input x.)

2. Functions do not always have to be written as $f(x)$. $x(t)$ is also a function. It tells us where a body is at different times t.

3. The derivative of $x(t)$ at time t is defined as:

$$\frac{dx}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}.$$



4. Let's see how to calculate the derivative of a simple function like $x(t) = t^2$. We must first calculate the difference in x at two slightly different values, t and $t + \Delta t$, while remembering that we choose Δt to be extremely small:

$$\Delta x = \left(t + \Delta t\right)^2 - t^2$$
$$= t^2 + \left(\Delta t\right)^2 + 2t\Delta t - t^2$$
$$\frac{\Delta x}{\Delta t} = \Delta t + 2t \Rightarrow \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = 2$$

5. In exactly the same way you can show that if $x(t) = t^n$ then:

$$\frac{dx}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = nt^{n-1}$$

This is an extremely useful result.

6. Let us apply the above to the function $x(t)$ which represents the distance moved by a body with constant acceleration (see lecture 2):

$$x(t) = x_0 + \mathrm{v}_0 t + \frac{1}{2}at^2$$

$$\frac{dx}{dt} = 0 + \mathrm{v}_0 + \frac{1}{2}a(2t) = \mathrm{v}_0 + a\,t$$

This clearly shows that $\frac{d\mathrm{v}}{dt} = 0 + a = a$ (acceleration is constant)

7. A stone dropped from rest increases its speed in the downward direction according to $\frac{d\mathrm{v}}{dt} = g \approx 9.8$ m/sec. This is true provided we are fairly close to the earth, otherwise the value of g decreases as we go further away from the earth. Also, note that if we measured distances from the ground up, then the acceleration would be negative.

8. A useful notation: write $\frac{d\mathrm{v}}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$. We call $\frac{d^2x}{dt^2}$ the second derivative of x with respect to t, or the rate of rate of change of x with respect to t.

9. It is easy to extend these ideas to a body moving in both the x and y directions. The position and velocity in 2 dimensions are:

$$\vec{r} = x(t)\hat{i} + y(t)\hat{j}$$

$$\vec{\mathrm{v}} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$= \mathrm{v}_x\hat{i} + \mathrm{v}_y\hat{j}$$

Here the unit vectors $\hat{i}$ and $\hat{j}$ are fixed, meaning that they do not depend upon time.

10. The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is defined as:

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

You can think of:

$$\vec{A}\cdot\vec{B} = (A)(B\cos\theta)$$

$$= (\text{length of } \vec{A}) \times (\text{projection of } \vec{B} \text{ on } \vec{A})$$

OR,

$$\vec{A}\cdot\vec{B} = (B)(A\cos\theta)$$

$$= (\text{length of } \vec{B}) \times (\text{projection of } \vec{A} \text{ on } \vec{B}).$$

Remember that for unit vectors $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = 1$ and $\hat{i}\cdot\hat{j} = 0$.

Summary of Lecture 4 – FORCE AND NEWTON'S LAWS

1. Ancient view: objects tend to stop if they are in motion; force is required to keep something moving. This was a natural thing to believe in because we see objects stop moving after some time; frictionless motion is possible to see only in rather special circumstances.

2. Modern view: objects tend to remain in their initial state; force is required to *change* motion. Resistance to changes in motion is called *inertia*. More inertia means it is harder to make a body accelerate or decelerate.

3. Newton's First Law: An object will remain at rest or move with constant velocity unless acted upon by a net external force. (A non-accelerating reference frame is called an inertial frame; Newton's First Law holds only in inertial frames.)
4. More force leads to more acceleration: $\Rightarrow \quad a \propto F$

5. The greater the mass of a body, the harder it is to change its state of motion. More mass means more inertia. In other words, more mass leads to less acceleration:

$$\Rightarrow \quad a \propto \frac{1}{m}$$

Combine both the above observations to conclude that:

$$a \propto \frac{F}{m}$$

6. Newton's Second Law: $a = \frac{F}{m}$ (or, if you prefer, write as $F = ma$).

7. $F = ma$ is one relation between three independent quantities (m, a, F). For it to be useful, we must have separate ways of measuring mass, acceleration, and force. Acceleration is measured from observing the rate of change of velocity; mass is a measure of the amount of matter in a body (e.g. two identical cars have twice the mass of a single one). Forces (due to gravity, a stretched spring, repulsion of two like charges, etc) will be discussed later.

8. Force has dimensions of $[\text{mass}] \times [\text{acceleration}] = M\,L\,T^{-2}$. In the MKS system the unit of force is the Newton. It has the symbol N where:
1 Newton = 1 kilogram.metre/second^2.

9. Forces can be internal or external. For example the mutual attraction of atoms within a block of wood are called internal forces. Something pushing the wood

is an external force. In the application of $F = ma$, remember that F stands for the total external force upon the body.

10. Forces are vectors, and so they must be added vectorially:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots$$

This means that the components in the $\hat{x}$ direction must be added separately, those in the $\hat{y}$ direction separately, etc.

11. Gravity acts directly on the mass of a body - this is a very important experimental observation due to Newton and does not follow from $F = ma$. So a body of mass m_1 experiences a force $F_1 = m_1 g$ while a body of mass m_2 experiences a force $F_2 = m_2 g$, where g is the acceleration with which any body (big or small) falls under the influence of gravity. (Galileo had established this important fact when he dropped different masses from the famous leaning tower of Pisa!)

12. The weight of a body W is the force which gravity exerts upon it, $W = mg$. Mass and weight are two completely different quantities. So, for example, if you used a spring balance to weigh a kilo of grapes on earth, the same grapes would weigh only 1/7 kilo on the moon.

13. Newton's Third Law: for every action there is an equal and opposite reaction. More precisely, $F_{AB} = -F_{BA}$, where F_{AB} is the force exerted by body *B* upon *A* whereas F_{BA} is the force exerted by body *A* upon *B*. Ask yourself what would happen if this was not true. In that case, a system of two bodies, even if it is completely isolated from the surroundings, would have a net force acting upon it because the net force acting upon both bodies would be $F_{AB} + F_{BA} \neq 0$.

14. If action and reaction are always equal, then why does a body accelerate at all? Students are often confused by this. The answer: in considering the acceleration of a body you must consider only the (net) force acting upon that body. So, for example, the earth pulls a stone towards it and causes it to accelerate because there is a net force acting upon the stone. On the other hand, by the Third Law, the stone also pulls the earth towards it and this causes the earth to accelerate towards the stone. However, because the mass of the earth is so large, we are only able to see the acceleration of the stone and not that of the earth.

Summary of Lecture 5 – APPLICATIONS OF NEWTON'S LAWS – I

1. An obvious conclusion from $F = ma$ is that if $F = 0$ then $a = 0$! How simple, yet how powerful ! This says that for any body that is not accelerating the *sum of all the forces acting upon it* must vanish.

2. Examples of systems in equilibrium: a stone resting on the ground; a pencil balanced on your finger; a ladder placed against the wall, an aircraft flying at a constant speed and constant height.

3. Examples of systems out of equilibrium: a stone thrown upwards that is at its highest point; a plane diving downwards; a car at rest whose driver has just stepped on the car's accelerator.

4. If you know the acceleration of a body, it is easy to find the force that causes it to accelerate. Example: An aircraft of mass m has position vector,

$$\vec{r} = (at + bt^3)\hat{i} + (ct^2 + dt^4)\hat{j}$$

What force is acting upon it?

SOLUTION:
$$\vec{F} = m\frac{d^2x}{dt^2}\hat{i} + m\frac{d^2y}{dt^2}\hat{j}$$
$$= 6\,b\,m\,t\,\hat{i} + m(2c + 12d\,t^2)\,\hat{j}$$

5. The other way around is not so simple: suppose that you know F and you want to find x. For this you must solve the equation,

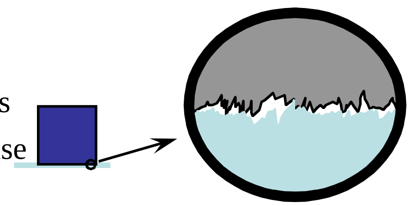
$$\frac{d^2x}{dt^2} = \frac{F}{m}$$

This may or may not be easy, depending upon F (which may depend upon both x as well as t if the force is not constant).

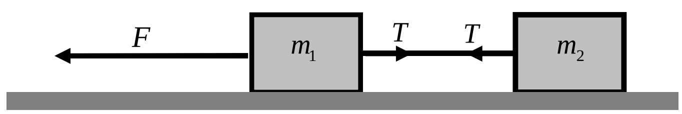
6. Ropes are useful because you can pull from a distance to change the direction of a force. The tension, often denoted by T, is the force you would feel if you cut the rope and grabbed the ends. For a massless rope (which may be a very good approximation in many situations) the tension is the same at every point along the rope. Why? Because if you take any small slice of the rope it weighs nothing (or very little). So if the force on one side of the slice was any different from the force force on the other side, it would be accelerating hugely. All this was for the "ideal rope" which has no mass and never breaks. But this idealization if often good enough.

7. We are all familiar with frictional force. When two bodies rub against each other, the frictional force acts upon each body separately opposite to its direction of motion (i.e it acts to slow down the motion). The harder you press two bodies against each other, the greater the friction. Mathematically, $\vec{F} = \mu \vec{N}$, where $\vec{N}$ is the force with which you press the two bodies against each other (normal force). The quantity μ is called the coefficient of friction (obviously!). It is large for rough surfaces, and small for smooth ones. Remember that $\vec{F} = \mu \vec{N}$ is an empirical relation and holds only approximately. This is obviously true: if you put a large enough mass on a table, the table will start to bend and will eventually break.

8. Friction is caused by roughness at a microscopic level - if you look at any surface with a powerful microscope you will see unevenness and jaggedness. If these big bumps are levelled somehow, friction will still not disappear because there will still be little bumps due to atoms. More precisely, atoms from the two bodies will interact each other because of the electrostatic interaction between their charges. Even if an atom is neutral, it can still exchange electrons and there will be a force because of surrounding atoms.



9. Consider the two blocks below on a frictionless surface:



We want to find the tension and acceleration: The total force on the first mass is $F - T$ and so $F - T = m_1 a$. The force on the second mass is simply T and so $T = m_2 a$. Solving the above, we get: $T = \dfrac{m_2 F}{m_1 + m_2}$ and $a = \dfrac{F}{m_1 + m_2}$.

10. There is a general principle by which you solve equilibrium problems. For equilibrium, the sum of forces in every direction must vanish. So $F_x = F_y = F_z = 0$. You may always choose the x, y, z directions according to your convenience. So, for example, as in the lecture problem dealing with a body sliding down an inclined plane, you can choose the directions to be along and perpendicular to the surface of the plane.

Summary of Lecture 6 – APPLICATIONS OF NEWTON'S LAWS – II

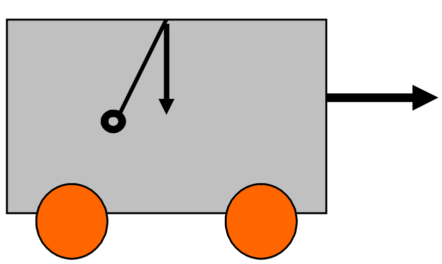
1. As a body moves through a body it displaces the fluid. it has to exert a force on the fluid to push it out of the way. By Newton's third law, the fluid pushes back on the body with an equal and opposite force. The direction of the fluid resistance force on a body is always opposite to the direction of the body's velocity relative to the fluid.

2. The magnitude of the fluid resistance force usually increases with the speed of the body through the fluid. Typically, $f = k\text{v}$ (an empirical law!). Imagine that you drop a ball bearing into a deep container filled with oil. After a while the ball bearing will approach its maximum (terminal) speed when the forces of gravity and friction balance each other: $mg = k\text{v}$ from which $\text{v}_{\text{final}} = mg/k$.

3. The above was a simple example of equilibrium under two forces. In general, while solving problems you should a)draw a diagram, b)define an origin for a system of coordinates, c)identify all forces (tension, normal, friction, weight, etc) and their x and y components, d)Apply Newton's law separately along the x and y axes. e) find the accelerations, then velocities, then displacements. This sounds very cook-book, and in fact it will occur to you naturally how to do this while solving actual problems.

4. Your weight in a lift: suppose you are in a lift that is at rest or moving at constant velocity. In either case $a=0$ and the normal force N and the force due to gravity are exactly equal, $N - Mg = 0 \Rightarrow N = Mg$. But if the lift is accelerating downwards then $Mg - N = Ma$ or $N = M(g-a)$. So now the normal force (i.e. the force with which the floor of the lift is pushing on you) is decreased. Note that if the lift is accelerating downwards with acceleration a (which it will if the cable breaks!) then $N=0$ and you will experience weightlessness just like astronauts in space do. Finally, if the lift is accelerating upwards then a is negative and you will feel heavier.

5. Imagine that you are in a railway wagon and want to know how much you are accelerating. You are not able to look out of the windows. A mass is hung from the roof. Find the acceleration of the car from the angle made by the mass.

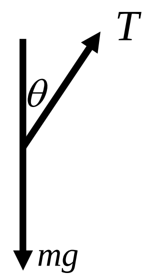


We first balance the forces vertically: $T\cos\theta = mg$

And then horizontally: $T\sin\theta = ma$

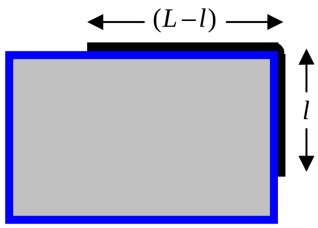
From these two equations we find that: $\tan\theta = \frac{a}{g}$

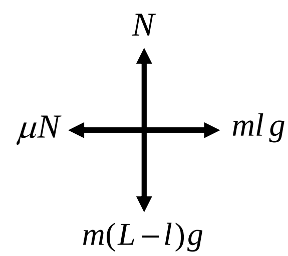
Note that the mass m doesn't matter - it cancels out!



6. Friction is a funny kind of force. It doe not make up its mind which way to act until some other force compels it to decide. Imagine a block lying on the floor. If you push it forward, friction will act backward. And if you push it to the left, friction will act to the right. In other words, the direction of the frictional force is always in the opposite direction to the applied force.

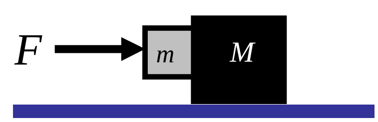
7. Let us solve the following problem: a rope of total length L and mass per unit length m is put on a table with a length l hanging from one edge. What should be l such that the rope just begins to slip?





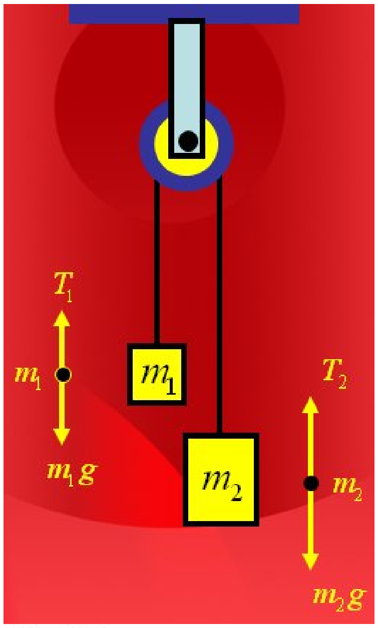
To solve this, look at the balance of forces in the diagram below: in the vertical direction, the normal force balances the weight of that part of the rope that lies on the table: $N = m(L-l)g$. In the horizontal direction, the rope exerts a force mlg to the right, which is counteracted by the friction that acts to the left. Therefore $\mu N = mlg$. Substituting N from the first equation we find that $l = \frac{\mu L}{\mu + 1}$. Note that if μ isvery small then even a small piece of string that hangs over the edge will cause the entire string to slip down.

9. In this problem, we would like to calculate the minimum force F such that the small block does not slip downwards. Clearly, since the 2 bodies move together, $F = (m+M)a$. This gives $a = \frac{F}{(m+M)}$. We want the friction μN to be at least as large as the downwards force, mg.



So, we put $N = ma = m\left(\frac{F}{(m+M)}\right)$ from which the minimum horizontal force needed to prevent slippage is $F = \frac{(m+M)g}{\mu}$.

There is an error in video lecture # 6 (video recording at 41:00) please consider the following problem with correct solution.
Consider two bodies of unequal masses m_1 and m_2 connected by the ends of a string, which passes over a frictionless pulley as shown in the diagram.



$$m_2 > m_1$$
$$T - m_1 g = m_1 a$$
$$m_2 g - T = m_2 a$$
$$m_2 g - m_1 g = (m_1 + m_2)a$$
$$a = \frac{(m_2 - m_1)}{(m_1 + m_2)} g$$

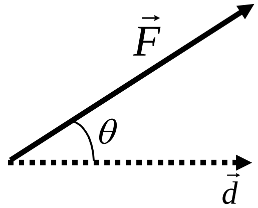
$$m_2 > m_1$$
$$T - m_1 g = m_1 a$$
$$a = \frac{T - m_1 g}{m_1} \qquad (i$$
$$m_2 g - T = m_2 a$$
$$a = \frac{m_2 g - T}{m_2} \qquad (ii$$

equating (i and (ii

$$\frac{T - m_1 g}{m_1} = \frac{m_2 g - T}{m_2}$$
$$m_2 T - m_1 m_2 g = m_1 m_2 g - m_1 T$$
$$m_2 T + m_1 T = m_1 m_2 g + m_1 m_2 g$$
$$T(m_2 + m_1) = 2 m_1 m_2 g$$
$$T = \frac{2 m_1 m_2 g}{(m_2 + m_1)}$$

Summary of Lecture 7 – WORK AND ENERGY

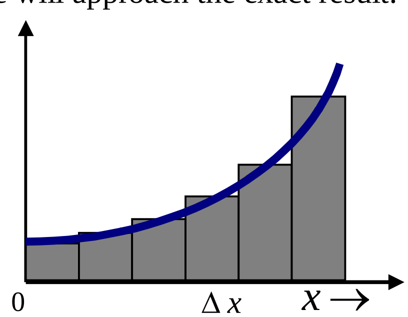
1. Definition of work: force applied in direction of displacement × displacement. This means that if the force F acts at an angle θ with respect to the direction of motion, then $W = \vec{F}\cdot\vec{d} = Fd\cos\theta$



2. a) Work is a scalar - it has magnitude but no direction.
   b) Work has dimensions: $M \times (L\ T^{-2}) \times L = M\ L^2 T^{-2}$
   c) Work has units: 1 Newton × 1 Metre ≡ 1 Joule (J)

3. Suppose you lift a mass of 20 kg through a distance of 2 metres. Then the work you do is 20 kg × 9.8 Newtons × 2 metres = 392 Joules. On the other hand, the force of gravity is directed opposite to the force you exert and the work done by gravity is -392 Joules.

4. What if the force varies with distance (say, a spring pulls harder as it becomes longer). In that case, we should break up the distance over which the force acts into small pieces so that the force is approximately constant over each bit. As we make the pieces smaller and smaller, we will approach the exact result:



Now add up all the little pieces of work:

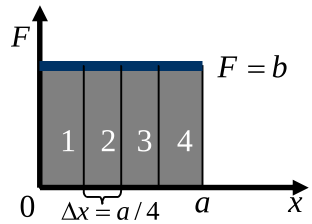
$$W = \Delta W_1 + \Delta W_2 + \cdots + \Delta W_N = F_1\Delta x + F_2\Delta x + \cdots + F_N\Delta x \equiv \sum_{n=1}^{N} F_n \Delta x$$

To get the exact result let $\Delta x \rightarrow 0$ and the number of intervals N $\rightarrow \infty$: $W = \lim_{\Delta x \rightarrow 0} \sum_{n=1}^{\infty} F_n \Delta x$

Definition: $W = \lim_{\Delta x \rightarrow 0} \sum_{n=1}^{\infty} F_n\ \Delta x \equiv \int_{x_i}^{x_f} F(x)dx$ is called the integral of F with respect to x from

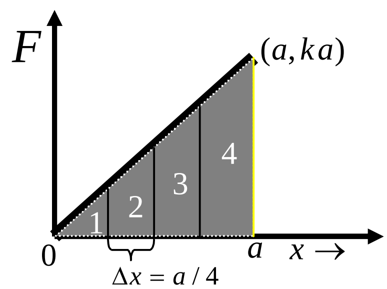
x_i to x_f. This quantity is the work done by a force, constant or non-constant. So if the force is known as a function of position, we can always find the work done by calculating the definite integral.

5. Just to check what our result looks like for a constant force, let us calculate W if $F = b$,



$$\frac{1}{4}a(b) + \frac{1}{4}a(b) + \frac{1}{4}a(b) + \frac{1}{4}a(b) = ab \qquad \therefore \int_0^a F dx = ab$$

6. Now for a less trivial case: suppose that *F=kx*, i.e. the force increases linearly with *x*.



Area of shaded region $= \frac{1}{2}(a)(ka) = k\frac{a^2}{2} \qquad \therefore \int_0^a F dx = k\frac{a^2}{2}$

7. Energy is the capacity of a physical system to do work:
   - it comes in many forms – mechanical, electrical, chemical, nuclear, etc
   - it can be stored
   - it can be converted into different forms
   - it can never be *created* or *destroyed*

8. Accepting the fact that energy is conserved, let us derive an expression for the kinetic energy of a body. Suppose a *constant force* accelerates a mass *m* from speed *0* to speed v over a *distance d*. What is the work done by the force? Obviously the answer is: $W = Fd$. But $F = ma$ and $\text{v}^2 = 2ad$. This gives $W = (ma)d = \frac{m\text{v}^2}{2d}d = \frac{1}{2}m\text{v}^2$. So, we conclude that the work done by the force has gone into creating kinetic energy. and that the amount of kinetic energy possessed by a body moving with speed v is $\frac{1}{2}m\text{v}^2$.

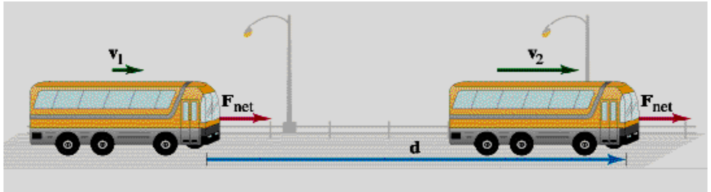
9. The work done by a force is just the force multiplied by the distance – it does not depend upon time. But suppose that the same amount of work is done in half the time. We then say that the *power* is twice as much.

We define:

$$\text{Power} = \frac{\text{Work done}}{\text{Time taken}}$$

If the force does not depend on time: $\dfrac{\text{Work}}{\text{Time}} = \dfrac{F\,\Delta x}{\Delta t} = F\,\text{v}$. Therefore, Power = F v.

10. Let's work out an example. A constant force accelerates a bus (mass m) from speed v_1 to speed v_2 over a distance d. What work is done by the engine?



Recall that for constant acceleration, $\text{v}_2^2 - \text{v}_1^2 = 2a\,(x_2 - x_1)$ where: v_2 = final velocity, x_2 = final position, v_1 = initial velocity, x_1 = initial position. Hence, $a = \dfrac{\text{v}_2^2 - \text{v}_1^2}{2d}$. Now calculate the work done: $W = Fd = mad = m\dfrac{\text{v}_2^2 - \text{v}_1^2}{2d}\,d = \dfrac{1}{2}m\text{v}_2{}^2 - \dfrac{1}{2}m\text{v}_1{}^2$. So the work done has resulted in an increase in the quantity $\dfrac{1}{2}m\text{v}^2$, which is kinetic energy.

Summary of Lecture 8 – CONSERVATION OF ENERGY

1. Potential energy is, as the word suggests, the energy "locked up" up somewhere and which can do work. An object can store energy as the result of its position. Stored energy of position is referred to as potential energy. **Potential energy** is the stored energy of position possessed by an object. It has the ability or capacity to do work like other forms of energies.Potential energy can be converted into kinetic energy, $\frac{1}{2}mv^2$. As I showed you earlier, this follows directly from Newton's Laws.

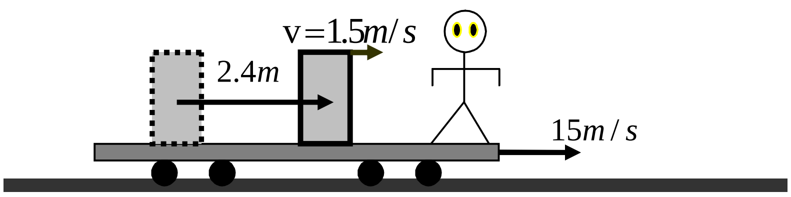
2. If you lift a stone of mass *m* from the ground up a distance *x*, you have to do work against gravity. The (constant) force is mg, and so $W = mgx$. By conservation of energy, the work done by you was transformed into gravitational potential energy whose values is exactly equal to *mgx*. Where is the energy stored? Answer: it is stored neither in the mass or in the earth - it is stored in the gravitational field of the combined system of stone+earth.

3. Suppose you pull on a spring and stretch it by an amount *x* away from its normal (equilibrium) position. How much energy is stored in the spring? Obviously, the spring gets harder and harder to pull as it becomes longer. When it is extended by length *x* and you pull it a further distance *dx*, the small amount of work done is $dW = Fdx = kxdx$. Adding up all the small bits of work gives the total work:

$$W = \int_0^x Fdx = \int_0^x kx\,dx = \frac{1}{2}kx^2$$

This is the work you did. Maybe you got tired working so hard. What was the result of your working so hard? Answer: this work was transformed into energy stored in the spring. The spring contains energy exactly equal to $\frac{1}{2}kx^2$.

4. Kinetic energy obviously depends on the frame you choose to measure it in. If you are running with a ball, it has zero kinetic energy with respect to you. But someone who is standing will see that it has kinetic energy! Now consider the following situation: a box of mass 12kg is pushed with a constant force so that so that its speed goes from zero to 1.5m/sec (as measured by the person at rest on the cart) and it covers a distance of 2.4m. Assume there is no friction.



mass of box $=12\ kg$

Let's first calculate the change in kinetic energy:

$$\Delta K = K_f - K_i = \frac{1}{2}(12kg)(1.5m/s)^2 - 0 = 13.5J$$

And then the (constant) acceleration:

$$a = \frac{v_f^2 - v_i^2}{2(x_f - x_i)} = \frac{(1.5m/s)^2 - 0}{2(2\cdot 4m)} = 0.469m/s^2$$

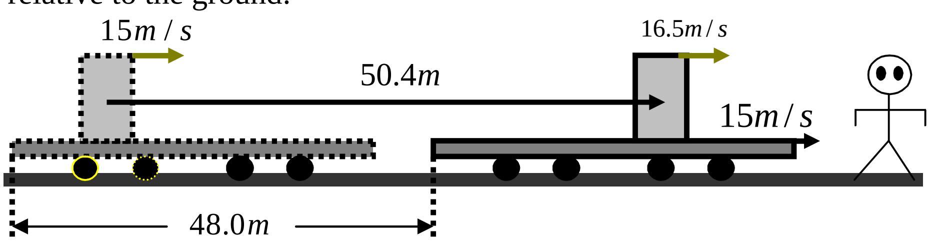
This acceleration results from a constant net force given by:

$$F = ma = (12kg)(0.469\,m/s^2) = 5.63N$$

From this, the work done on the crate is:

$$W = F\Delta x = (5.63N)(2.4m) = 13.5J \quad (\text{same as } \Delta K = 13.5J\ !)$$

5. Now suppose there is somebody standing on the ground, and that the trolley moves at 15 m/sec relative to the ground:



Let us repeat the same calculation:

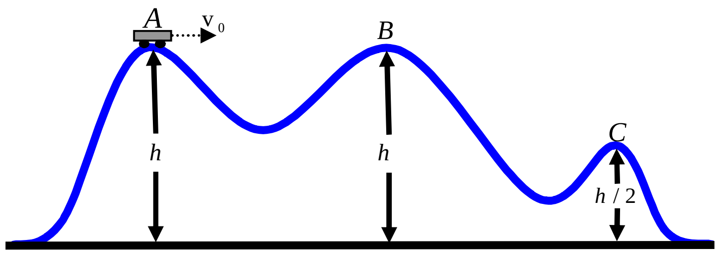
$$\Delta K' = K_f' - K_i' = \frac{1}{2}mv_f^{\ 2} - \frac{1}{2}mv_i^{\ 2}$$

$$= \frac{1}{2}(12kg)(16.5m/s)^2 - \frac{1}{2}(12kg)(15.0m/s)^2 \ = 284J$$

This example clearly shows that work and energy have different values in different frames.

6. The total mechanical energy is: $E_{mech} = KE + PE$. If there is no friction then E_{mech} is conserved. This means that the sum does not change with time. For example: a ball is thrown upwards at speed v_0. How high will it go before it stops? The loss of potential energy is equal to the gain of potential energy. Hence, $\frac{1}{2}mv_0^2 = mgh \ \Rightarrow \ h = \frac{v_0^2}{2g}$.

Now look at the smooth, frictionless motion of a car over the hills below:



Even though the motion is complicated, we can use the fact that the total energy is a constant to get the speeds at the points B,C,D:

At point A: $\frac{1}{2}mv_A^{\ 2}+mgh=\frac{1}{2}mv_B^{\ 2}+mgh \quad \Rightarrow \quad v_B=v_C$

At point C: $\frac{1}{2}mv_A^{\ 2}+mgh=\frac{1}{2}mv_C^{\ 2}+mg\frac{h}{2} \quad \Rightarrow \quad v_C=\sqrt{v_A^{\ 2}+gh}$

At point D: $\frac{1}{2}mv_A^{\ 2}+mgh=\frac{1}{2}mv_D^{\ 2} \quad \Rightarrow \quad v_D=\sqrt{v_A^{\ 2}+2gh}$

7. Remember that potential energy has meaning only for a force that is conservative. A conservative force is that for which the work done in going from point A to point B is independent of the path chosen. Friction is an example of a non-conservative force and a potential energy cannot be defined. For a conservative force, $F=-\frac{dV}{dx}$. So, for a spring, $V=\frac{1}{2}kx^2$ and so $F=-kx$.

8. Derivation of $F=-\frac{dV}{dx}$: If the particle moves distance Δx in a potential V, then change in PE is ΔV where, $\Delta V=-F\ \Delta x$. From this, $F=-\frac{\Delta V}{\Delta x}$. Now let $\Delta x\rightarrow 0$. Hence, $F=-\lim_{\Delta x\rightarrow 0}\frac{\Delta V}{\Delta x}=-\frac{dV}{dx}$.

Summary of Lecture 9 – MOMENTUM

1. Momentum is the "quantity of motion" possessed by a body. More precisely, it is defined as:

$$\textbf{Mass of the body} \times \textbf{Velocity of the body}.$$

The dimensions of momentum are MLT^{-1} and the units of momentum are kg-m/s.

2. Momentum is a vector quantity and has both magnitude and direction, $p = m\vec{v}$. We can easily see that Newton's Second Law can be reexpressed in terms of momentum. When I wrote it down originally, it was in the form $m\vec{a} = \vec{\mathbf{F}}$. But since $\frac{d\vec{v}}{dt} = \vec{a}$, this can also be written as $\frac{d\vec{\mathbf{p}}}{dt} = \vec{\mathbf{F}}$ (new form). In words, the rate of change of momentum of a body equals the total force acting upon it. Of course, the old and new are exactly the same,

$$\frac{d\vec{\mathbf{p}}}{dt} = \frac{d(m\vec{\mathbf{v}})}{dt} = m\frac{d\vec{\mathbf{v}}}{dt} = m\vec{\mathbf{a}} = \vec{\mathbf{F}}.$$

3. When there are many particles, then the total momentum $\vec{\mathbf{P}}$ is,

$$\vec{\mathbf{P}} = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 + \cdots \vec{\mathbf{p}}_N$$

$$\frac{d}{dt}\vec{\mathbf{P}} = \frac{d\vec{\mathbf{p}}_1}{dt} + \frac{d\vec{\mathbf{p}}_2}{dt} + \cdots \frac{d\vec{\mathbf{p}}_N}{dt}$$

$$= \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots \vec{\mathbf{F}}_N = \vec{\mathbf{F}}$$

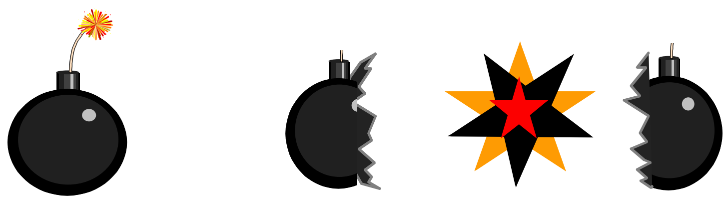
This shows that when there are several particles, the rate at which the total momentum changes is equal to the total force. It makes sense!

4. A very important conclusion of the above is that if the sum of the total external forces vanishes, then the total momentum is conserved, $\sum \vec{F}_{ext} = 0 \Rightarrow \frac{d}{dt}\vec{\mathbf{P}} = 0.$ This is quite independent of what sort of forces act between the bodies - electric, gravitational, etc. - or how complicated these are. We shall see why this is so important from the following examples.

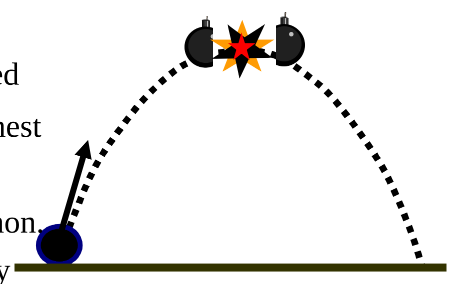
5. Two balls, which can only move along a straight line, collide with each other. The initial momentum is $P_i = m_1 u_1 + m_2 u_2$ and the final momentum is $P_f = m_1 v_1 + m_2 v_2$. Obviously one ball exerts a force on the other when they collide, so its momentum changes. But, from the fact that there is no external force acting on the balls, $P_i = P_f$, or $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$.

6. A bomb at rest explodes into two fragments. Before the explosion the total momentum is

zero. So obviously it is zero after the explosion as well, $\mathbf{P}_f = \mathbf{0}$. During the time that the explosion happens, the forces acting upon the pieces are very complicated and changing rapidly with time. But when all is said and done, there are two pieces flying away with a total zero final momentum $\mathbf{P}_f = m_1 v_1 + m_2 v_2$. Hence $m_1 v_1 = -m_2 v_2$. In other words, the fragments fly apart with equal momentum but in opposite directions. The centre-of-mass stays at rest. So, knowing the velocity of one fragment permits knowing the velocity of the other fragment.



7. If air resistance can be ignored, then we can do some interesting calculations with what we have learned. So, suppose a shell is fired from a cannon with a speed 10 m/s at an angle 60^0 with the horizontal. At the highest point in its path it explodes into two pieces of equal masses. One of the pieces retraces its path to the cannon. Let us find the velocity of the other piece immediately after the explosion.



Solution: After the explosion: $P_{1x} = -5\dfrac{M}{2}$ (why?). But $P_{1x} + P_{2x} = P_x = M \times 10\cos 60$

$$\Rightarrow P_{2x} = 5M + 5\frac{M}{2}. \text{ Now use: } P_{2x} = \frac{M}{2} v_{2x} \Rightarrow v_{2x} = 15\, m/s.$$

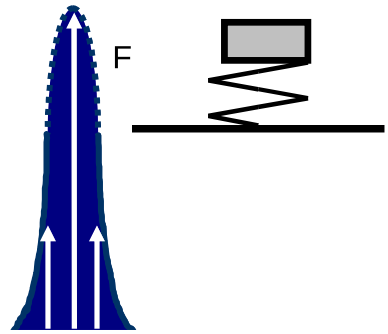
8. When you hit your thumb with a hammer it hurts, doesn't it? Why? Because a large amount of momentum has been destroyed in a short amount of time. If you wrap your thumb with foam, it will hurt less. To understand this better, remember that force is the rate of change of momentum: $F = \dfrac{dp}{dt} \Rightarrow dp = Fdt$. Now define the **impulse** I as:

force × time over which the force acts.

If the force changes with time between the limits , then one should define I as,

$I = \int_{t_1}^{t_2} F dt$. Since $\int_{t_1}^{t_2} F dt = \int_{p_i}^{p_f} dp$, therefore $I = p_f - p_i$. In words, the change of momentum equals the impulse, which is equal to the area under the curve of force versus time. Even if you wrap your thumb in foam, the impulse is the same. But the force is definitely not!

9. Sometimes we only know the force numerically (i.e. there is no expression like F=something). But we still know what the integral means: it is the area under the curve of force versus time. The curve here is that of a hammer striking a table. Before the hammer strikes, the force is zero, reaches a peak, and goes back to zero.



Summary of Lecture 10 – COLLISIONS

1. Collisions are extremely important to understand because they happen all the time - electrons collide with atoms, a bat with a ball, cars with trucks, star galaxies with other galaxies,...In every case, the sum of the initial momenta equals the sum of the final momenta. This follows directly from Newton's Second Law, as we have already seen.

2. Take the simplest collision: two bodies of mass m_1 and m_2 moving with velocities u_1 and u_2. After the collision they are moving with velocities v_1 and v_2. From momentum conservation,

$$m_1u_1 + m_2u_2 = m_1\mathrm{v}_1 + m_2\mathrm{v}_2 \Rightarrow m_1(u_1 - \mathrm{v}_1) = m_2(\mathrm{v}_2 - u_2)$$

This is as far as we can go. There are two unknowns but only one equation. However, if the collision is elastic then,

$$\tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2 = \tfrac{1}{2}m_1\mathrm{v}_1^2 + \tfrac{1}{2}m_2\mathrm{v}_2^2 \Rightarrow \tfrac{1}{2}m_1(u_1^{\,2} - \mathrm{v}_1^{\,2}) = \tfrac{1}{2}m_2(\mathrm{v}_2^{\,2} - u_2^{\,2}).$$

Combine the two equations above,

$$u_1 + \mathrm{v}_1 = \mathrm{v}_2 + u_2 \Rightarrow u_1 - u_2 = \mathrm{v}_2 - \mathrm{v}_1.$$

In words, this says that in an elastic collision the relative speed of the incoming particles equals the relative speed of the outgoing particles.

3. One can solve for v_1 and v_2 (please do it!) easily and find that:

$$\mathrm{v}_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$

$$\mathrm{v}_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2$$

Notice that if $m_1 = m_2$, then $\mathrm{v}_1 = u_2$ and $\mathrm{v}_2 = u_1$. So this says that after the collision, the bodies will just reverse their velocities and move on as before.

4. What if one of the bodies is much heavier than the other body, and the heavier body is at rest? In this case, $m_2 >> m_1$ and $u_2 = 0$. e can immediately see that $\mathrm{v}_1 = -u_1$ and $\mathrm{v}_2 = 0$. This makes a lot of sense: the heavy body continues to stay at rest and the light body just bounces back with the same speed. In the lecture, you saw a demonstration of this!

5. And what if the lighter body (rickshaw) is at rest and is hit by the heavier body (truck)? In this case, $m_2 << m_1$ and $u_2 = 0$. From the above equation we see that $\mathrm{v}_1 = u_1$ and $\mathrm{v}_2 = 2u_1$. So the truck's speed is unaffected, but the poor rickshaw is thrust in the direction of the truck at twice the truck's speed!

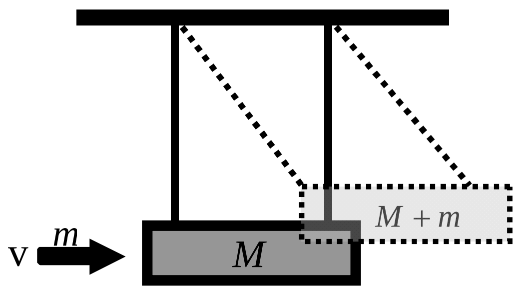
6. Sometimes we wish to slow down particles by making them collide with other particles.

In a nuclear reactor, neutrons can be slowed down in this way. let's calculate the fraction by which the kinetic energy of a neutron of mass m_1 decreases in a head-on collision with an atomic nucleus of mass m_2 that is initially at rest:

Solution: $\frac{K_i - K_f}{K_i} = 1 - \frac{K_f}{K_i} = 1 - \frac{v_f^2}{v_i^2}$

For a target at rest: $v_f = (\frac{m_1 - m_2}{m_1 + m_2})v_i$ $\quad \therefore \frac{K_i - K_f}{K_i} = \frac{4m_1 m_2}{(m_1 + m_2)^2}$.

7. A bullet with mass m, is fired into a block of wood with mass M, suspended like a pendulum and makes a completely inelastic collision with it. After the impact, the block swings up to a maximum height y. What is the initial speed of the bullet?



Solution:
By conservation of momentum in the direction of the bullet,

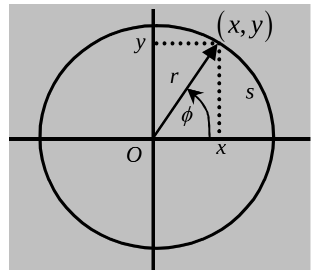
$m\text{v} = (m + M)V, \Rightarrow \text{v} = \frac{(m + M)}{m}V$

The block goes up by distance y, and so gains potential energy. Now we can use the conservation of energy to give, $\frac{1}{2}(m + M)V^2 = (m + M)gy,$ where V is the velocity acquired by the block+bullet in the upward direction just after the bullet strikes. Now use $V = \sqrt{2gy}$. So finally, the speed of the bullet is: $\text{v} = \frac{(m + M)}{m}\sqrt{2gy}$.

8. In 2 or 3 dimensions, you must apply conservation of momentum in each direction separately. The equation $\vec{P}_i = \vec{P}_f$ looks as if it is one equation, but it is actually 3 separate equations: $p_{ix} = p_{fx}, p_{iy} = p_{fy}, p_{iz} = p_{fz}$. On the other hand, suppose you had an elastic collision. In that case you would have only one extra equation coming from energy conservation, not three.

9. What happens to energy in an inelastic collision? Let's say that one body smashes into another body and breaks it into 20 pieces. To create 20 pieces requires doing work against the intermolecular forces, and the initial kinetic energy is used up for this.

Summary of Lecture 11 – ROTATIONAL KINEMATICS



1. Any rotation is specified by giving two pieces of information:
   a) The point about which the rotation occurs, i.e. the origin.
   b) The angle of rotation is denoted by ϕ in the diagram.

2. The arc length = radius $\times$ angular displacement, or $s = r\phi$. Here ϕ is measured in radians. The maximum value of ϕ is 2π radians, which corresponds to 360 degrees or one full revolution. From this it follows that 1 radian $= 57.3^0$ or 1 radian $= 0.159$ revolution. Obviously, if $\phi = 2\pi$, then $s = 2\pi r$, which is the total circumference.

3. Suppose that there is a particle located at the tip of the radius vector. Now we wish to describe the rotational *kinematics* of this particle, i.e. describe its motion as goes around the circle. So, suppose that the particle moves from angle ϕ_1 to ϕ_2 in time $t_2 - t_1$. Then, the *average angular speed* $\bar{\omega}$ is defined as, $\bar{\omega} = \dfrac{\phi_2 - \phi_1}{t_2 - t_1} = \dfrac{\Delta\phi}{\Delta t}$. Suppose that we look at $\bar{\omega}$ over a very short time. Then, $\omega = \lim\limits_{\Delta t \to 0} \dfrac{\Delta\phi}{\Delta t} = \dfrac{d\phi}{dt}$ is called the instantaneous angular speed.

4. To familarize ourselves with the notion of angular speed, let us compute ω for a clock second, minute and hour hands:

$$\omega_{\text{second}} = \frac{2\pi}{60} = 0.105\ rad/s,$$

$$\omega_{\text{minute}} = \frac{2\pi}{60 \times 60} = 1.75 \times 10^{-3}\ rad/s,$$

$$\omega_{\text{hour}} = \frac{2\pi}{60 \times 60 \times 12} = 1.45 \times 10^{-4}\ rad/s.$$

5. Just as we defined accelaration for linear motion, we also define acceleration for circular motion:

$$\bar{\alpha} \equiv \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t} \qquad \text{(average angular speed)}$$

Hence, $\alpha = \lim\limits_{\Delta t \to 0} \dfrac{\Delta\omega}{\Delta t}$ becomes $\alpha = \dfrac{d\omega}{dt} = \dfrac{d}{dt}\dfrac{d\phi}{dt} = \dfrac{d^2\phi}{dt^2}$ (angular acceleration). Let us see what this means for the speed with which a particle goes around. Now use $s = r\phi$. Differentiate with respect to time t: $\dfrac{ds}{dt} = r\dfrac{d\phi}{dt}$. The rate of change of arc length s is clearly what we should call the circular speed, v. So $v = r\omega$. Since r is held fixed, it follows that $\frac{dv}{dt} = r\frac{d\omega}{dt}$. Now define $a_T = \frac{dv}{dt}$. Obviously, $a_T = r\alpha$. Here T stands for tangential, i.e. in the direction of increasing *s*.

6. Compare the formulae for constant linear and angular accelerations:

LINEAR	ANGULAR
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\phi = \phi_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\phi - \phi_0)$

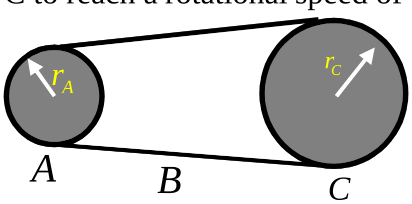
Why are they almost identical even though they describe two totally different physical situations. Answer: because the mathematics is identical!

7. The angular speed of a car engine is increased from 1170 rev/min to 2880 rev/min in 12.6 s. a)Find the average angular acceleration in rev/min^2. (b) How many revolutions does the engine make during this time?

SOLUTION: this is a straightforward application of the formulae in point 5 above.

$$\alpha = \frac{\omega_f - \omega_i}{t} = 8140\,\text{rev/min}^2, \qquad \phi = \omega_i t + \frac{1}{2}\alpha t^2 = 425\,\text{rev.}$$

8. Wheel A of radius r_A = 10.0 cm is coupled by a chain B to wheel C of radius r_C = 25.0 cm. Wheel A increases its angular speed from rest at a uniform rate of 1.60 rad/s2. Determine the time for wheel C to reach a rotational speed of 100 rev/min.

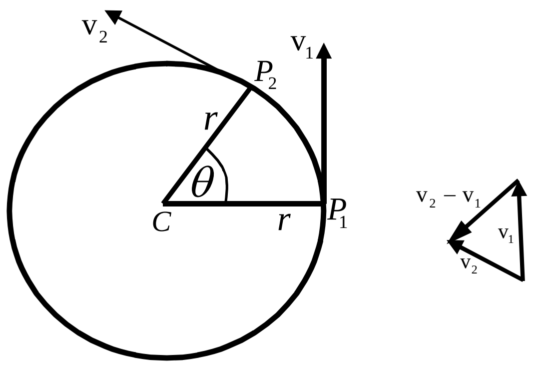


SOLUTION: Obviously every part of the chain moves with the same speed and so $v_A = v_C$. Hence $r_A\omega_A = r_C\omega_C \Rightarrow \omega_A = \frac{r_C\omega_C}{r_A}$. From the definition of acceleration,

$\alpha = \frac{\omega_A - 0}{t}$. From this, $t = \frac{\omega_A}{\alpha} = \frac{r_C\omega_C}{r_A\alpha} = 16.4\ s.$

9. Imagine a disc going around. All particles on the disc will have same 'ω' and 'α' but different 'v' and 'a'. Clearly 'ω' and 'α' are simpler choices !!

10. Now consider a particle going around a circle at constant speed. You might think that constant speed means no acceleration. Bu this is wrong! It is changing its direction and accelerating. This is called "centripetal acceleration", meaning acceleration directed towards the centre of the circle. Look at the figure below:



Note that the distance between points P_1 and P_2 is $\Delta r = \mathrm{v}\Delta t \approx r\theta$. Similarly, $\Delta \mathrm{v} \approx \mathrm{v}\theta \Rightarrow \bar{a} = \frac{\Delta \mathrm{v}}{\Delta t} \approx \frac{\mathrm{v}\theta}{r\theta/\mathrm{v}} = \frac{\mathrm{v}^2}{r}$. More generally, $a = \lim_{\Delta t \to 0} \frac{\Delta \mathrm{v}}{\Delta t} = \frac{\mathrm{v}^2}{r}$. In vector form, $\vec{a}_R = -\frac{\mathrm{v}^2}{r}\hat{r}$. The negative sign indicates that the acceleration is towards the centre.

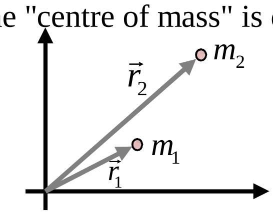
12. Vector Cross Products: The vector crossproduct of two vectors is defined as: $\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$ where $\hat{n}$ is a unit vector that is perpendicular to both $\vec{A}$ and $\vec{B}$.
Apply this definition to unit vectors in 3-dimensions: $\hat{i}\times\hat{j} = \hat{k},\ \hat{k}\times\hat{i} = \hat{j},\ \hat{j}\times\hat{k} = \hat{i}$.

13. Some key properties of the crossproduct:
   - $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$
   - $\vec{A}\times\vec{A} = 0$
   - $\left(\vec{A}+\vec{B}\right)\times\vec{C} = \left(\vec{A}\times\vec{B}\right) + \left(\vec{A}\times\vec{C}\right)$
   - $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$

14. The cross product is only definable in 3 dimensions and has no meaning in 2-d. This is unlike the dot product which as a meaning in any number of dimensions.

Summary of Lecture 12 – PHYSICS OF MANY PARTICLES

1. A body is made of a collection of particles. We would like to think of this body having a "centre". For two masses the "centre of mass" is defined as: $\vec{r}_{cm} \equiv \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$.



In 2 dimensions (i.e. a plane) this is actually two equations:

$x_{cm} = \dfrac{m_1x_1 + m_2x_2}{m_1 + m_2}$ and $y_{cm} = \dfrac{m_1y_1 + m_2y_2}{m_1 + m_2}$. These give the coordinates of the centre of mass of the two-particle system.

2. Example: one mass is placed at $x = 2cm$ and a second mass, equal to the first, is placed at $x = 6cm$. The cm position lies halfway between the two as you can see from:

$$x_{cm} = \frac{mx_1 + mx_2}{m + m} = \frac{2m + 6m}{2m} = 4cm.$$

Note that there is no physical body that is actually located at $x_{cm} = 4cm$! So the centre of mass can actually be a point where there is no matter. Now suppose that the first mass is three times bigger than the first:

$$x_{cm} = \frac{(3m)x_1 + mx_2}{3m + m} = \frac{2(3m) + 6m}{4m} = 3cm$$

This shows that the cm lies closer to the heavier body. This is always true.

3. For N masses the obvious generalization of the centre of mass position is the following:

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_2 + \cdots + m_N} = \frac{1}{M}\left(\sum_{n=1}^{n=N} m_n\vec{r}_n\right).$$

In words, this says that the following: choose any origin and draw vectors $\vec{r}_1, \vec{r}_2, \cdots \vec{r}_N$ that connect to the masses $m_1, m_2, \cdots m_N$. Heavier masses get more importance in the sum. So suppose that m_2 is much larger than any of the others. If so, $\vec{r}_{cm} \approx \dfrac{m_2\vec{r}_2}{m_2} = \vec{r}_2$. Hence, the cm is very close to the position vector of m_2.

4. For symmetrical objects, it is easy to see where the cm position lies: for a sphere or circle it lies at the centre; for a cylinder it is on the axis halfway between the two faces, etc.

5. Our definition of the cm allows Newton's Second Law to be written for entire collection

of particles:

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}\left(\sum m_n \vec{v}_n\right)$$

$$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\left(\sum m_n \vec{a}_n\right)$$

$$\therefore \quad M\,\vec{a}_{cm} = \sum \vec{F}_n = \sum\left(\vec{F}_{ext} + \vec{F}_{int}\right) \qquad \text{use } \sum \vec{F}_{int} = 0$$

$$\Rightarrow \quad \sum \vec{F}_{ext} = M\,\vec{a}_{cm} \text{ (the sum of external forces is what causes acceleration)}$$

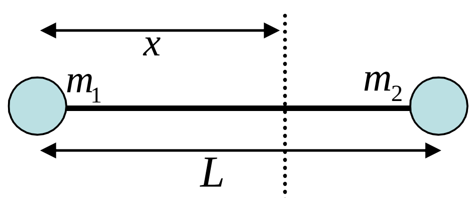
In the above we have used Newton's Third Law as well: $\vec{F}_{12} + \vec{F}_{21} = 0$ etc.

6. Consider rotational motion now for a rigid system of N particles. Rigid means that all particles have a fixed distance from the origin. The kinetic energy is,

$$K = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_3 v_3^2 + \cdots$$
$$= \frac{1}{2}m_1 r_1^2\omega^2 + \frac{1}{2}m_2 r_2^2\omega^2 + \frac{1}{2}m_3 r_3^2\omega^2 + \cdots$$
$$= \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

Now suppose that we define the "moment of inertia" $I \equiv \sum m_i r_i^2$. Then clearly the kinetic energy is $K = {}^1\!/_2 I\omega^2$. How similar this is to $K = \frac{1}{2}Mv^2$!

7. To familiarize ourselves with I, let us consider the following: Two particles m_1 and m_2 are connected by a light rigid rod of length L. Neglecting the mass of the rod, find the rotational inertia I of this system about an axis perpendicular to the rod and at a distance x from m_1.



Answer: $I = m_1 x^2 + m_2\left(L - x\right)^2$. Of course, this was quite trivial. Now we can ask a more interesting question: For what x is I the largest? Now, near a maximum, the slope of a function is zero. So calculate $\frac{dI}{dx}$ and then put it equal to zero:

$$\therefore \quad \frac{dI}{dx} = 2m_1 x - 2m_2\left(L - x\right) = 0 \quad \Rightarrow \quad x_{\max} = \frac{m_2 L}{m_1 + m_2}.$$

8. Although matter is made up of discrete atoms, even if one takes small pieces of any body, there are billions of atoms within it. So it is useful to think of matter as being continuously distributed. Since a sum $\sum$ becomes an integral $\int$, it is obvious that the new definitions of I and $\vec{R}_{cm}$ becom:

$$I \equiv \int r^2 dm \quad \text{and} \quad \vec{R}_{cm} \equiv \frac{1}{M}\int \vec{r}dm.$$

9. A simple application: suppose there is a hoop with mass distributed uniformly over it. The moment of inertia is: $I = \int r^2 dm = R^2 \int dm = MR^2$.

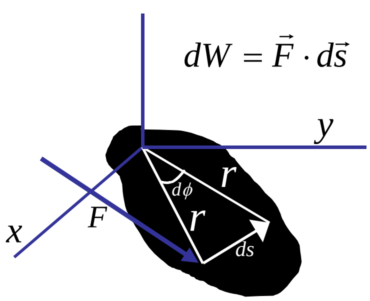
10. A less trivial application: instead of a hoop as above, now consider a solid plate:

$$I = \int r^2 dm \quad (dm = 2\pi r dr \rho_0)$$

$$= \int_0^R 2\pi r^3 dr \rho_0 = \frac{1}{2}(\pi R^2 \rho_0)R^2 = \frac{1}{2} M R^2$$

11. You have seen that it is easier to turn things (e.g. a nut, when changing a car's tyre after a puncture) when the applied force acts at a greater distance. This is because the *torque* τ is greater. We define $\vec{\tau} = \vec{r} \times \vec{F}$ from the magnitude is $\tau = rF\sin\theta$. Here θ is the angle between the radius vector and the force.

12. Remember that when a force $\vec{F}$ acts through a distance $d\vec{r}$ it does an amount of work equal to $\vec{F} \cdot d\vec{r}$. Now let us ask how much work is done when a torque acts through a certain angle as in the diagram below:



The small amount of work done is:

$$dW = \vec{F} \cdot d\vec{s} = F\cos\theta\, ds = \left(F\cos\theta\right)\left(r d\phi\right) = \tau d\phi$$

Add the contributions coming from from all particles,

$$dW_{net} = \left(F_1 \cos\theta_1\right) r_1 d\phi + \left(F_2 \cos\theta_2\right) r_2 d\phi + \cdots + \left(F_n \cos\theta_n\right) r_n d\phi$$

$$= \left(\tau_1 + \tau_2 + \cdots + \tau_n\right) d\phi$$

$$\therefore dW_{net} = \left(\sum \tau_{ext}\right) d\varphi = \left(\sum \tau_{ext}\right) \omega dt \qquad .....(1)$$

Now consider the change in the kinetic energy K,

$$dK = d\left(\frac{1}{2} I\omega^2\right) = I\omega d\omega = (I\alpha)\omega dt \qquad ......(2)$$

By conservation of energy, the change in K must equal the work done, and so:

$$dW_{net} = dK \quad \Rightarrow \quad \sum \tau_{ext} = I\alpha$$

In words this says that the total torque equals the moment of inertia times the angular acceleration. This is just like Newton's second law, but for rotational motion !

13.A comparison between linear and rotational motion quantities and formulae:

LINEAR	ROTATIONAL
x, M	ϕ, I
$\text{v} = \frac{dx}{dt}$	$\omega = \frac{d\phi}{dt}$
$\text{a} = \frac{d\text{v}}{dt}$	$\alpha = \frac{d\omega}{dt}$
$F = M\text{a}$	$\tau = I\alpha$
$K = \frac{1}{2} M\text{v}^2$	$K = \frac{1}{2} I\omega^2$
$W = \int F dx$	$W = \int \tau d\phi$

14. Rotational and translational motion can occur simultaneously. For example a car's wheel rotates and translates. In this case the total kinetic energy is clearly the sum of the energies of the two motions: $K = \frac{1}{2} M\, \text{v}_{\text{cm}}^2 + \frac{1}{2} I_{cm}\omega^2$.

15. It will take a little work to prove the following fact that I simply stated above: for a system of N particles, the total kinetic energy divides up neatly into the kinetic energy of rotation and translation. Start with the expression for kinetic energy and write $\vec{\text{v}}_{\text{cm}} + \vec{\text{v}}_i'$ where $\vec{\text{v}}_i'$ is the velocity of a particle with respect to the cm frame,

$$K = \sum \frac{1}{2} m_i \text{v}_i{}^2 = \sum \frac{1}{2} m_i \vec{\text{v}}_i \cdot \vec{\text{v}}_i$$

$$= \sum \frac{1}{2} m_i \left(\vec{\text{v}}_{\text{cm}} + \vec{\text{v}}_i'\right) \cdot \left(\vec{\text{v}}_{\text{cm}} + \vec{\text{v}}_i'\right)$$

$$= \sum \frac{1}{2} m_i \left(\text{v}_{\text{cm}}^2 + 2\vec{\text{v}}_{\text{cm}} . \vec{\text{v}}_{\text{i}}' + \text{v}_{\text{i}}'^2\right)$$

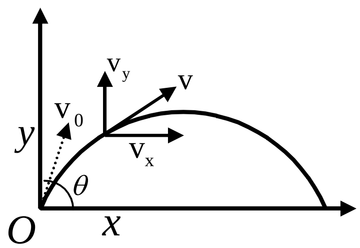
Now, $\sum m_i \vec{\mathrm{v}}_{\mathrm{cm}} \cdot \vec{\mathrm{v}}'_{\mathrm{i}} = \vec{\mathrm{v}}_{\mathrm{cm}} \cdot \sum m_i \vec{\mathrm{v}}'_i = \vec{\mathrm{v}}_{\mathrm{cm}} \cdot \sum \vec{p}'_i = 0$. Why? because the total momentum is xero in the cm frame! So this brings us to our result that,

$$K = \sum \frac{1}{2} m_i \mathrm{v}_{\mathrm{cm}}^2 + \sum \frac{1}{2} m_i \mathrm{v}_{\mathrm{i}}'^2$$
$$= \frac{1}{2} M \mathrm{v}_{\mathrm{cm}}^2 + \sum \frac{1}{2} m_i r_{\mathrm{i}}'^2 \omega^2.$$

Summary of Lecture 13 – ANGULAR MOMENTUM

1. Recall the definition of angular momentum: $\vec{L} = \vec{r} \times \vec{p}$. The magnitude can be written in several different but equivalent ways,
   (a) $L = r p \sin\theta$
   (b) $L = (r\sin\theta)\, p = r_\perp p$
   (c) $L = r(p\sin\theta) = r\, p_\perp$

2. Let us use this definition to calculate the angular momentum of a projectile thrown from the ground at an angle θ. Obviously, initial angular momentum is zero (why?).



We know what the projectile's coordinates will be at time t after launch,

$$x = (\mathrm{v}_0 \cos\theta)t, \quad y = (\mathrm{v}_0 \sin\theta)t - \frac{1}{2}gt^2$$

as well as the velocity components,

$$\mathrm{v}_x = \mathrm{v}_0 \cos\theta,\ \mathrm{v}_y = \mathrm{v}_0 \sin\theta - gt.$$

Hence, $\vec{L} = \vec{r} \times \vec{p} = (x\hat{i} + y\hat{j}) \times (\mathrm{v}_x \hat{i} + \mathrm{v}_y \hat{j}) m = m(x\mathrm{v}_y - y\mathrm{v}_x)\hat{k}$

$$= m\left(\frac{1}{2}gt^2\mathrm{v}_0\cos\theta - gt^2\mathrm{v}_0\cos\theta\right)\hat{k} = -\frac{m}{2}gt^2\mathrm{v}_0\cos\theta\,\hat{k}.$$

In the above, $\hat{k} = \hat{i} \times \hat{j}$ is a unit vector perpendicular to the paper. You can see here that the angular momentum increases as t^2.

2. Momentum changes because a force makes it change. What makes angular momentum change? Answer: torque. Here is the definition again: $\vec{\tau} = \vec{r} \times \vec{F}$. Now let us establish a very important relation between torque and rate of change of L.
   Begin:
   $\vec{L} = \vec{r} \times \vec{p}$. At a slightly later time, $\vec{L} + \Delta\vec{L} = (\vec{r} + \Delta\vec{r}) \times (\vec{p} + \Delta\vec{p})$

$$= \vec{r} \times \vec{p} + \vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p} + \Delta\vec{r} \times \Delta\vec{p}$$

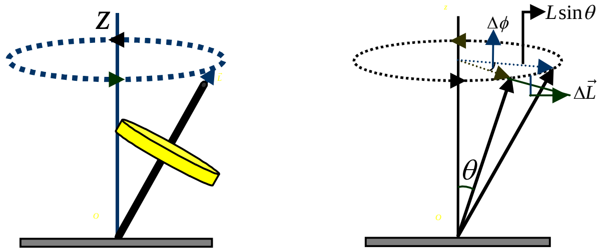
   By subtracting, $\Delta\vec{L} = \vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p}$ $\dfrac{\Delta\vec{L}}{\Delta t} = \dfrac{\vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p}}{\Delta t} = \vec{r} \times \dfrac{\Delta\vec{p}}{\Delta t} + \dfrac{\Delta\vec{r}}{\Delta t} \times \vec{p}.$

Now divide by the time difference and then take limit as $\Delta t \to 0$:

$$\because \quad \lim_{\Delta t \to 0} \frac{\Delta \vec{L}}{\Delta t} = \frac{d\vec{L}}{dt} \quad \therefore \quad \frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

But $\frac{d\vec{r}}{dt}$ is $\vec{v}$ and $\vec{p} = m\vec{v}$! Also, $\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = m(\vec{v} \times \vec{v}) = 0$. So we arrive at $\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$. Now use Newton's Law, $\vec{F} = \frac{d\vec{p}}{dt}$. Hence we get the fundamental equation $\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$, or $\frac{d\vec{L}}{dt} = \vec{\tau}$. So, just as a particle's momentum changes with time because of a force, a particle's *angular* momentum changes with time because of a torque.

3. As you saw in the lecture, the spinning top is an excelllent application of $\frac{d\vec{L}}{dt} = \vec{\tau}$.



Start from $\vec{\tau} = \vec{r} \times \vec{F}$ where $\vec{F} = m\vec{g} \;\therefore\; \tau = Mgr\sin\theta$. But $\vec{\tau}$ is perpendicular to $\vec{L}$ and so it cannot change the magnitude of $\vec{L}$. Only the direction changes. Since $\Delta\vec{L} = \vec{\tau}\,\Delta t$, you can see from the diagram that $\Delta\phi = \frac{\Delta L}{L\sin\theta} = \frac{\tau\,\Delta t}{L\sin\theta}$. So the precession speed ω_P is: $\omega_P = \frac{\Delta\phi}{\Delta t} = \frac{\tau}{L\sin\theta} = \frac{Mgr\sin\theta}{L\sin\theta} = \frac{Mgr}{L}$. As the top slows down due to friction and L decreases, the top precesses faster and faster.

4. Now consider the case of many particles. Choose any origin with particles moving with respect to it. We want to write down the total angular momentum,

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \cdots + \vec{L}_N = \sum_{n=1}^{N} \vec{L}_n$$

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}_1}{dt} + \frac{d\vec{L}_2}{dt} + \cdots + \frac{d\vec{L}_N}{dt} = \sum_{n=1}^{N} \frac{d\vec{L}_n}{dt}$$

Since $\frac{d\vec{L}_n}{dt} = \vec{\tau}_n$, it follows that $\frac{d\vec{L}}{dt} = \sum_{n=1}^{N} \vec{\tau}_n$. Thus the time rate of change of the total angular momentum of a system of particles equals the net torque acting on the system. I showed earlier that internal forces cancel. So also do internal torques, as we shall see.

5. The torque on a system of particles can come both from external and internal forces. For example, there could be charged particles which attract/repel each other while they are all in an external gravitational field. Mathematically, $\sum \vec{\tau} = \sum \vec{\tau}_{\text{int}} + \sum \vec{\tau}_{ext}$. Now, if the forces between two particles not only are equal and opposite but are also directed along the line joining the two particles, then can easily show that the total internal torque, $\sum \vec{\tau}_{\text{int}} = 0$. Take the case of two particles,

$$\sum \vec{\tau}_{\text{int}} = \vec{\tau}_1 + \vec{\tau}_2 = \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21}$$

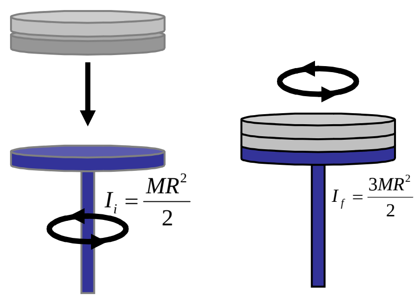
But $\vec{F}_{12} = -\vec{F}_{21} = F\,\hat{r}_{12}, \quad \therefore \sum \vec{\tau}_{\text{int}} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} = \vec{r}_{12} \times (F\,\hat{r}_{12}) = F\,(\vec{r}_{12} \times \hat{r}_{12}) = 0$

Thus net external torque acting on a system of particles is equal to the time rate of change of the of the total angular momentum of the system.

6. It follows from $\dfrac{d\vec{L}}{dt} = \vec{\tau}$ that if no net external torque acts on the system, then the angular momentum of the system does not change with the time: $\dfrac{d\vec{L}}{dt} = 0 \quad \Rightarrow \quad \vec{L} =$ a constant.

This is simple but extremely important. Let us apply this to the system shown here. Two stationary discs, each with $I = \dfrac{1}{2}MR^2$, fall on top of a rotating disc. The total angular momentum is unchanged so, $I_i\omega_i = I_f\omega_f \;\Rightarrow\; \omega_f = \omega_i\left(\dfrac{I_i}{I_f}\right)$.

Hence, $\omega_f = \omega_i\left(\dfrac{MR^2}{2} \times \dfrac{2}{3MR^2}\right) = \dfrac{1}{3}\omega_i$.



7. You shoud be aware of the similarities and differences between the equations for linear and rotational motion: $\vec{F} = \dfrac{d\vec{p}}{dt} \Leftrightarrow \vec{\tau} = \dfrac{d\vec{L}}{dt}$. One big difference is that for momentum, $\vec{p} = m\vec{v}$, it does not matter where you pick the origin. But $\vec{L}$ definitely depends on the choice of the origin. So changing $\vec{r}$ to $\vec{c} + \vec{r}$ changes $\vec{L}$ to $\vec{L}'$:

$$\vec{L}' = \vec{r}' \times \vec{p} = (\vec{c} + \vec{r}) \times \vec{p} = \vec{c} \times \vec{p} + \vec{L}.$$

8. Linear and angular acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r}) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$
$$= \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$$

So the acceleration has a tangential and radial part, $\vec{a} = \vec{a}_T + \vec{a}_R$.

Summary of Lecture 14 – EQUILIBRIUM OF RIGID BODIES

1. A rigid body is one where all parts of the body are fixed relative to each other (for example, a pencil). Fluids and gases are non-rigid.

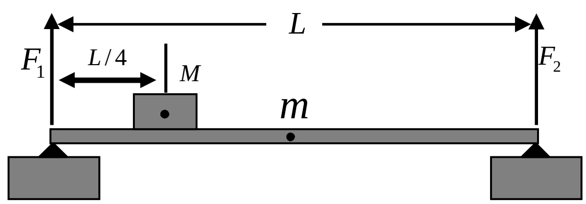
2. The translational motion of the centre of mass of a rigid body is governed by:

$$\frac{d\vec{P}}{dt} = \vec{F} \text{ where } \vec{F} = \sum \vec{F}_{ext} \text{ is the } net \text{ external force.}$$

Similarly, for rotational motion, $\frac{d\vec{L}}{dt} = \vec{\tau}$ where $\vec{\tau} = \sum \vec{\tau}_{ext}$ is the net external torque.

3. A rigid body is in mechanical equilibrium if **both** the linear momentum $\vec{P}$ and angular momentum $\vec{L}$ have a constant value. i.e., $\frac{d\vec{P}}{dt} = 0$ **and** $\frac{d\vec{L}}{dt} = 0$. **Static equilibrium** refers to $\vec{P} = 0$ and $\vec{L} = 0$.

4. As an example of static equilibrium, consider a beam resting on supports:



We want to find the forces F_1 and F_2 with which the supports push on the rod in the upwards direction. First, balance forces in the vertical y direction:

$$\sum F_y = F_1 + F_2 - Mg - mg = 0$$

Now demand that the total torque vanishes:

$$\sum \tau_y = (F_1)(0) + (F_2)(L) - (Mg)(L/4) - (mg)(L/2) = 0$$

From these two conditions you can solve for F_1 and F_2,

$$F_1 = \frac{(3M + 2m)g}{4}, \quad \text{and} \quad F_2 = \frac{(M + 2m)g}{4}.$$

5. Angular momentum and torque depend on where you choose the origin of your coordinates. However, I shall now prove that for a body in equilibrium, the choice of origin does not matter. Let's start with the origin O and calculate the torque about O,

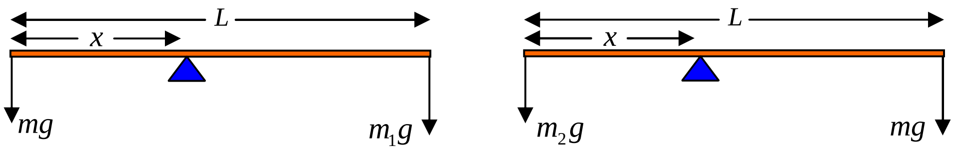
$$\vec{\tau}_O = \vec{\tau}_1 + \vec{\tau}_2 + \cdots + \vec{\tau}_N = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots + \vec{r}_N \times \vec{F}_N$$

Now, if we take a second point P, then all distances will be measured from P and each vector will be shifted by an amount $\vec{r}_P$. Hence the torque about P is,

$$\begin{aligned}\vec{\tau}_P &= \left(\vec{r}_1 - \vec{r}_P\right)\times\vec{F}_1 + \left(\vec{r}_2 - \vec{r}_P\right)\times\vec{F}_2 + \cdots + \left(\vec{r}_N - \vec{r}_P\right)\times\vec{F}_N \\ &= \left[\vec{r}_1\times\vec{F}_1 + \vec{r}_2\times\vec{F}_2 + \cdots + \vec{r}_N\times\vec{F}_N\right] - \left[\vec{r}_P\times\vec{F}_1 + \vec{r}_P\times\vec{F}_2 + \cdots + \vec{r}_P\times\vec{F}_N\right] \\ &= \vec{\tau}_O - \left[\vec{r}_P\times\left(\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N\right)\right] \\ &= \vec{\tau}_O - \left[\vec{r}_P\times\left(\sum\vec{F}_{ext}\right)\right]\end{aligned}$$

but $\sum\vec{F}_{ext} = 0,$ for a body in translational equilibrium $\therefore \quad \vec{\tau}_P = \vec{\tau}_O$.

6. Let us use the equilibrium conditions to do something of definite practical importance. Consider the balance below which is in equilibrium when two known weights are hung as shown. We want to know m in terms of m_1 and m_2.



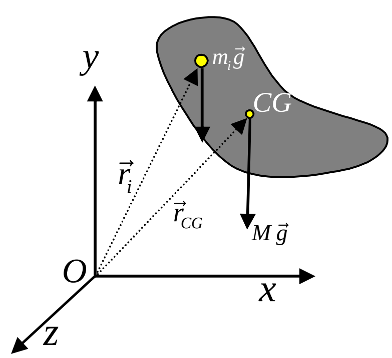
Taking the torques about the knife edge in the two cases, we have:

$$mg\,x = m_1 g\left(L - x\right) \text{ and } m_2\,g\,x = mg\left(L - x\right)$$

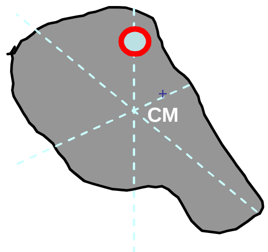
$$\Rightarrow \frac{m}{m_2} = \frac{m_1}{m} \text{ or } m = \sqrt{m_1 m_2}.$$

Remarkably, we do not need the values of x or L.

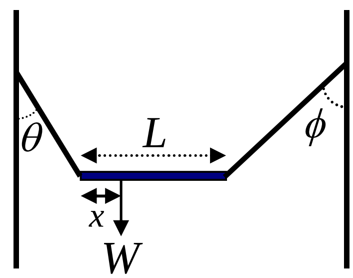
7. **Centre of Gravity.** The centre of gravity is the average location of the weight of an object. This is not quite the same as the centre of mass of a body (see lecture 12) but suppose the gravitational acceleration $\vec{g}$ has the same value at all points of a body. Then: 1) The weight is equal to $M\,\vec{g},$ and 2)the centre of gravity coincides with the centre of mass. Remember that weight is force, so the CG is really the centre of gravitational force acting on the body. The net force on the whole body = sum of forces over all individual particles, $\sum\vec{F} = \sum m_i\,\vec{g}$. If $\vec{g}$ has the same value at all points of the body, then $\sum\vec{F} = \vec{g}\sum m_i = M\,\vec{g}$. So the net torque about the origin O is $\sum\vec{\tau} = \sum\left(\vec{r}_i\times m_i\vec{g}\right) = \sum\left(m_i\vec{r}_i\times\vec{g}\right)$ Hence, $\sum\vec{\tau} = M\,\vec{r}_{cm}\times\vec{g} = \vec{r}_{cm}\times M\,\vec{g}$. So the torque due to gravity about the centre of mass of a body (i.e. at $\vec{r}_{cm} = 0$) is zero !!



8. In the demonstration I showed, you saw how to find the CG of an irregular object by simply suspending it on a pivot,



9. Let's solve a problem of static equilibrium: A non-uniform bar of weight W is suspended at rest in a horizontal position by two light cords. Find the distance x from the left-hand end to the center of gravity.



Solution : Call the tensions T_1 and T_2 . Put the forces in both directions equal to zero,

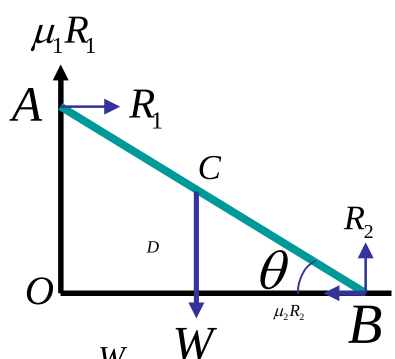
a) $T_2 \sin\phi - T_1 \sin\theta = 0$ (horizontal)

b) $T_2 \cos\phi + T_1 \cos\theta - W = 0$ (vertical) $\Rightarrow T_2 = \dfrac{W}{\sin(\theta+\phi)}$

The torque about any point must vanish. Let us choose that point to be one end of the bar,

$$-Wx + (T_2 \cos\phi)L = 0 \Rightarrow x = \frac{(T_2\cos\phi)L}{W} = \frac{L\cos\phi}{\sin(\theta+\phi)}$$

10.Here is another problem of the same kind: find the least angle θ at which the rod can lean to the horizontal without slipping.



Solution : Considering the translational equilibrium of the rod, $R_1 = \mu_2 R_2$ and $R_2 + \mu_1 R_1 = W$. This gives, $R_2 = \dfrac{W}{(1+\mu_1\mu_2)}$. Now consider rotational equilibrium about the point A: $R_2 \times OB = W \times OD + \mu_2 R_2 \times OA$

or, $R_2 \times AB\cos\theta = W \times \dfrac{AB\cos\theta}{2} + \mu_2 R_2 \times AB\sin\theta.$

This gives $\cos\theta\left(R_2 - \dfrac{W}{2}\right) = \mu_2 R_2 \sin\theta$ from which $\tan\theta = \dfrac{R_2 - \frac{W}{2}}{\mu_2 R_2}$ with $R_2 = \dfrac{W}{(1+\mu_1\mu_2)}$.

Using this value of R_2, we get $\tan\theta = \dfrac{1\text{-}\mu_1\mu_2}{2\mu_2}$.

10. **Types of Equilibrium.** In the lecture you heard about:

a)Stable equilibrium: object returns to its original position if displaced slightly.

b)Unstable equilibrium: object moves farther away from its original position if displaced slightly.

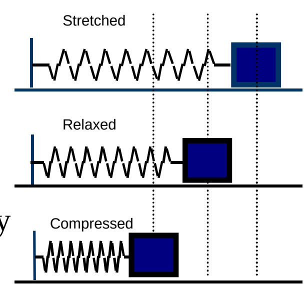
c)Neutral equilibrium: object stays in its new position if displaced slightly.

Summary of Lecture 15 – OSCILLATIONS: I

1. An oscillation is any self-repeating motion. This motion is characterized by:
   a) The period T, which is the time for completing one full cycle.
   b) The frequency $f = 1/T$, which is the number of cycles per second. (Another frequently used symbol is ν).
   c) The amplitude A, which is the maximum displacement from equilibrium (or the size of the oscillation).

2. Why does a system oscillate? It does so because a force is always directed towards a central equilibrium position. In other words, the force always acts to return the object to its equilibrium position. So the object will oscillate around the equilibrium position. The restoring force depends on the displacement $F_{\text{restore}} = -k\,\Delta x,$ where Δx is the distance away from the equilibrium point, the negative sign shows that the force acts towards the equilibrium point, and k is a constant that gives the strength of the restoring force.

3. Let us understand these matters in the context of a spring tied to a mass that can move freely over a frictionless surface. The force $F(x) = -kx$ (or $-k\Delta x$ because the extension Δx will be called x for short). In the first diagram x is positive, x is negative in the second, and zero in the middle one. The energy stored in the spring, $U(x) = \frac{1}{2}kx^2,$ is positive in the first and third diagrams and zero in the middle one. Now we will use Newton's second law to derive a differential euation that describes the motion of the mass: From $F(x) = -kx$ and $ma = F$ it follows that $m\frac{d^2x}{dt^2} = -kx,$ or $\frac{d^2x}{dt^2} + \omega^2 x = 0$ where $\omega^2 \equiv \frac{k}{m}$. This is the equation of motion of a simple harmonic oscillator (SHO) and is seen widely in many different branches of physics. Although we have derived it for the case of a mass and spring, it occurs again and again. The only difference is that $\omega,$ which is called the oscillator frequency, is defined differently depending on the situation.



4. In order to solve the SHO equation, we shall first learn how to differentiate some elementary trignometric functions. So let us first learn how to calculate $\frac{\mathrm{d}}{\mathrm{dt}}\cos\omega t$ starting from the basic definition of a derivative:

Start : $x(t) = \cos\omega t$ and $x(t+\Delta t) = \cos\omega(t+\Delta t)$. Take the difference:

$$x(t+\Delta t) - x(t) = \cos\omega(t+\Delta t) - \cos\omega t$$
$$= -\sin\omega\Delta t \sin(\omega t + \omega\Delta t/2)$$
$$\approx -\omega\Delta t \sin\omega t \quad \text{as } \Delta t \text{ becomes very small.}$$
$$\therefore \frac{d}{dt}\cos\omega t = -\omega\sin\omega t.$$

(Here you should know that $\sin\theta \approx \theta$ for small θ, easily proved by drawing triangles.)

You should also derive and remember a second important result:

$$\frac{d}{dt}\sin\omega t = \omega\cos\omega t\ .$$

(Here you should know that $\cos\theta \approx 1$ for small θ.)

5. What happens if you differentiate twice?

$$\frac{d^2}{dt^2}(\sin\omega t) = \omega\frac{d}{dt}\cos\omega t = -\ \omega^2\sin\omega t$$
$$\frac{d^2}{dt^2}(\cos\omega t) = -\omega\frac{d}{dt}\sin\omega t = -\ \omega^2\cos\omega t.$$

So twice differentiating either $\sin\omega t$ or $\cos\omega t$ gives the same function back!

6. Having done all the work above, now you can easily see that any function of the form $x(t) = a\cos\omega t + b\sin\omega t$ satisfies $\frac{d^2x}{dt^2} = -\omega^2 x$. But what do ω, a, b represent ?

a) The significance of ω becomes clear if your replace t by $t + \frac{2\pi}{\omega}$ in either $\sin\omega t$ or $\cos\omega t$. You can see that $\cos\omega\left(t+\frac{2\pi}{\omega}\right) = \cos(\omega t + 2\pi) = \cos\omega t$. That is, the function merely repeats itself after a time $2\pi/\omega$. So $2\pi/\omega$ is really the period of the motion T, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$. The frequency ν of the oscillator is the number of complete vibrations per unit time: $\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ so $\omega = 2\pi\nu = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$. Sometimes ω is also called the angular frequency. Note that $\dim[\omega] = T^{-1}$, from it is clear that the unit of ω is radian/second.

b) To understand what a and b mean let us note that from $x(t) = a\cos\omega t + b\sin\omega t$ it follows that $x(0) = a$ and that $\frac{\mathrm{d}}{\mathrm{dt}}x(t) = -\omega a\sin\omega t + \omega b\cos\omega t = \omega b$ (at $t = 0$). Thus, a is the initial position, and b is the initial velocity divided by ω.

c) To understand what a and b mean let us note that from $x(t) = a\cos\omega t + b\sin\omega t$ it follows that $x(0) = a$ and that $\frac{\mathrm{d}}{\mathrm{dt}}x(t) = -\omega a\sin\omega t + \omega b\cos\omega t = \omega b$ (at $t = 0$). Thus, a is the initial position, and b is the initial velocity divided by ω.

d) The solution can also be written as: $x(t) = x_m\cos(\omega t + \phi)$. Since cos and sin never beome bigger than 1, or less than -1, it follows that $-x_m \le x \le +x_m$. For obvious reason x_m is called the amplitude of the motion. The frequency of the simple harmonic motion is independent of the amplitude of the motion.

e) The quantity $\theta = \omega t + \phi$ is called the phase of the motion. The constant ϕ is called the *phase constant.* A different value of ϕ just means that the origin of time has been chosen differently.

7. **Energy of simple harmonic motion**. Put $\phi = 0$ for convenience, and so imagine a mass whose position oscillates like $x = x_m\cos\omega t$. Let us first calculate the potential energy:

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2\cos^2\omega t.$$

Now calculate the kinetic energy:

$$K = \frac{1}{2}m\mathrm{v}^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}m\omega^2 x_m^2\sin^2\omega t = \frac{1}{2}kx_m^2\sin^2\omega t$$

The sum of potential + kinetic is:

$$E = K + U = \frac{1}{2}kx_m^2\cos^2\omega t + \frac{1}{2}kx_m^2\sin^2\omega t$$
$$= \frac{1}{2}kx_m^2\left(\cos^2\omega t + \sin^2\omega t\right) = \frac{1}{2}kx_m^2.$$

Note that this is independent of time and energy goes from kinetic to potential, then back to kinetic etc.

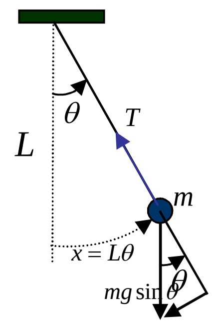
8. From the above, you can see that $\mathrm{v} = \frac{dx}{dt} = \pm\sqrt{\frac{k}{m}\left(x_m^2 - x^2\right)}$. From this it is clear that the speed is maximum at $x = 0$ and that the speed is zero at $x = \pm x_m$.

9. Putting two springs in parallel makes it harder to stretch them, and $k_{eff} = k_1 + k_2$. In series they are easier to stretch, and $k_{eff} = \left(\frac{k_1 k_2}{k_1 + k_2}\right)$. So a mass will oscillate faster in the first case as compared to the second.

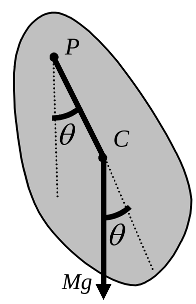
Summary of Lecture 16 – OSCILLATIONS: II

1. In this chapter we shall continue with the concepts developed in the previous chapter that relate to simple harmonic motion and the simple harmonic oscillator (SHO). It is really very amazing that the SHO occurs again and again in physics, and in so many different branches.

2. As an example illustrating the above, consider a mass suspended a string. From the diagram, you can see that $F = -mg\sin\theta$. For small values of θ we know that $\sin\theta \approx \theta$. Using $x = L\theta$ (length of arc), we have $F = -mg\theta = -mg\dfrac{x}{L} = -\left(\dfrac{mg}{L}\right)x$. So now we have a restoring force that is proportional to the distance away from the equilibrium point. Hence we have a SHO with $\omega = \sqrt{g/L}$. What if we had not made the small θ approximation? We would still have an oscillator (i.e. the motion would be self repeating) but the solutions of the differential equation would be too complicated to discuss here.



3. If you take a common object (like a piece of cardboard) and pivot it at some point, it will oscillate when disturbed. But this is not the simple pendulum discussed above because all the mass is not concentrated at one point. So now let us use the ideas of torque and angular momentum discussed earlier for many particle systems. You can see that $\tau = -Mgd\sin\theta$. For small θ, $\sin\theta \approx \theta$ and so $\tau = -Mgd\theta$. But we also know that $\tau = I\alpha$ where I is the moment of inertia and α is the angular acceleration, $\alpha = \dfrac{d^2\theta}{dt^2}$. Hence, we have $I\dfrac{d^2\theta}{dt^2} = -Mgd\theta$, or, $\dfrac{d^2\theta}{dt^2} = -\left(\dfrac{Mgd}{I}\right)\theta$. From this we immediately see that the oscillation frequency is $\omega = \sqrt{\dfrac{Mgd}{I}}$. Of course, we have used the small angle approximation over here again. Since all variables except I are known, we can use this formula to tell us what I is about any point. Note that we can choose to put the pivot at any point on the body. However, if you put the pivot exactly at the centre of mass then it will not oscillate. Why? Because there is no restoring force and the torque vanishes at the cm position, as we saw earlier.



4. Suppose you were to put the pivot at point P which is at a distance L from the centre of mass of the irregular object above. What should L be so that you get the same formula as for a simple pendulum?

Answer: $T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{I}{Mgd}} \Rightarrow L = \frac{I}{Md}$

P is then called the centre of gyration - when suspended from this point it appears as if all the mass is concentrated at the cm position.

5. Sum of two simple harmonic motions of the same period along the same line:

$$x_1 = A_1 \sin\omega t \text{ and } x_2 = A_2 \sin(\omega t + \phi)$$

Let us look at the sum of x_1 and x_2,

$$x = x_1 + x_2 = A_1 \sin\omega t + A_2 \sin(\omega t + \phi)$$
$$= A_1 \sin\omega t + A_2 \sin\omega t \cos\phi + A_2 \sin\phi \cos\omega t$$
$$= \sin\omega t\left(A_1 + A_2 \cos\phi\right) + \cos\omega t\left(A_2 \sin\phi\right)$$

Let $A_1 + A_2 \cos\phi = R\cos\theta$ and $A_2 \sin\phi = R\sin\theta$. Using some simple trigonometry, you can put x in the form, $x = R\sin(\omega t + \theta)$. It is easy to find R and θ:

$R = \sqrt{A_1^2 + A_2^2 + A_1 A_2 \cos\phi}$ and $\tan\theta = \frac{A_2 \sin\phi}{A_1 + A_2 \cos\phi}$.

Note that if $\phi = 0$ then $R = \sqrt{A_1^2 + A_2^2 + A_1 A_2} = \sqrt{(A_1 + A_2)^2} = A_1 + A_2$ and $\tan\theta = 0$ $\Rightarrow \theta = 0$. So we get $x = (A_1 + A_2)\sin\omega t$. This is an example of *constructive interference*. If $\phi = \pi$ then $R = \sqrt{A_1^2 + A_2^2 - A_1 A_2} = \sqrt{(A_1 - A_2)^2} = A_1 - A_2$ and $\tan\theta = 0$ $\Rightarrow \theta = 0$. Now we get $x = (A_1 - A_2)\sin\omega t$. This is *destructive interference.*

6. Composition of two simple harmonic motions of the same period but now at right angles to each other:

Suppose $x = A\sin\omega t$ and $y = B\sin(\omega t + \phi)$. These are two independent motions. We can write $\sin\omega t = \frac{x}{A}$ and $\cos\omega t = \sqrt{1 - x^2 / A^2}$.

From this, $\frac{y}{B} = \sin\omega t \cos\phi + \sin\phi \cos\omega t = \frac{x}{A}\cos\phi + \sin\phi\sqrt{1 - x^2 / A^2}$. Now square and rearrange terms to find:

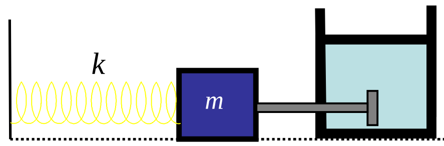
$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - 2\frac{xy}{AB}\cos\phi = \sin^2\phi$$

This is the equation for an ellipse (see questions at the end of this section).

7. If two oscillations of different frequencies at right angles are combined, the resulting motion is more complicated. It is not even periodic unless the two frequencies are in the ratio of integers. This resulting curve are called Lissajous figures. Specifically, if $x = A\sin\omega_x t$ and $y = B\sin\left(\omega_y t + \phi\right)$, then periodic motion requires $\frac{\omega_x}{\omega_y} =$ integers. You should look up a book for more details.

8. **Damped harmonic motion**: Typically the frictional force due to air resistance, or in a liquid, is proportional to the speed. So suppose that the damping force $= -b\frac{dx}{dt}$ (why negative sign?). Now apply Newton's law to a SHO that is damped: $-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$

   Rearrange slightly to get the equation for a damped SHO: $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$



   Its solution for $\frac{k}{m} \geq \left(\frac{b}{2m}\right)^2$ is $x = x_m e^{-bt/2m}\cos\left(\omega' t + \phi\right)$. The frequency is now changed: $\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$. The damping causes the amplitude to decrease with time and when $bt/2m = 1,$ the amplitude is $1/e \approx\ 1/2.7$ of its initial value.

9. **Forced oscillation and resonance.** There is a characteristic value of the driving frequency ω at which the amplitude of oscillation is a maximum. This condition is called resonance. For negligible damping resonance occurs at $\omega = \omega_0$. Here ω_0 is the natural frequency of the system and is given by $\omega_0 = \sqrt{\frac{k}{m}}$. The equation of motion is: $m\frac{d^2x}{dt^2} + kx = F_0\cos\omega t$. You should check that this is solved by putting $x = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}\cos\omega t$ (just subsitute into the equation and see!). Note that the amplitude "blows up" when $\omega \to \omega_0$. This is because we have no damping term here. With damping, the amplitude is large when $\omega \to \omega_0$ but remains finite.

Summary of Lecture 17 – PHYSICS OF MATERIALS

1. **Elasticity :** the property by virtue of which a body tends to regain its original shape and size when external forces are removed. If a body completely recovers its original shape and size , it is called perfectly elastic. Quartz, steel and glass are very nearly elastic.

2. **Plasticity :** if a body has no tendency to regain its original shape and size , it is called perfectly plastic. Common plastics, kneaded dough, solid honey, etc are plastics.

3. **Stress** characterizes the strength of the forces causing the stretch, squeeze, or twist. It is defined usually as force/unit area but may have different definitions to suit different situations. We distinguish between three types of stresses:
   a) If the deforming force is applied along some linear dimension of a body, the stress is called *longitudinal stress* or *tensile stress* or *compressive stress.*
   b) If the force acts normally and uniformly from all sides of a body, the stress is called *volume stress.*
   c) If the force is applied tangentially to one face of a rectangular body, keeping the other face fixed, the stress is called tangential or shearing stress.

4. Strain: When deforming forces are applied on a body, it undergoes a change in shape or size. The fractional (or relative) change in shape or size is called the strain.

$$\text{Strain} = \frac{\text{change in dimension}}{\text{original dimension}}$$

Strain is a ratio of similar quantities so it has no units. There are 3 different kinds of strain:

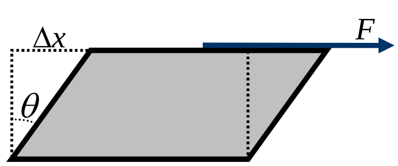
a) *Longitudinal (linear) strain* is the ratio of the change in length (ΔL) to original length (l), i.e., the linear strain $= \dfrac{\Delta l}{l}$.

b) *Volume strain* is the ratio of the change in volume (ΔV) to original volume (V)

Volume strain $= \dfrac{\Delta V}{V}$.

c) *Shearing strain :* The angular deformation (θ) in radians is called shearing stress.

For small θ the shearing strain $\equiv \theta \approx \tan\theta = \dfrac{\Delta x}{l}$.



5. Hooke's Law: for small deformations, stress is proportional to strain.

$$\text{Stress} = \text{E} \times \text{Strain}$$

The constant E is called the modulus of elasticity. E has the same units as stress because strain is dimensionless. There are three moduli of elasticity.

(a) Young's modulus (Y) for linear strain:

$$\text{Y} \equiv \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{F/A}{\Delta l / l}$$

(b) Bulk Modulus (B) for volume strain: Let a body of volume V be subjected to a uniform pressure ΔP on its entire surface and let ΔV be the corresponding decrease in its volume. Then,

$$\text{B} \equiv \frac{\text{Volume Stress}}{\text{Volume Strain}} = -\frac{\Delta P}{\Delta V / V}.$$

$1/B$ is called the compressibility. A material having a small value of B can be compressed easily.

(c) Shear Modulus (η) for shearing strain: Let a force F produce a strain θ as in the diagram in point 4 above. Then,

$$\eta \equiv \frac{\text{shearing stress}}{\text{shearing strain}} = \frac{F/A}{\theta} = \frac{F}{A \tan\theta} = \frac{Fl}{A\Delta x}.$$

6. When a wire is stretched, its length increases and radius decreases. The ratio of the lateral strain to the longitudinal strain is called Poisson's ratio, $\sigma = \frac{\Delta r / r}{\Delta l / l}$. Its value lies between 0 and 0.5.

7. We can calculate the work done in stretching a wire. Obviously, we must do work against a force. If x is the extension produced by the force F in a wire of length l, then $F = \frac{YA}{l}x$. The work done in extending the wire through Δl is given by,

$$W = \int_0^{\Delta l} F dx = \frac{YA}{l}\int_0^{\Delta l} x dx = \frac{YA}{l}\frac{(\Delta l)^2}{2}$$

$$= \frac{YA}{l}\frac{(\Delta l)^2}{2} = \frac{1}{2}(Al)\left(\frac{Y\Delta l}{l}\right)\left(\frac{\Delta l}{l}\right) = \frac{1}{2}\times \text{volume} \times \text{stress} \times \text{strain}$$

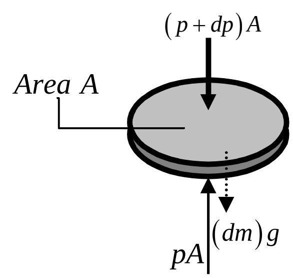
Hence, Work / unit volume $= \frac{1}{2}\times \text{stress}\times\text{strain}$. We can also write this as,

$$W = \frac{1}{2}\left(\frac{YA\Delta l}{l}\right)\Delta l = \frac{1}{2}\times \text{load} \times \text{extension}.$$

8. A fluid is a substance that can flow and does not have a shape of its own. Thus all liquids and gases are fluids. Solids possess all the three moduli of elasticity whereas a fluid possess only the bulk modulus (B). A fluid at rest cannot sustain a tangential force. If such force is applied to a fluid, the different layers simply slide over one another. Therefore the forces acting on a fluid at rest have to be normal to the surface. This implies that the free surface of a liquid at rest, under gravity, in a container, is horizontal.

9. The normal force per unit area is called pressure, $P = \frac{\Delta F}{\Delta A}$. Pressure is a scalar quantity. Its unit is Newtons/metre^2, or Pascal (Pa). Another scalar is density, $\rho = \frac{\Delta m}{\Delta V}$, where Δm is the mass of a small piece of the material and ΔV is the volume it occupies.

10. Let us calculate how the pressure in a fluid changes with depth. So take a small element of fluid volume submerged within the body of the fluid: $dm = \rho dV = \rho A dy \quad \therefore \ (dm)g = \rho g A dy$
Now let us require that the sum of the forces on the fluid element is zero: $pA - (p+dp)A - \rho g A dy = 0 \quad \Rightarrow \frac{dp}{dy} = -\rho g.$



Note that we are taking the origin ($y = 0$) at the bottom of the liquid. Therefore as the elevation increases (dy positive), the pressure decreases (dp negative). The quantity ρg is the weight per unit volume of the fluid. For liquids, which are nearly incompressible, ρ is practically constant.

$$\therefore \rho g = \text{constant} \Rightarrow \frac{dp}{dy} = \frac{\Delta p}{\Delta y} = \frac{p_2 - p_1}{y_2 - y_1} = -\rho g \Rightarrow \ p_2 - p_1 = -\rho g\left(y_2 - y_1\right).$$

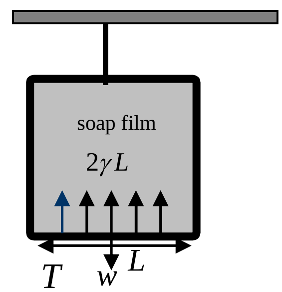
11. **Pascal's Principle**: Pressure applied to an enclosed fluid is transmitted to every portion of the fluid and to the walls of the containing vessel. This follows from the above: if h is the height below the liquid's surface, then $p = p_{ext} + \rho g h$. Here p_{ext} is the pressure at the surface of the liquid, and so the difference in pressure is $\Delta p = \Delta p_{ext} + \Delta(\rho g h)$. Therefore, $\Delta(\rho g h) = 0 \ \Rightarrow \Delta p = \Delta p_{ext}$. (I have used here the fact that liquids are incompressible). So the pressure is everywhere the same.

Summary of Lecture 18 – PHYSICS OF FLUIDS

1. A fluid is matter that has no definite shape and adjusts to the container that it is placed in. Gases and liquids are both fluids. All fluids are made of molecules. Every molecules attracts other molecules around it.

2. Liquids exhibit surface tension. A liquid has the property that its free surface tends to contract to minimum possible area and is therefore in a state of tension. The molecules of the liquid exerts attractive forces on each other, which is called cohesive forces. Deep inside a liquid, a molecule is surrounded by other molecules in all directions. Therefore there is no net force on it. At the surface, a molecule is surrounded by only half as many molecules of the liquid, because there are no molecules above the surface.

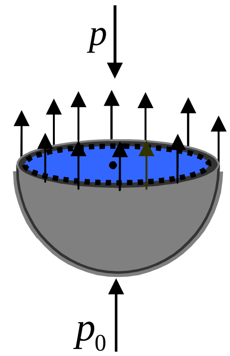
3. The force of contraction is at right angles to an imaginary line of unit length, tangential to the surface of a liquid, is called its surface tension: $\gamma = \frac{F}{L}$. Here F is the force exerted by the "skin" of the liquid. The SI unit of the surface tension is N/m.



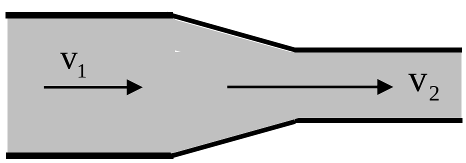
4. Quantitative measurement of surface tension: let w be the weight of the sliding wire, $T =$ force with which you pull the wire downward. Obviously, $T + w = F =$ net downward force. Since film has both front and back surfaces, the force F acts along a total length of $2L$. The surface tension in the film is defined as, $\gamma = \frac{F}{2L} \quad \Rightarrow F = 2\gamma L$.
   Hence, the surface tension is $\gamma = \frac{w+T}{2L}$.

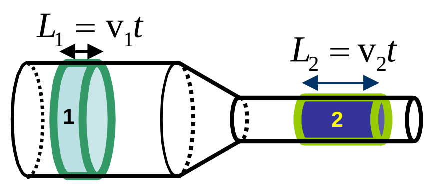
5. Let's ask how much work is done when we stretch the skin of a liquid. If we move the sliding wire through a displacement Δx, the work done is $F\Delta x$. Now F is a conservative force, so there is potential energy $\Delta U = F\Delta x = \gamma L \Delta x$ where L is the length of the surface layer $L\Delta x = \Delta A =$ change in area of the surface. Thus $\gamma = \frac{\Delta U}{\Delta A}$. So we see that surface tension is the surface potential energy per unit area !

7. The surface tension causes a pressure difference between the inside and outside of a soap bubble or a liquid drop. A soap bubble consists of two spherical surface films with a thin layer of liquid between them. Let p = pressure exerted by the upper half, and p_0 = external pressure. $\therefore$ force exerted due to surface tension is $2(2\pi r\gamma)$ (the "2" is for two surfaces). In equilibrium the net forces must be equal: $\pi r^2 p = 2(2\pi r\gamma) + \pi r^2 p_0$. So the excess pressure is $p - p_0 = \frac{4\gamma}{r}$. For a liquid drop, the difference is that there is only surface and so, excess pressure $= p - p_0 = \frac{2\gamma}{r}$.



8. From the fact that liquids are incompressible, equal volumes of liquid must flow in both sections in time t, i.e. $V_1/t = V_2/t \Rightarrow V_1 = V_2$. But you can see that $V_1 = A_1L_1 = A_1 v_1 t$ and similarly that $V_2 = A_2L_2 = A_2 v_2 t$. Hence $A_1 v_1 = A_2 v_2$. This means that liquid will flow faster when the tube is narrower, and slower where it is wider.





9. **Bernoulli's Equation :** When a fluid flows steadily, it obeys the equation:

$$p + \frac{1}{2}\rho v^2 + \rho g y = \text{ constant}$$

This famous equation, due to Daniel Bernoulli about 300 years ago, tells you how fast a fluid will flow when there is also a gravitational field acting upon. For a derivation, see any of the suggested references. In the following, I will only explain the meaning of the various terms in the formula and apply it to a couple of situations.

10. Let us apply this to water flowing in a pipe whose crossection decreases along its length. (A_1 is area of the wide part, etc).There is no change in the height so $y_1 = y_2$ and the gravitational potential cancels,

$$p_1 + \frac{1}{2}\rho v_1^2 + \cancel{\rho g y_1} = p_2 + \frac{1}{2}\rho v_2^2 + \cancel{\rho g y_2} \Rightarrow p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2.$$

Now, since the liquid is incompressible, it flows faster in the narrower part:

$$A_1 v_1 = A_2 v_2 \Rightarrow v_2 = \left(\frac{A_1}{A_2}\right) v_1 \Rightarrow p_2 = p_1 - \frac{1}{2}\rho\left(v_2^2 - v_1^2\right).$$

This means that the pressure is smaller where the fluid is flowing faster! This is exactly why an aircraft flies: the wing shape is curved so that when the aircraft moves through the air, the air moves faster on the top part of the wing than on the lower part. Thus, the air pressure is lower on the top compared to the bottom and there is a net pressure upwards. This is called lift.

Summary of Lecture 19 – PHYSICS OF SOUND

1. Sound waves correspond to longitudinal oscillations of density. So if sound waves are moving from left to right, as you look along this direction you will find the density of air greater in some places and less in others. Sound waves carry energy. The minimum energy that humans can hear is about 10^{-12} watts per cm^3 (This is called I_0, the threshold of hearing.)

2. To measure the intensity of sound, we use *decibels* as the unit. Decibels (db) are a relative measure to compare the intensity of different sounds with one another,

$$R \equiv \text{relative intensity of sound } I \;=\; \log_{10}\frac{I}{I_0} \text{ (decibels)}$$

Typically, on a street without traffic the sound level is about 30db, a pressure horn creates about 90db, and serious ear damage happens around 120db.

3. A sound wave moving in the x direction with speed v is described by

$$\rho(x,t) = \rho_m \sin\frac{2\pi}{\lambda}(x - \mathrm{v}t)$$

where $\rho(x,t)$ is the density of air at a point x at time t. Let us understand various aspects of this formula.

a) Suppose that as time t increases, we move in such a way as to keep $x - \mathrm{v}t$ constant. So if at $t = 0$ the value of x is 0.23 (say), then at $t = 1$ the value of x would be v+0.23, etc. In other words, to keep the density $\rho(x,t)$ constant, we would have to move with the speed of sound , i.e. v.

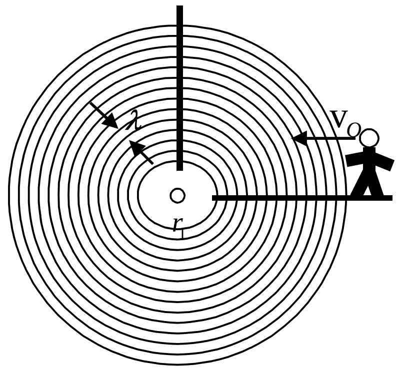
b) In the expression for $\rho(x,t)$, replace x by $x + \lambda$. What happens? Answer: nothing, because $\sin\frac{2\pi}{\lambda}(x + \lambda - \mathrm{v}t) = \sin\frac{2\pi}{\lambda}(x - \mathrm{v}t)$. This is why we call λ the "wavelength", meaning that length after which a wave repeats itself.

c) In the expression for $\rho(x,t)$, replace t by $t + T$ where $T = \lambda/\mathrm{v}$. What happens? Again the answer: nothing. T is called the time period of the sound wave, meaning that time after which it repeats itself. The frequency is the number of cycles per second and is obviously related to T through $\nu = 1/T$.

d) It is also common to introduce the *wavenumber* k the and *angular frequency* ω:

$$k = \frac{2\pi}{\lambda} \quad \text{and} \quad \omega = \frac{2\pi}{T} = 2\pi\nu$$

4. **Doppler Effect**: The relative motion between source and observer causes the observer to receive a frequency that is different from that emitted by the source. One must distinguish between two cases:



Moving observer, source at rest. If the observer was at rest, the number of waves she would receive in time t would be t/T (or $\mathrm{v}t/\lambda$). But if she is moving towards the source with speed v_0 (as in the above figure), the additional number of waves received is obviously $\mathrm{v}_0 t/\lambda$. By definition,

$$\nu' = \text{ frequency actually heard} = \frac{\text{number of waves received}}{\text{unit time}}$$

$$\therefore \nu' = \frac{\frac{\mathrm{v}t}{\lambda} + \frac{\mathrm{v}_0 t}{\lambda}}{t} = \frac{\mathrm{v}+\mathrm{v}_0}{\lambda} = \frac{\mathrm{v}+\mathrm{v}_0}{\mathrm{v}/\nu} = \nu\frac{\mathrm{v}+\mathrm{v}_0}{\mathrm{v}}$$

We finally conclude that the frequency actuall heard is $\nu' = \nu\left(1+\frac{\mathrm{v}_0}{\mathrm{v}}\right)$. So as the observer runs towards the source, she hears a higher frequency (higher pitch).

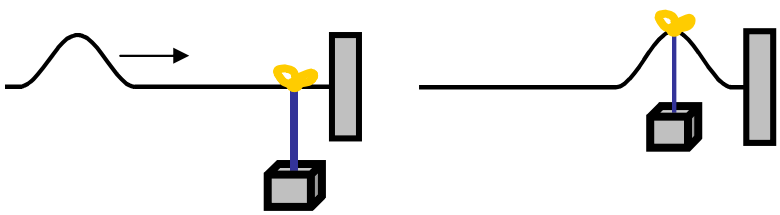
Moving source, observer at rest : As the source runs towards the observed, more waves will have to packed together. Each wavelength is reduced by $\frac{\mathrm{v}_s}{\nu}$. So the wavelength seen by the observer is $\lambda' = \frac{\mathrm{v}}{\nu} - \frac{\mathrm{v}_s}{\nu}$. From this, the frequency that she hears is $\nu' = \frac{\mathrm{v}}{\lambda} = \frac{\mathrm{v}}{(\mathrm{v}-\mathrm{v}_s)/\nu} = \nu\frac{\mathrm{v}}{(\mathrm{v}-\mathrm{v}_s)}$.

Moving source and moving observer : $\nu' = \nu\frac{\mathrm{v}+\mathrm{v}_o}{\mathrm{v}-\mathrm{v}_s}$. As you can easily see, the above two results are special cases of this.

Summary of Lecture 20 – WAVE MOTION

1. Wave motion is any kind of self-repeating (periodic, or oscillatory) motion that transports energy from one point to another. Waves are of two basic kinds:
   (a) **Longitudinal Waves**: the oscillation is parallel to the direction of wave travel. Examples: sound, spring, "P-type" earthquake waves.
   (b) **Transverse Waves**: the oscillation is perpendicular to the direction of wave travel. Examples: radio or light waves, string, "S-type" earthquake waves.

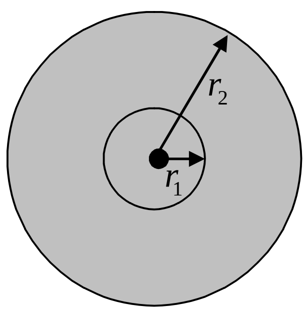
2. Waves transport energy, not matter. Taking the vibration of a string as an example, each segment of the string stays in the same place, but the work done on the string at one end is transmitted to the other end. Work is done in lifting the mass at the other end below.



3. The height of a wave is called the amplitude. The average power (or intensity) in a wave is proportional to the square of the amplitude. So if $a(t) = a_0 \sin(\omega t - kx)$ is a wave of some kind, then a_0 is the amplitude and $I \propto a_0^2$.

4. A sound source placed at the origin will radiate sound waves in all directions equally. These are called spherical waves. For spherical waves the amplitude $\propto \frac{1}{r}$ and so the power $\propto \frac{1}{r^2}$.
   We can easily see why this is so. Consider a source of sound and draw two spheres:
   Let P_1 be the total radiated power and I_1 the intensity at r_1, etc. All the power (and energy) that crosses r_1 also crosses r_2 since none is lost in between the two. We have that,

   $4\pi r_1^2 I_1 = P_1$ and $4\pi r_2^2 I_2 = P_2$. But $P_1 = P_2 = P$, and so $\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}$ or $I \propto \frac{1}{r^2}$.



5. We have encountered waves of the kind $y(x,t) = y_0 \sin(kx - \omega t)$ in the previous lecture. Obviously $y(0,0) = 0$. But what if the wave is not zero at $x = 0$, $t = 0$? Then it could be represented by $y(x,t) = y_m \sin(kx - \omega t - \phi)$, where $kx - \omega t - \phi$ is called the phase and ϕ is called the phase constant. Note that you can rewrite $y(x,t)$ either as,

   a) $y(x,t) = y_m \sin\left[k\left(x - \frac{\phi}{k}\right) - \omega t\right],$

   or as, b) $y(x,t) = y_m \sin\left[kx - \omega\left(t + \frac{\phi}{\omega}\right)\right].$

   The two different ways of writing the same expression can be interpreted differently. In (a) x has effectively been shifted to $x - \frac{\phi}{k}$ whereas in (b) t has been shifted to $t + \frac{\phi}{\omega}$. So the phase constant only moves the wave forward or backward in space or time.

6. When two sources are present the total amplitude at any point is the sum of the two separate amplitudes, $y(x,t) = y_1(x,t) + y_2(x,t)$. Now you remember that the power is proportional to the *square* of the amplitude, so $P \propto (y_1 + y_2)^2$. This is why *interference* happens. In the following we shall see why. Just to make things easier, suppose the two waves have equal amplitude. So lets take the two waves to be:

   $$y_1(x,t) = y_m \sin(kx - \omega t - \phi_1) \quad \text{and} \quad y_2(x,t) = y_m \sin(kx - \omega t - \phi_2)$$

   The total amplitudes is: $y(x,t) = y_1(x,t) + y_2(x,t)$

   $$= y_m\left[\sin(kx - \omega t - \phi_1) + \sin(kx - \omega t - \phi_2)\right]$$

   Now use the trigonometric formula, $\sin B + \sin C = 2\sin\frac{1}{2}(B + C) \times \cos(B - C)$ to get,

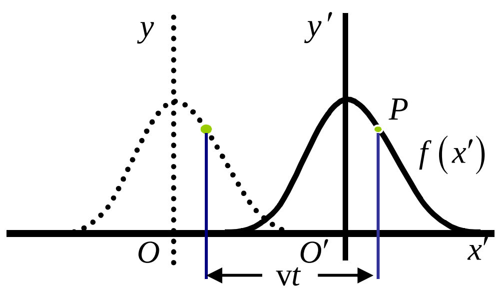
   $$y(x,t) = y_m\left[\sin(kx - \omega t - \phi_1) + \sin(kx - \omega t - \phi_2)\right]$$

   $$= \left[2 y_m \cos\left(\frac{\Delta\phi}{2}\right)\right] \times \sin(kx - \omega t - \phi').$$

   Here $\Delta\phi = \phi_2 - \phi_1$ is the difference of phases, and $\phi' = \frac{(\phi_1 + \phi_2)}{2}$ is the sum. So what do we learn from this? That if $\phi_2 = \phi_1$, then the two waves are in phase and the resultant amplitude is maximum (because $\cos 0 = 1$). And that if $\phi_2 = \phi + \pi$, then the two waves are out of phase and the resultant amplitude is minimum (because $\cos \pi / 2 = 0$). The two waves have interfered with each other and have increased/decreased their amplitude in these two extreme cases. In general $\cos\left(\frac{\Delta\phi}{2}\right)$ will be some number that lies between -1 and +1.

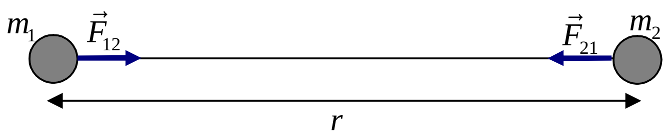
7. There was no time in the lecture to prove it, but you can look up any good book to find a proof for the formaula that the speed of sound in a medium is: $\text{v} = \sqrt{\dfrac{B}{\rho}}$ where B is the bulk modulus and ρ is the density of the medium.

8. **The speed of a pulse.** A pulse is a burst of energy (sound, electromagnetic, heat,...) and could have any shape. Mathematically any pulse has the form $y(x,t) = f(x - \text{v}t)$. Here f is any function (e.g. sin, cos, exp,....). Note that at time $t = 0$, $y(x,0) = f(x)$ and the shape would look as on the left in the diagram below. At a late time t, it will look just the same, but shifted to the right. In other words at time t, $y(x,t) = f(x')$ where $x' = x - \text{v}t$. Fix your attention on any one point of the curve and follow it as the pulse moves to the right. From $x - \text{v}t = \text{constant}$ it follows that $\dfrac{dx}{dt} - \text{v} = 0$, or $\text{v} = \dfrac{dx}{dt}$. This is called the *phase velocity* because we derived it using the constancy of phase.

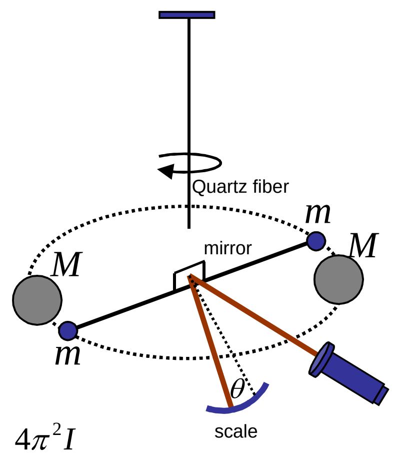


Summary of Lecture 21 – GRAVITY

1. Newton's law of universal gravitation states that the force of attraction between two masses m_1 and m_2 is $F \propto \frac{m_1 m_2}{r^2}$ and is directed along the line joining the two bodies. Putting in a constant of proportionality, $F = G\frac{m_1 m_2}{r^2}$. Now let's be a bit careful of the direction of the force. Looking at the diagram below, $\vec{F}_{21}$ = Force on m_2 by m_1, $\vec{F}_{12}$ = Force on m_1 by m_2 , $\left|\vec{F}_{12}\right| = \left|\vec{F}_{21}\right| = F$. By Newton's Third Law, $\vec{F}_{12} = -\vec{F}_{21}$.



2. The gravitational constant G is a very small quantity and needs a very sensitive experiment. An early experiment to find G involved suspending two masses and measuring the attractive force. From the figure you can see that the is gravitational torque is $2\left(\frac{GmM}{r^2}\right)\frac{L}{2}$. A thread provides the restoring torque $\kappa\theta$. The deflection θ can be measured by observing the beam of the light reflected from the small mirror. In equilibrium the torques balance, $\frac{GmML}{r^2} = \kappa\theta$. Hence $G = \frac{\kappa\theta r^2}{GmML}$. How to find κ? It can be found from observing the period of free oscillations, $T = 2\pi\sqrt{\frac{I}{\kappa}} \Rightarrow \kappa = \frac{4\pi^2 I}{T^2}$ with $I = \frac{mL^2}{2}$. The modern value is $G = 6.67259 \times 10^{-11} N.m^2/kg^2$



3. The magnitude of the force with which the Earth attracts a body of mass m towards its centre is $F = \frac{GmM_E}{R_E^{\,2}}$, where $R_E = 6400\,km$ is the radius of the Earth and M_E is the mass. The material does not matter - iron, wood, leather, etc. all feel the force in proportion to their masses. If the body can fall freely, then it will accelerate. So, $F = mg = \frac{GmM_E}{R_E^{\,2}}$.

We measure g, the acceleration due to gravity, as $9.8m/s^2$. From this we can immediately deduce the Earth's mass: $M_E = \frac{gR_E{}^2}{G} = 5.97\times 10^{24} kg$. What a remarkable achievement!

We can do still more: the volume of the Earth $= V_E = \frac{4}{3}\pi R_E{}^3 = 1.08\times 10^{21} m^3$. Hence the] density of the Earth $= \rho_E = \frac{V_E}{M_E} = 5462\,\text{kg}\,m^{-3}$. So this is 5.462 times greater than the density of water and tells us that the earth must be quite dense inside.

4. The **gravitational potential** is an important quantity. It is the work done in moving a unit mass from infinity to a given point R, and equals $V(r) = -\frac{GM}{R}$.

   Proof: Conservation of energy says, $dV = -Fdr \Rightarrow \int_{V(R)}^{0} dV = -\int_{R}^{\infty} drF(r)$

   Integrate both sides: $0 - V(R) = GM\int_R^\infty \frac{dr}{r^2} = -GM\left[\frac{1}{r}\right]_R^\infty, \qquad \therefore V(R) = -\frac{GM}{R}$

5. Using the above formula, let us calculate the change in potential energy ΔU when we raise a body of mass m to a height h above the Earth's surface.

$$\Delta U = GMm\left(\frac{1}{R_E} - \frac{1}{R_E + h}\right) = GMm\left(1 - \frac{1}{1 + h/R_E}\right) = GMm\left(1 - \left(1 + h/R_E\right)^{-1}\right)$$

   Now suppose that the distance h is much smaller than the Earth's radius. So, for $h \ll R_E$, $\left(1 + h/R_E\right)^{-1} = 1 - h/R_E$. So we find $\Delta U = GMm\left(1 - \left(1 - h/R_E\right)\right) = m\left(\frac{GM}{R_E}\right)h = mgh.$

6. We can use the expression for potential energy and the law of conservation of energy to find the minimum velocity needed for a body to escape the Earth' gravity. Far away from the Earth, the potential energy is zero, and the smallest value for the kinetic energy is zero. Requiring that $\left(KE + PE\right)_{r=R} = \left(KE + PE\right)_{r=\infty}$ gives $\frac{1}{2}mv_e^2 - \frac{GMm}{R_E} = 0 + 0$. From this, $v_e = \sqrt{\frac{2GM}{R_E}} = \sqrt{2gR_E}$. Putting in some numbers we find that for the Earth $v_e = 11.2$ km/s and for the Sun $v_e = 618$ km/s. For a Black Hole, the escape velocity is so high that nothing can escape, even if it could move with the speed of light! (Nevertheless, Black Holes can be observed because when matter falls into them, a certain kind of radiation is emitted.)

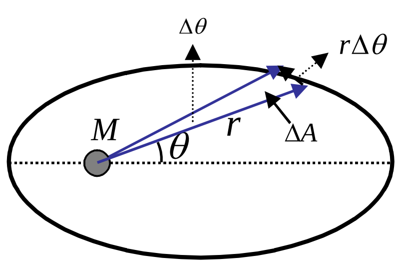
7. **Satellite problems**: A satellite is in circular orbit over the Earth's surface. The condition for equilibrium, $\frac{mv_o^2}{r}=\frac{GMm}{r^2}\Rightarrow v_o=\sqrt{\frac{GM}{r}}$. If R_E is the Earth's radius, and h is the height of the satellite above the ground, then $r=R_E+h$. Hence, $v_o=\sqrt{\frac{GM}{R_E+h}}$. For $h<<R_E$, we can approximate $v_o=\sqrt{\frac{GM}{R_E}}=\sqrt{gR_E}$. We can easily calculate the time for one complete revolution, $T=\frac{2\pi}{\omega}=\frac{2\pi r}{v_o}=2\pi r\sqrt{\frac{r}{GM}}=\frac{2\pi}{\sqrt{GM}}r^{3/2}$. This gives the important result, observed by Kepler nearly 3 centuries ago that $T^2=\frac{4\pi^2}{GM}r^3$, or $T^2\propto r^3$.

8. What is the total energy of a satellite moving in a circular orbit around the earth? Clearly, it has two parts, kinetic and potential. Remeber that the potential energy is negative. So, $E=KE+PE=\frac{1}{2}mv_o^2-\frac{GM_Em}{r}$. But, $v_o^2=\frac{GM}{r}$ as we saw earlier and therefore, $E=\frac{1}{2}\frac{GMm}{r}-\frac{GMm}{r}=-\frac{1}{2}\frac{GMm}{r}$. Note that the magnitude of the potential energy is larger than the kinetic energy. If it wasn't, the satellite would not be bound to the Earth!

9. A famous discovery of the astronomer Johann Kepler some 300 years ago says that the line joining a planet to the Sun sweeps out equal areas in equal intervals of time. We can easily see this from the conservation of angular momentum. Call ΔA the area swept out in time Δt. Then from the diagram below you can see that $\Delta A=\frac{1}{2}r(r\Delta\theta)$. Divide this by Δt and then take the limit where it becomes very small,

$$\frac{dA}{dt}=\lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t}=\frac{1}{2}r^2\left(\lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}\right)=\frac{1}{2}r^2\omega=\frac{L}{2m}.$$

Since L is a constant, we have proved one of Kepler's laws (with so little effort)!



Summary of Lecture 22 – ELECTROSTATICS I

1. Like charges repel, unlike charges attract. But by how much? Coulomb's Law says that this depends both upon the strength of the two charges and the distance between them. In mathematical terms, $F \propto \frac{q_1 q_2}{r^2}$ which can be converted into an equality, $F = k\frac{q_1 q_2}{r^2}$. The constant of proportionality will take different values depending upon the units we choose. In the MKS system, charge is measured in Coulombs (C) and $k = \frac{1}{4\pi\varepsilon_0}$ with $\varepsilon_0 = 8.85\times10^{-12} C^2/Nm^2$ and hence $k = 8.99\times10^9 Nm^2/C^2$.

2. The situation is quite similar to that of gravity, except that electric charges and not masses are the source of force. In vector form, $\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12}$ is the force exerted by 2 on 1, where the unit vector is $\hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}}$. On the other hand, $\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{21}$ is the force exerted by 1 on 2. By Newton's Third Law, $\vec{F}_{12} = -\vec{F}_{21}$. For many charges, the force on charge 1 is given by, $\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \cdots$

3. Charge is quantized. This means that charge comes in certain units only. So the size of a charge can only be 0, $\pm e, \pm 2e, \pm 3e, \cdots$ where $e = 1.602\times10^{-19} C$ is the value of the charge present on a proton. By definition we call the charge on a proton positive. This makes the charge on an electron negative.

4. Charge is conserved. This means that charge is never created or destroyed. Equivalently, in any possible situation, the total charge at an earlier time is equal to the charge at a later time. For example, in any of the reactions below the initial charge = final charge:

   $e^- + e^+ \to \gamma + \gamma$ (electron and positron annihilate into neutral photons)

   $\pi^0 \to \gamma + \gamma$ (neutral pion annihilates into neutral photons)

   $H^2 + H^2 \to H^3 + p$ (two deuterons turn into tritium and proton)

5. **Field :** this a quantity that has a definite value at any point in space and at any time. The simplest example is that of a scalar field, which is a *single number* for any value of x, y, z, t. Examples: temperature inside a room $T(x, y, z, t)$, density in a blowing wind $\rho(x, y, z, t)$, $\cdots$ There are also *vector fields*, which comprise of three numbers at each value of x, y, z, t. Examples: the velocity of wind, the pressure inside a fluid, or even a sugarcane field. In

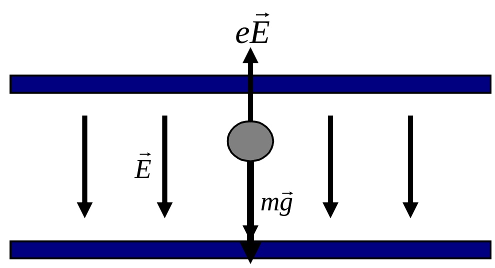
every case, there are 3 numbers: $\vec{V}(x,y,z,t)=\{V_1(x,y,z,t),V_2(x,y,z,t),V_3(x,y,z,t)\}$.

6. The electric field is also an example of a vector field, and will be the most important for our purpose. It is defined as the force on a unit charge. Or, since we don't want the charge to disturb the field it is placed in, we should properly define it as the force on a "test" charge q_0, $E\equiv\frac{F}{q_0}$. Here q_0 is very very small. The electric field due to a point charge can be calculated by considering two charges. The force between them is $F=\frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}$ and so $E=\frac{F}{q_0}=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$. A way to visualize E fields is to think of lines starting on positive charges and ending on negative charges. The number of lines leaving/entering gives the amount of charge.

7. Typical values for the maginitude of the electric field E :

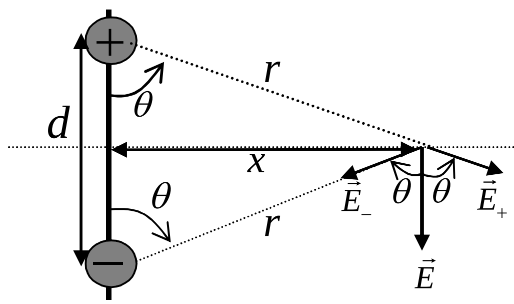
Inside an atom-	10^{11}	N/C
Inside TV tube-	10^{5}	N/C
In atmosphere-	10^{2}	N/C
Inside a wire-	10^{-2}	N/C

8. Measuring charge. One way to do this is to balance the gravitational force pulling a charged particle with mass m with the force exerted on it by a known electric field (see below). For equilibrium, the two forces must be equal and so $mg=qE$. The unknown charge q can then be found from $q=\frac{mg}{E}$.



9. Given several charges, one can find the total electric field at any point as the sum of the fields produced by the charges at that point individually, $\vec{E}=\vec{E}_1+\vec{E}_2+\vec{E}_3+\cdots\cdots$ or $\vec{E}=\sum_i\vec{E}_i=k\sum_i\frac{q_i}{r_i^2}\hat{r}_i \quad (i=1,2,3,\cdots)$. Here $\hat{r}_i$ is the unit vector pointing from the charge to the point of observation.

10. Let us apply the principle we have just learned to a system of two charges $+q$ and $-q$ which are separated by a distance d (see diagram). Then, $\vec{E} = \vec{E}_+ + \vec{E}_-$. Just to make things easier (not necessary; one can do it for any point) I have taken a point that lies on the x-axis. The magnitudes of the electric field due to the two charges are equal; $E_+ = E_- = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{x^2 + (d/2)^2}$. The vertical components cancel out, and the net electric field is directed downwards with magnitude, $E = E_+\cos\theta + E_-\cos\theta = 2E_+\cos\theta$. From the diagram below you can see that $\cos\theta = \frac{d/2}{\sqrt{x^2 + (d/2)^2}}$. Substituting this, we find: $E = 2\frac{1}{4\pi\varepsilon_0}\frac{q}{x^2 + (d/2)^2}\frac{d/2}{\sqrt{x^2 + (d/2)^2}} = \frac{1}{4\pi\varepsilon_0}\frac{qd}{\left[x^2 + (d/2)^2\right]^{3/2}}$.



11. The result above is so important that we need to discuss it further. In particular, what happens if we are very far away from the dipole, meaning $x >> d$? Let us first define the *dipole moment* as the product of the charge $\times$ the separation between them $p = qd$. Then, $E = \frac{1}{4\pi\varepsilon_0}\frac{p}{x^3}\frac{1}{\left[1 + (d/2x)^2\right]^{3/2}} = \frac{1}{4\pi\varepsilon_0}\frac{p}{x^3}\left[1 + \left(\frac{d}{2x}\right)^2\right]^{-3/2} = \frac{1}{4\pi\varepsilon_0}\frac{p}{x^3}$. In the above, $d/2x$ has been neglected in comparison to 1. So finally, we have found that for $x >> d$, $E = \frac{1}{4\pi\varepsilon_0}\frac{p}{x^3}$.

12. It is easy to find the torque experienced by an electric dipole that is placed in a uniform electric field: The magnitude is $\tau = F\frac{d}{2}\sin\theta + F\frac{d}{2}\sin\theta = Fd\sin\theta$, and the direction is perpendicular and into the plane. Here θ is the angle between the dipole and the electric field. So $\tau = (qE)d\sin\theta = pE\sin\theta$.

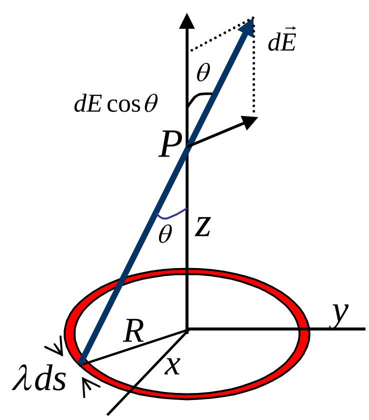
Summary of Lecture 23 – ELECTROSTATICS II

1. In the last lecture we learned how to calculate the electric field if there are any number of point charges. But how to calculate this when charges are continuously distributed over some region of space? For this, we need to break up the region into little pieces so that each piece is small enough to be like a point charge. So, $\vec{E} = \Delta\vec{E}_1 + \Delta\vec{E}_2 + \Delta\vec{E}_3 + \cdots$, or $\vec{E} = \sum \Delta\vec{E}_i$ is the total electric field. Remember that $\vec{E}$ is a vector that can be resolved into components, $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$. In the limit where the pieces are small enough, we can write it as an integral, $\vec{E} = \int d\vec{E}$ (or $E_x = \int dE_x$, $E_y = \int dE_y$, $E_z = \int dE_z$)

2. Charge Density: when the charges are continuously distributed over a region - a line, the surface of a material, or inside a sphere - we must specify the *charge density*. Depending upon how many dimensions the region has, we define:
   (a) For linear charge distribution: $dq = \lambda ds$
   (b) For surface charge distribution: $dq = \sigma dA$
   (c) For volume charge distribution: $dq = \rho dV$

   The dimensions of λ, σ, ρ are determined from the above definitions.

3. As an example of how we work out the electric field coming from a continuous charge distribution, let us work out the electric field from a uniform ring of charge at the point P.



The small amount of charge λds gives rise to an electric field whose magnitude is

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda ds}{r^2} = \frac{\lambda ds}{4\pi\varepsilon_0\left(z^2 + R^2\right)}$$

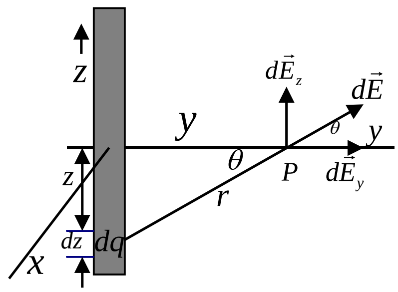
The component in the z direction is $dE_z = dE\cos\theta$ with $\cos\theta = \frac{z}{r} = \frac{z}{\left(z^2 + R^2\right)^{1/2}}$.

So $dE_z = \dfrac{z\lambda ds}{4\pi\varepsilon_0\left(z^2+R^2\right)^{3/2}}$. Since s, which is the arc length, does not depend upon z or R,

$$E_z = \frac{z\lambda}{4\pi\varepsilon_0\left(z^2+R^2\right)^{3/2}}\int ds = \frac{z\lambda(2\pi R)}{4\pi\varepsilon_0\left(z^2+R^2\right)^{3/2}} = \frac{qz}{4\pi\varepsilon_0\left(z^2+R^2\right)^{3/2}}.\ \text{Answer!!}$$

Note that if you are very far away, the ring looks like a point: $E_z = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{z^2}, \quad (z >> R)$.

4. As another example, consider a continuous distribution of charges along a wire that lies along the z-axis, as shown below. We want to know the electric field at a distance x from the wire. By symmetry, the only non-cancelling component lies along the y-axis.



Applying Coulomb's law to the small amount of charge λdz along the z axis gives,

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dz}{y^2+z^2}$$

the component along the y direction is $dE_y = dE\cos\theta.$ Integrating this gives,

$$E_y = \int dE = \int_{z=-\infty}^{z=\infty}\cos\theta dE = 2\int_{z=0}^{z=\infty}\cos\theta dE = \frac{\lambda}{2\pi\varepsilon_0}\int_{z=0}^{z=\infty}\cos\theta\frac{dz}{y^2+z^2}.$$

The rest is just technical: to solve the integral, put $z = y\tan\theta \Rightarrow dz = y\sec^2\theta d\theta$. And so,

$E = \dfrac{\lambda}{2\pi\varepsilon_0 y}\displaystyle\int_{\theta=0}^{\theta=\pi/2}\cos\theta d\theta = \dfrac{\lambda}{2\pi\varepsilon_0 y}$. Now, we could have equally well taken the x axis. The only thing that matters is the distance from the wire, and so the answer is better written as:

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}.$$

5. The **flux** of any vector field is a particularly important concept. It is the measure of the "flow" or penetration of the field vectors through an imaginary fixed surface. So, if there is a uniform electric field that is normal to a surface of area A, the flux is $\Phi = EA$. More generally, for any surface, we divide the surface up into little pieces and take the

component of the electric field normal to each little piece, $\Phi_E = \sum \vec{E}_i \cdot \Delta\vec{A}_i$. If the pieces are made small enough, then in this limit, $\Phi = \int \vec{E} \cdot d\vec{A}$.

5. Let us apply the above concept of flux to calculate the flux leaving a sphere which has a charge at its centre. The electric field at any point on the sphere has magnitude equal to $\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$ and it is directed radially outwards. Let us now divide up the surface of the sphere into small areas. Then $\Phi = \sum E\Delta A = E\sum \Delta A = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\left(4\pi r^2\right)$. So we end up with the important result that the flux leaving this closed surface is $\Phi = \frac{q}{\varepsilon_0}$.

6. **Gauss's Law**: the total electric flux leaving a closed surface is equal to the charge enclosed by the surface divided by ε_0. We can express this directly in terms of the mathematics we have learned, $\Phi \equiv \int \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_0}$. Actually, we have already seen why this law is equivalent to Coulomb's Law in point 5 above, but let's see it again. So, applying Gauss's Law to a sphere containing charge, $\varepsilon_0 \int \vec{E} \cdot d\vec{A} = \varepsilon_0 \int E\,dA = q_{enclosed}$. If the surface is a sphere, then E is constant on the surface and $\varepsilon_0 E \int dA = q$ and from this $\varepsilon_0 E\left(4\pi r^2\right) = q \Rightarrow E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$. This is Coulomb's Law again, but the power of Gauss's law is that it holds for any shape of the (closed) surface and for any distribution of charge.

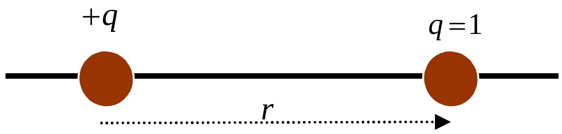
7. Let us apply Gauss's Law to a hollow sphere that has charges only on the surface. At any distance r from the centre, Gauss's Law is $\varepsilon_0 E\left(4\pi r^2\right) = q_{enclosed}$. Now, if we are inside the sphere then $q_{enclosed} = 0$ and there is no electric field. But if we are outside, then the total charge is $q_{enclosed} = q$ and $E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$, which is as if all the charge was concentrated at the centre.

8. Unfortunately, it will not be possible for me to prove Gauss's Law in the short amount of time and space available but the general method can be outlined as follows: take any volume and divide it up into little cubes. Each little cube may contain some small amount of charge. Then show that for each little cube, Gauss's Law follows from Coulomb's Law. Finally, add up the results. For details, consult any good book on electromagnetism.

Summary of Lecture 24 – ELECTRIC POTENTIAL ENERGY

1. You are already familiar with the concept of gravitational potential energy. When you lift a weight, you have to do work against the downwards pull of the Earth. That work is stored as potential energy. Suppose a force $\vec{F}$ acts on something and displaces it by $d\vec{s}$. Then the work done is $\vec{F}\cdot d\vec{s}$. The work done in going from point a to point b (call it W_{ab}) is then got by adding together the little bits of work, $W_{ab}=\int_a^b \vec{F}\cdot d\vec{s}$. The change in potential energy is defined as $\Delta U = U_b - U_a = -W_{ab}$. Remember always that we can only define the potential U at a point if the force is conservative.

2. The electrostatic force is conservative and can be represented by a potential. Let us see how to calculate the potential. So consider two charges separated by a distance as below.



Let us take the point a very far from the fixed charge q, and the unit charge at the point b to be at a distance R from q. Then the work you did in bringing the unit charge from infinity to R is, $W_{ab}=\int_{\infty}^{R}(-qE)dr = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{R} dr\frac{q}{r^2} = -\frac{1}{4\pi\varepsilon_0}q\left(\frac{1}{R}-\frac{1}{\infty}\right) = -\frac{q}{4\pi\varepsilon_0}\frac{1}{R}$. Since the charges repel each other, it is clear that you had to do work in pushing the two charges closer together. So where did the negative sign come from? Answer: the force you exert on the unit charge is directed *towards* the charge q, i.e. is in the negative direction. This is why $\vec{F}\cdot d\vec{s} = (-qE)dr$. Now $\Delta U = U(R) - U(\infty) = \frac{q}{4\pi\varepsilon_0}\frac{1}{R}$. If we take the potential at ∞ to be zero, then the electric potential due to a charge q at the point r is $U(r)=\frac{q}{4\pi\varepsilon_0}\frac{1}{r}$.

Remember that we know how to calculate the force given the potential: $F=-\frac{dU}{dr}$. Apply this here and you see that $F=\frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}$, (which also has the correct repulsive sign).

3. From the above, it is quite obvious that the potential energy of two charges q_1, q_2 is, $U(r)=\frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r}$. Compare this with the formula for gravitational energy, $U(r)=-G\frac{m_1m_2}{r}$. What is the difference? From here, you can see that the gravitational force is always negative (which means attractive), whereas the electrostatic force can be both attractive

or repulsive because we have both + and - charges in nature.

3. The electric potential (or simply potential) is the energy of a unit charge in an electric field.

So, in our MKS units, the unit of potential is $1\ \dfrac{\text{Joule}}{\text{Coulomb}} = 1$ Volt. Another useful unit is "electron volt" or eV. The definition is:

$$\textit{One electron - volt} = \textit{energy gained by moving one electron charge through one Volt}$$

$$= \left(1.6\times10^{-19}C\right)\times 1V = 1.6\times10^{-19}J$$

It is useful to note that 1 Kev $= 10^{3}eV$ (kilo-electron-volt)

1 Mev $= 10^{6}eV$ (million-electron-volt)

1 Gev $= 10^{9}eV$ (giga-electron-volt)

1 Tev $= 10^{12}eV$ (tera-electron-volt)

4. Every system seeks to minimize its potential energy (that is why a stone falls down!). So, positive charges accelerate toward regions of lower potential, but negative charges accelerate toward regions of higher potential. Note that only the potential difference matters - even if a charge is placed in a region where there is a high potential, it will not want to move unless there is some other place where the potential is higher/lower.

5. Given a system of charges, we can always compute the force - and hence the potential - that arises from them. Here are some important general statements:

a)Potentials are more positive in regions which have more positive charge.

b)The electric potential is a scalar quantity (a scalar field, actually).

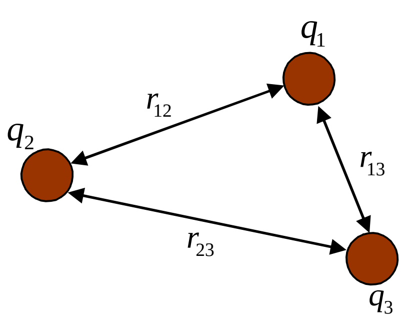
c)The electric potential determines the force through $F = -\dfrac{dU}{dr}$, and hence the electric field because $F = qE$.

d)The electric potential exists only because the electrostatic force is conservative.

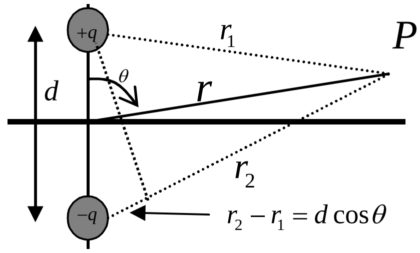
6. To compute the potential at a point, the potentials arising from charges 1, 2, $\cdots$ N must be added up:

$$V = V_1 + V_2 + \cdots V_N = \sum_{i=1}^{N} V_i = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i}.$$ Here r_i is the distance of the i'th charge from the point where the potential is being calculated or measured. As an example, the potential from the three charges is:

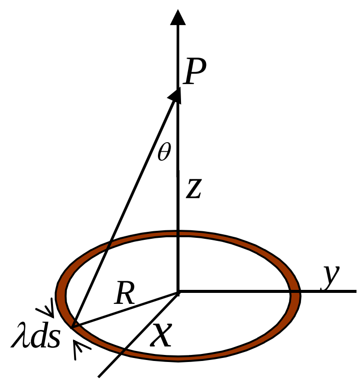
$$\mathrm{V}(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_{12}} + \frac{1}{4\pi\varepsilon_0}\frac{q_1q_3}{r_{13}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2q_3}{r_{23}}.$$



7. Let us apply these concepts to the dipole system considered earlier. With two charges, $V_P = V_1 + V_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} + \frac{-q}{r_2}\right) = \frac{q}{4\pi\varepsilon_0}\frac{r_2 - r_1}{r_1 r_2}$. We are particularly interested in the situation where $r >> d$. from the diagram you can see that $r_2 - r_1 \approx d\cos\theta$ and that $r_1 r_2 \approx r^2$. Hence, $V \approx \frac{q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$. So we have calculated the potential at any θ with such little difficulty. Note that $V = 0$ at $\theta = \frac{\pi}{2}$.



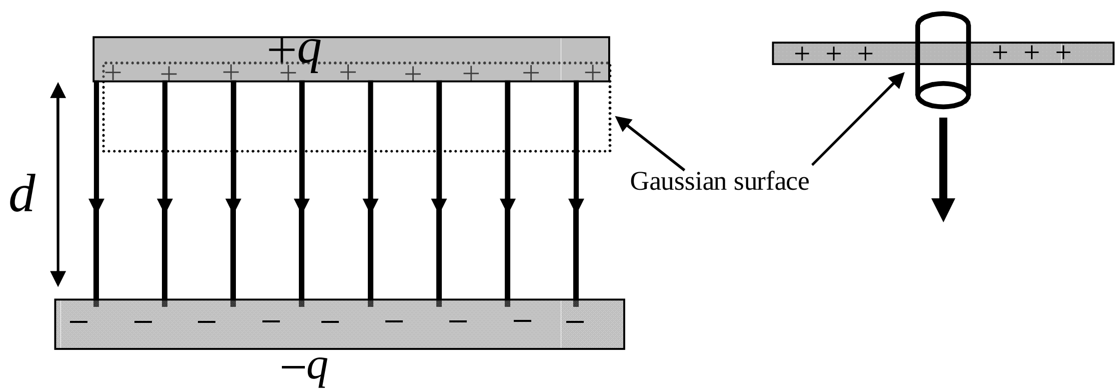
8. Now let us calculate the potential which comes from charges that are uniformly spread over a ring. This is the same problem as in the previous lecture, but simpler. Give the small amount of potential coming from the small amount of charge $dq = \lambda ds$ some name, $dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$. Then obviously $V = \int dV = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{R^2 + z^2}}$.



Summary of Lecture 25 – CAPACITORS AND CURRENTS

1. Two conductors isolated from one another and from their surroundings, form a *capacitor*. These conductors may be of any shape and size, and at any distance from each other. If a potential difference is created between the conductors (say, by connecting the terminals of a battery to them), then there is an electric field in the space between them. The electric field comes from the charges that have been pushed to the plates by the battery. The amount of charge pushed on to the conductors is proportional to the potential difference between the battery terminals (which is the same as between the capacitor plates). Hence, $Q \propto V$. To convert this into an equality, we write $Q = CV$. This provides the definition of capacitance, $C = \frac{Q}{V}$.

2. Using the above definition, let us calculate the capacitance of two parallel plates separated by a distance d as in the figure below.

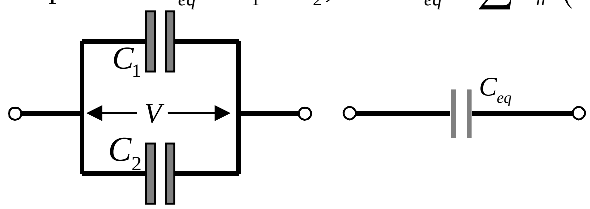


Recall Gauss's Law: $\Phi \equiv \int \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_0}$. Draw any Gaussian surface. Since the electric field is zero above the top plate, the flux through the area A of the plate is $\Phi = EA = \frac{Q}{\varepsilon_0}$, where Q is the total charge on the plate. Thus, $E = \frac{Q}{\varepsilon_0 A}$ is the electric field in the gap between the plates. The potential difference is $V = \frac{E}{d}$, and so $C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}$. You can see that the capacitance will be large if the plates are close to each other, and if the plates have a large area. We have simplified the calculation here by assuming that the electric field is strictly directed downwards. This is only true if the plates are infinitely long. But we can usually neglect the side effects. Note that any arrangement with two plates forms a capacitor: plane, cylindrical, spherical, etc. The capacitance depends upon the geometry, the size of plates and the gap between them.

3. One can take two (or more) capacitors in various ways and thus change the amount of charge they can contain. Consider first two capacitors connected in parallel with each other. The same voltage exists across both. For each capacitor, $q_1 = C_1V,\ q_2 = C_2V$ where V is the potential between terminals *a* and *b.* The total charge is:

$$Q = q_1 + q_2 = C_1V + C_2V = (C_1 + C_2)V$$

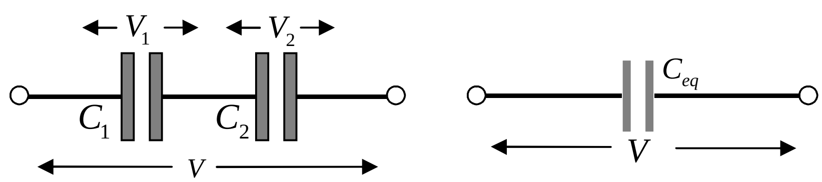
Now, let us define an "effective" or "equivalent" capacitance as $C_{eq} = \frac{Q}{V}$. Then we can immediately see that for 2 capacitors $C_{eq} = C_1 + C_2$, and $C_{eq} = \sum C_n$ $\left(\text{for } n \text{ capacitors}\right)$.



4. We can repeat the analysis above when the capacitors are put in series. Here the difference is that now we must start with $V = V_1 + V_2$, where V_1 and V_2 are the voltages across the two. Clearly the same charge had to cross both the capacitors. Hence,

$$V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2} = Q(\frac{1}{C_1} + \frac{1}{C_2}).$$

From our definition, $C_{eq} = \frac{Q}{V}$, it follows that $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$. The total capacitance is now less than if they were in parallel. In general, $\frac{1}{C_{eq}} = \sum \frac{1}{C_n}$ $\left(\text{for } n \text{ capacitors}\right)$.



5. When a battery is connected to a capacitor, positive and negative charges appear on the opposite plates. Some energy has been transferred from the battery to the capacitor, and now been stored in it. When the capacitor is discharged, the energy is recovered. Now let us calculate the energy required to charge a capacitor from zero to V volts.

Begin: the amount of energy required to transfer a small charge dq to the plates is $dU = \text{v}dq,$ where v is the voltage at a time when the charge is $q = C\text{v}$. As time goes on, the total charge increases until it reaches the final charge Q (at which point the voltage becomes V). So,

$$dU = \mathrm{v}dq = \frac{q}{C}dq \quad \Rightarrow \quad U = \int dU = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{2C} = \frac{1}{2}CV^2.$$

But where in the capacitor is the energy stored? Answer, it is present in the electric field in the volume between the two plates. We can calculate the energy density:

$$u = \frac{\text{energy stored in capacitor}}{\text{volume of capacitor}} = \frac{U}{Ad} = \frac{\frac{1}{2}CV^2}{Ad} = \frac{\varepsilon_0}{2}\left(\frac{V}{d}\right)^2 = \frac{1}{2}\varepsilon_0 E^2.$$

In the above we have used $C = \frac{\varepsilon_0 A}{d}$, derived earlier. The important result here is that $u \propto E^2$. Turning it around, wherever there is an electric field, there is energy available.

6. **Dielectrics.** Consider a free charge $+Q$. Around it is an electric field, $E = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{Q}{r^2}$.

Now suppose this charge is placed among water molecules. These molecules will polarise, i.e. the centre of positive charge and centre of negative charge will be slightly displaced. The negative part of the water molecule will be attracted toward the positive charge $+Q$. So, in effect, the electric field is weakened by $\frac{1}{\varepsilon_r}$ and becomes, $\frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{Q}{r^2}$. Here I have introduced a new quantity ε_r called "dielectric constant". This is a number that is usually bigger than one and measures the strength of the polarization induced in the material. For air, $\varepsilon_r = 1.0003$ while $\varepsilon_r \approx 80$ for pure water. The effect of a dielectric is to increase the capacitance of a capacitor: if the air between the plates of a capacitor is replaced by a dielectric, $C = \frac{\varepsilon_0 A}{d} \rightarrow \varepsilon_r \frac{\varepsilon_0 A}{d}$.

Summary of Lecture 26 – ELECTRIC POTENTIAL ENERGY

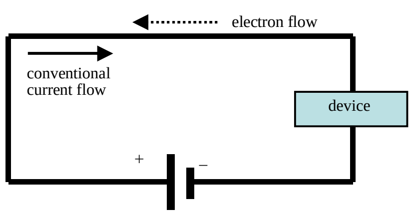
1. Electric current is the flow of electrical charge. If a small amount of charge dq flows in time dt, then the current is $i = \frac{dq}{dt}$. If the current is constant in time, then in time t, the current that flows is $q = i \times t$. The unit of charge is ampere, which is define as:

$$1 \text{ ampere} = \frac{1 \text{ coulomb}}{\text{second}}$$

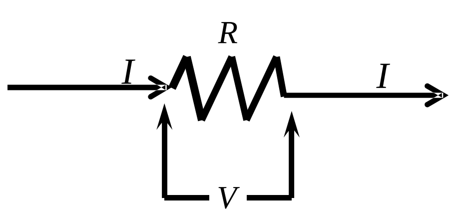
A car's battery supplies upto 50 amperes when starting the car, but often we need to deal with smaller values:

$$1 \text{ milliampere} = 1 \text{ ma} = 10^{-3} A$$
$$1 \text{ microampere} = 1\ \mu A = 10^{-6} A$$
$$1 \text{ nanoampere} = 1 \text{ n}A = 10^{-9} A$$
$$1 \text{ picoampere} = 1 \text{ p}A = 10^{-12} A$$

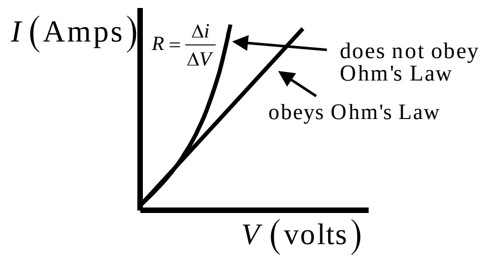
2. The direction of current flow is the direction in which positive charges move. However, in a typical wire, the positive charges are fixed to the atoms and it is really the negative charges (electrons) that move. In that case the direction of current flow is reversed.



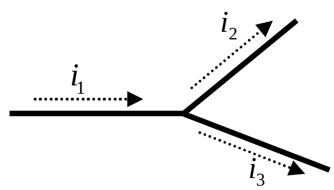
3. Current flows because something forces it around a circuit. That "something" is EMF, electromotive force. But remember that we are using bad terminology and that EMF is not a force - it is actually the difference in electric potentials between two parts of a circuit. So, in the figure below, $V = V_a - V_b$ is the EMF which causes current to flow in the resistor. How much current? Generally, the larger V is , the more current will flow and we expect $I \propto V$. In general this relation will not be completely accurate but when it holds, we say Ohm's Law applies: $I = \frac{V}{R}$. Here, $R = \frac{V}{I}$ is called the resistance.



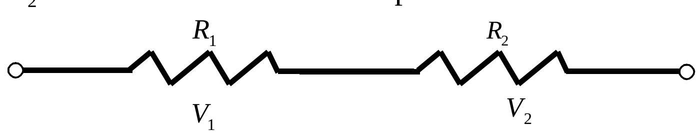
4. Be careful in understanding Ohm's Law. In general the current may depend upon the applied voltage in a complicated way. Another way of saying this is that the resistance may depend upon the current. Example: when current passes through a resistor, it gets hot and its resistance increases. Only when the graph of current versus voltage is a straight line does Ohm's Law hold. Else, we can only define the "incremental resistance".



5. Charge is always conserved, and therefore current is conserved as well. This means that when a current splits into two currents the sum remains constant, $i_1 = i_2 + i_3$.



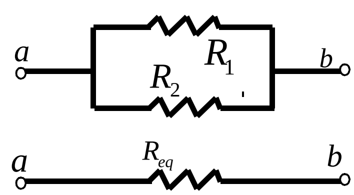
6. When resistors are put in series with each other, the same current flows through both. So, $V_1 = iR_1$ and $V_2 = iR_2$. The total potential drop across the pair is $V = V_1 + V_2 = i(R_1 + R_2)$. $\Rightarrow R_{eq} = R_1 + R_2$. So resistors in series add up.



7. Resistors can also be put in parallel. This means that the same voltage V is across both. So the currents are $i_1 = \dfrac{V}{R_1},\ i_2 = \dfrac{V}{R_2}$.

Since $i = i_1 + i_2$ it follows that $i = \dfrac{V}{R_{eq}} = \dfrac{V}{R_1} + \dfrac{V}{R_2}$

$$\Rightarrow \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \quad \text{or} \quad R_{eq} = \frac{R_1 R_2}{R_1 + R_2}.$$



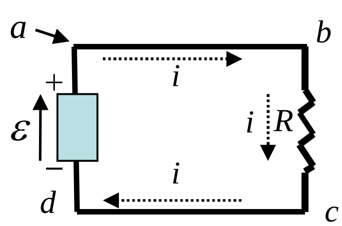
This makes sense: with two possible paths the current will find less resistance than if only one was present.

8. When current flows in a circuit work is done. Suppose a small amount of charge dq is moved through a potential difference V. Then the work done is $dW = Vdq = V\,idt$. Hence

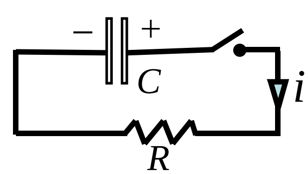
$Vidt = i^2 Rdt$ (because $i = \frac{V}{R}$). The rate of doing work, i.e. power, is $P = \frac{dW}{dt} = i^2 R.$

This is an important formula. It can also be written as $P = \frac{V^2}{R}$, or as $P = iV$. The unit of power is: $1 \text{ volt-ampere} = 1\frac{\text{joule}}{\text{coulomb}} \cdot \frac{\text{coulomb}}{\text{second}} = 1\frac{\text{joule}}{\text{second}} = 1 \text{ watt.}$

9. Kirchoff's Law: The sum of the potential differences encountered in moving around a closed circuit is zero. This law is easy to prove: since the electric field is conservative, therefore no work is done in taking a charge all around a circuit and putting it back where it was. However, it is very useful in solving problems. As a trivial example, consider the circuit below. The statement that, starting from any point a we get back to the same potential after going around is: $V_a - iR + \mathcal{E} = V_a$. This says $-iR + \mathcal{E} = 0.$



10. We can apply Kirchoff's Law to a circuit that consists of a resistor and capacitor in order to see how current flows through it. Since $q = VC$, we can see that $\frac{q}{C} + iR = 0$. Now differentiate with respect to time to find $\frac{1}{C}\frac{dq}{dt} + \frac{di}{dt}R = 0,$ or $\frac{di}{dt} = -\frac{1}{RC}i$. This equation has solution: $i = i_0 e^{-\frac{t}{RC}}$. The product RC is called the time constant τ, and it gives the time by which the current has fallen to $1/e \approx 1/2.7$ of the initial value.

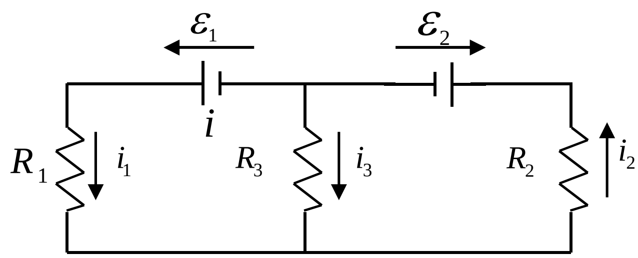


A reminder about the exponential function, $e^x \equiv 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

From this, $\frac{d}{dx}e^x = 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots = e^x$ Similarly, $\frac{d}{dx}e^{-x} = -e^{-x}$

11. Circuits often have two or more loops. To find the voltages and currents in such situations, it is best to apply Kirchoff's Law. In the figure below, you see that there are 3 loops and you can see that:

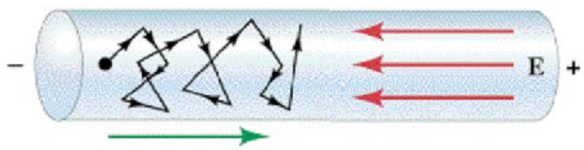
$$i_1 + i_3 = i_2$$
$$\mathcal{E}_1 - i_1 R_1 + i_3 R_3 = 0$$
$$-i_3 R_3 - i_2 R_2 + \mathcal{E}_2 = 0$$



Make sure that you understand each of these, and then check that the solution is:

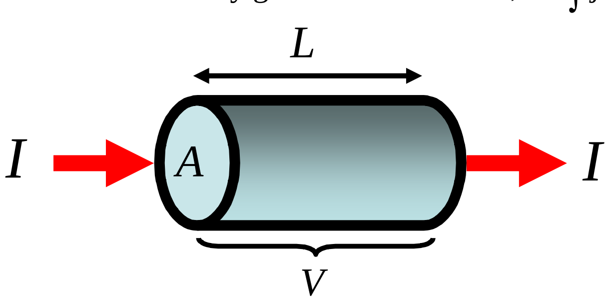
$$i_1 = \frac{\mathcal{E}_1 (R_2 + R_3) - \mathcal{E}_2 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} \qquad i_2 = \frac{\mathcal{E}_1 R_3 - \mathcal{E}_2 (R_1 + R_3)}{R_1 R_2 + R_2 R_3 + R_1 R_3} \qquad i_3 = \frac{-\mathcal{E}_1 R_2 - \mathcal{E}_2 R_1}{R_1 R_2 + R_2 R_3 + R_1 R_3}$$

12. A charge inside a wire moves under the influence of the applied electric field and suffers many collisions that cause it to move on a highly irregular, jagged path as shown below.

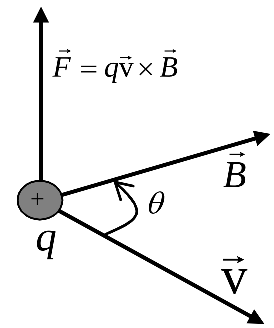


Nevertheless, it moves on the average to the right at the "drift velocity" (or speed).

13. Consider a wire through which charge is flowing. Suppose that the number of charges per unit volume is n. If we multiply n by the crossectional area of the wire A and the length L, then the charge in this section of the wire is $q = (nAL)e$. If the drift velocity of the charges is $\mathrm{v_d}$, then the time taken for the charge to move through the wire is $t = \frac{L}{\mathrm{v_d}}$. hence the current is $i = \frac{q}{t} = \frac{nALe}{L/\mathrm{v_d}} = nAe\mathrm{v_d}$. From this we can calculate the drift velocity of the charges in terms of the measured current, $\mathrm{v_d} = \frac{i}{nAe}$. The current density, which is the current per unit crossectional area is defined as $j = \frac{i}{A} = ne\mathrm{v_d}$. If j varies inside a volume, then we can easily generalize and write, $i = \int \vec{j} \cdot d\vec{A}$.



Summary of Lecture 27 – THE MAGNETIC FIELD



1. The magnetic field exerts a force upon any charge that moves in the field. The greater the size of the charge, and the faster it moves, the larger the force. The direction of the force is perpendicular to both the direction of motion and the magnetic field. If θ is the angle between $\vec{v}$ and $\vec{B}$, then $F = qvB\sin\theta$ is the magnitude of the force. This vanishes when $\vec{v}$ and $\vec{B}$ are parallel ($\theta = 0$), and is maximum when they are perpendicular.

2. The unit of magnetic field that is used most commonly is the *tesla*. A charge of one coulomb moving at 1 metre per second perpendicularly to a field of one tesla experiences a force of 1 newton. Equivalently,

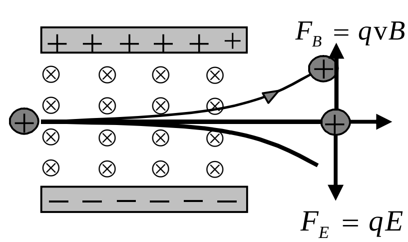
$$1 \text{ tesla} = 1 \frac{\text{newton}}{\text{coulomb} \cdot \text{meter/second}} = 1 \frac{\text{newton}}{\text{ampere} \cdot \text{meter}} = 10^4 \text{ gauss (CGS unit)}$$

In order to have an appreciation for how much a tesla is, here are some typical values of the magnetic field in these units:

Earth's surface	10^{-4} T
Bar magnet	10^{-2} T
Powerful electromagnet	1 T
Superconducting magnet	5 T

3. When both magnetic and electric fields are present at a point, the total force acting upon a charge is the vector sum of the electric and magnetic forces, $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$. This is known as the *Lorentz Force.* Note that the electric force and magnetic force are very different. The electric force is non-zero even if the charge is stationary, and it is in the same direction as $\vec{E}$.

4. The Lorentz Force can be used to select charged particles of whichever velocity we want. In the diagram below, particles enter from the left with velocity v. They experience a force due to the perpendicular magnetic field, as well as force downwards because of an electric field. Only particles with speed $v = E/B$ are undeflected and keep going straight.



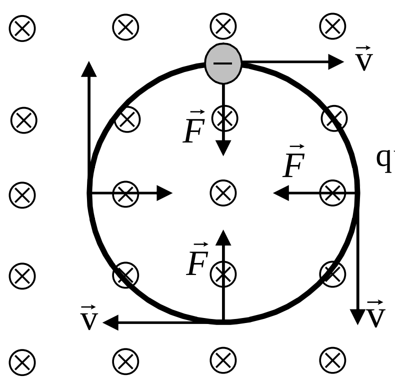
$qE = qvB \Rightarrow v = \frac{E}{B}$ velocity selector !!

5. A magnetic field can be strong enough to lift an elephant, but it can never increase or decrease the energy of a particle. Proof: suppose the magnetic force $\vec{F}$ moves a particle through a displacement $d\vec{r}$. Then the small amount of work done is,

$$\begin{aligned} dW = \vec{F}\cdot d\vec{r} &= q(\vec{\mathrm{v}}\times\vec{\mathrm{B}})\cdot d\vec{r} \\ &= q(\vec{\mathrm{v}}\times\vec{\mathrm{B}})\cdot\frac{d\vec{r}}{dt}dt \\ &= q(\vec{\mathrm{v}}\times\vec{\mathrm{B}})\cdot\vec{\mathrm{v}}\,dt = 0. \end{aligned}$$

   Basically the force and direction of force are orthogonal, and hence there can be no work done on the particle or an increase in its energy.

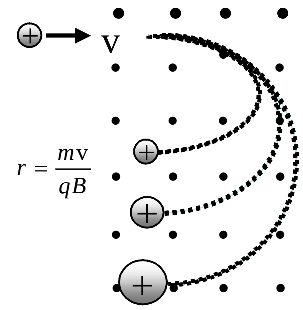
6. A magnetic field bends a charged particle into a circular orbit because the particle feels a force that is directed perpendicular to the magnetic field. As we saw above, the particle cannot change its speed, but it certainly does change direction! So it keeps bending and bending until it makes a full circle. The radius of orbit can be easily calculated: the magnetic and centrifugal forces must balance each other for equilibrium. So, $\mathrm{qv}B = m\frac{\mathrm{v}^2}{r}$ and we find that $r = \frac{m\mathrm{v}}{qB}$. A strong B forces the particle into a tighter orbit, as you can see. We can also calculate the angular frequency, $\omega = \frac{\mathrm{v}}{r} = \frac{qB}{m}$. This shows that a strong B makes the particle go around many times in unit time. There are a very large number of applications of these facts.



$$\mathrm{qv}B = m\frac{\mathrm{v}^2}{r} \Rightarrow r = \frac{m\mathrm{v}}{qB}.$$

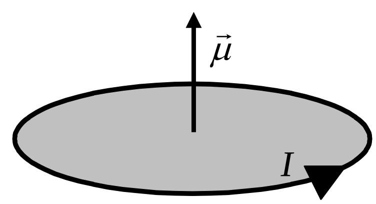
7. The fact that a magnetic field bends charged particles is responsible for shielding the earth from harmful effects of the "solar wind". A large number of charged particles are released from the sun and reach the earth. These can destroy life. Fortunately the earth's magnetic field deflects these particles, which are then trapped in the "Van Allen" belt around the earth.

8. The mass spectrometer is an extremely important equipment that works on the above principle. Ions are made from atoms by stripping away one electron. Then they pass through a velocity selector so that they all have the same speed. In a beam of many different ions, the heavier ones bend less, and lighter ones more, when they are passed through a B field.

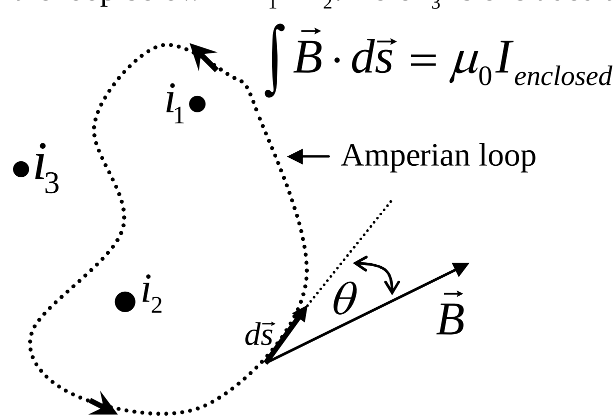


9. A wire carries current, and current is flowing charges. Since each charge experiences a force when placed in a magnetic field, you might expect the same for the current. Indeed, that is exactly the case, and we can easily calculate the force on a wire from the force on individual charges. Suppose N is the total number of charges and they are moving at the average (or drift) velocity $\vec{\text{v}}_\text{d}$. Then the total force is $\vec{F} = Ne\vec{\text{v}}_d \times \vec{B}$. Now suppose that the wire has length $L,$ crossectional area $A,$ and it has n charges per unit volume. Then clearly $N = nAL,$ and so $\vec{F} = nALe\vec{\text{v}}_\text{d} \times \vec{B}$. Remember that the current is the charge that flows through the wire per unit time, and so $nAe\vec{\text{v}}_\text{d} = \vec{I}$. We get the important result that the force per unit length on the wire is $\vec{F} = \vec{I} \times \vec{B}$.

10. A current that goes around a loop (any shape) produces a magnetic field. We define the *magnetic moment* as the product of current and area, $\vec{\mu} = IA\hat{z}$. Here A is the area of the loop and I the current flowing around it. The direction is perpendicular to the plane of the loop, as shown.



11. Magnetic fields are produced by currents. Every small bit of current produces a small amount of the B field. Ampere's Law, illustrated below, says that if one goes around a loop (of any shape or size) then the integral of the B field around the loop is equal to the enclosed current. In the loop below $I = I_1 + I_2$. Here I_3 is excluded as it lies outside.

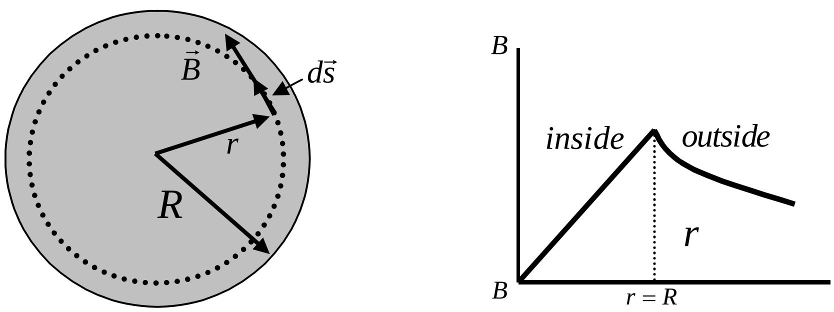


12. Let us apply Ampere's Law to a circular loop of radius r outside an infinitely long wire carrying current I through it. The magnetic field goes around in circles, and so $\vec{B}$ and $d\vec{s}$ are both in the same direction. Hence, $\int \vec{B}\cdot d\vec{s} = B\int ds = B(2\pi r) = \mu_0 I$. We get the important result that $B = \dfrac{\mu_0 I}{2\pi r}$.

13. Assuming that the current flows uniformly over the crossection, we can use Ampere's Law to calculate the magnetic field at distance r, where r now lies inside the wire.

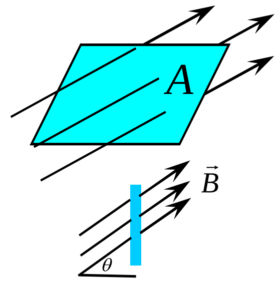
$$B(2\pi r) = \mu_0 I\left(\frac{\pi r^2}{\pi R^2}\right) \Rightarrow B = \frac{\mu_0 I r}{2\pi R^2}$$

Here is a sketch of the B field inside and outside the wire as a function of distance r.

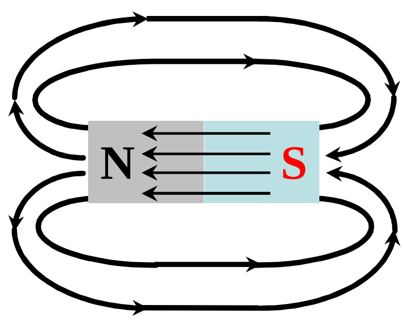


Summary of Lecture 28 – ELECTROMAGNETIC INDUCTION

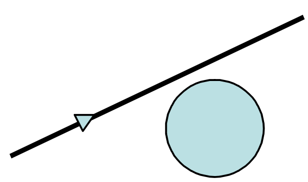
1. Earlier we had defined the flux of any vector field. For a magnetic field, this means that flux of a uniform magnetic field (see figure) is $\Phi = B_{\perp} A = BA\cos\theta$. If the field is not constant over the area then we must add up all the little pieces of flux: $\Phi_B = \int \vec{B}\cdot d\vec{A}$. The dimension of flux is magnetic field $\times$ area, and the unit is called weber, where $1\text{ weber} = 1\text{ tesla}\cdot\text{metre}^2$.



2. A fundamental law of magnetism states that the net flux through a closed surface is always zero, $\Phi_B = \int \vec{B}\cdot d\vec{A} = 0$. Note that this is very different from what you learned earlier in electrostatics where the flux is essentially the electric charge. There is no such thing as a magnetic charge! What we call the magnetic north (or south) pole of a magnet are actually due to the particular electronic currents, not magnetic charges. In the bar magnet below, no matter which closed surface you draw, the amount of flux leaving the surface is equal to that entering it.



Example : A sphere of radius R is placed near a long, straight wire that carries a steady current I. The magnetic field generated by the current is B. Find the total magnetic flux passing through the sphere.



Answer: zero, of course!

3. Faraday's Law for Induced EMF: when the magnetic flux changes in a circuit, an electro-motive force is induced which is proportional to the rate of change of flux. Mathematically, $\varepsilon = -\frac{d\Phi_B}{dt}$ where ε is the induced emf. If the coil consists of N turns, then $\varepsilon = -N\frac{d\Phi_B}{dt}$.
How does the flux through a coil change? Consider a coil and magnet. We can:
    a) move the magnet,
    b) change the size and shape of the coil by squeezing it,
    c) move the coil.

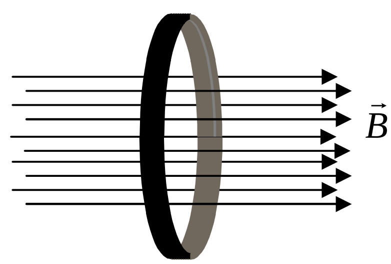
In all cases, the flux through the coil changes and $\frac{d\Phi_B}{dt}$ is non-zero leading to an induced emf.

Example: A flexible loop has a radius of 12cm and is in a magnetic field of strength 0.15T. The loop is grasped at points A and B and stretched until it closes. If it takes 0.20s to close the loop, find the magnitude of the average induced emf in it during this time.

Solution: Here the loop area changes, hence the flux. So the induced emf is:

$$\varepsilon = -\frac{d\Phi_B}{dt} \approx -\left[\frac{\text{final flux} - \text{initial flux}}{\text{time taken}}\right] = -\left[\frac{0 - \pi(0.12)^2 \times 0.15}{0.2}\right] = 0.034 \text{ Volts.}$$

Example : A wire loop of radius 0.30m lies so that an external magnetic field of +0.30T is perpendicular to the loop. The field changes to -0.20T in 1.5s. Find the magnitude of the average induced EMF in the loop during this time.



Solution: Again, we will find the initial and final fluxes first and then divide by the time taken for the change. Use $\Phi = BA = B\pi r^2$ to calculate the flux.

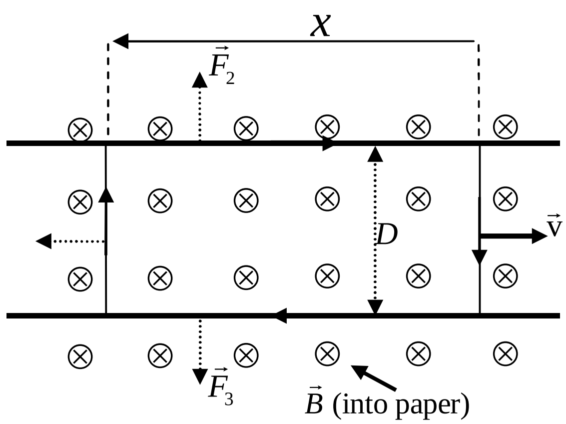
$$\Phi_i = 0.30 \times \pi \times (0.30)^2 = 0.085 \text{ Tm}^2$$

$$\Phi_f = -0.20 \times \pi \times (0.30)^2 = -0.057 \text{ Tm}^2$$

$$\varepsilon = \frac{\Delta\Phi}{\Delta t} = \frac{\Phi_f - \Phi_i}{\Delta t} = \frac{0.085 - 0.057}{1.5} = 0.095 \text{ V}$$

4. Remember that the electromotive force is not really a force but the difference in electric potentials between two points. In going around a circular wire, where the electric field is constant as a function of angle, the emf is $\varepsilon = E\left(2\pi r\right)$. More generally, for any size or shape of a closed circuit, $\varepsilon = \int \vec{E}\cdot d\vec{s}$. So Faraday's Law reads: $\int \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt}$.

Example : A conducting wire rests upon two parallel rails and is pulled towards the right with speed v. A magnetic field B is perpendicular to the plain of the rails as shown below. Find the current that flows in the circuit.



Solution: As the wire is pulled to the right, the area of the circuit increases and so the flux increases. Measure x as above so that $x = \text{v}t$, i.e. x keeps increasing as we pull. The flux at any value of x is $\Phi_B = BDx$, and so,

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{d\left(BDx\right)}{dt} = -BD\frac{dx}{dt} = BD\text{v}.$$

To calculate the current, we simply use Ohm's Law: $I = \frac{\varepsilon}{R} = \frac{BD\text{v}}{R}$. Here R is the resistance of the circuit.

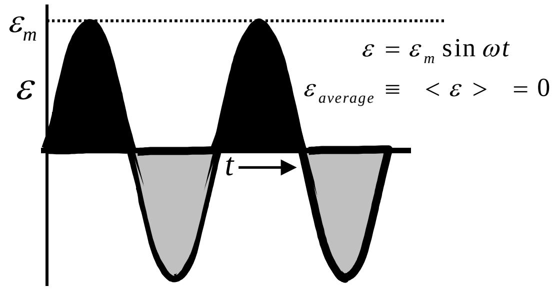
Example : Find the power dissipated in the above circuit using I^2R, and then by directly calculating the work you do by pulling the wire.

Solution: Clearly $I^2R = \frac{B^2D^2\text{v}^2}{R}$ is the power dissipated, as per usual formula. Now let us calculate the force acting upon the piece of wire (of length D) that you are pulling. From the formula for the force on a wire, $\vec{F} = I\vec{L}\times\vec{B}$, the magnitude is $F = IBD = \frac{B^2D^2\text{v}}{R}$. So, the power is $P = F\text{v} = \frac{B^2D^2\text{v}^2}{R}$. This is exactly the value calculated above!

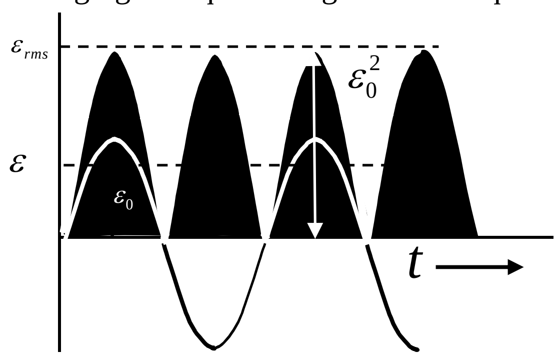
5. **Lenz's Law :** The direction of any magnetic induction effect is such as to oppose the cause of the effect. Imagine a coil wound with wire of finite resistance. If the magnetic field decreases, the induced EMF is positive. This produces a positive current. The magnetic field produced by the current opposes the decrease in flux. Of course, because of finite resistance in loop, the induced current cannot completely oppose the change in flux.

Summary of Lecture 29 – ALTERNATING CURRENT

1. Alternating current (AC) is current that flows first in one direction along a wire, and then in the reverse direction. The most common AC is sinuisoidal in which the current (and voltage) follow a sine function, as in the graph below. The average value is zero because the current flows for the same time in one direction as in the other.



However, the square of any AC wave is always positive. Thus its average is not zero. As we shall see, upon averaging the square we get half the square of the peak value.



The calculation follows: if there is a sine wave of amplitude (height) equal to one and frequency ω, ($\omega = 2\pi / T$, T=time period), then the squared amplitude is $\sin^2 \omega t$ and its average is:

$$< \sin^2 \omega t > \equiv \frac{1}{T}\int_0^T dt \sin^2 \omega t = \frac{1}{T}\int_0^T dt (\frac{1-\cos 2\omega t}{2}) = \frac{1}{T}\int_0^T dt \frac{1}{2} = \frac{1}{2}$$

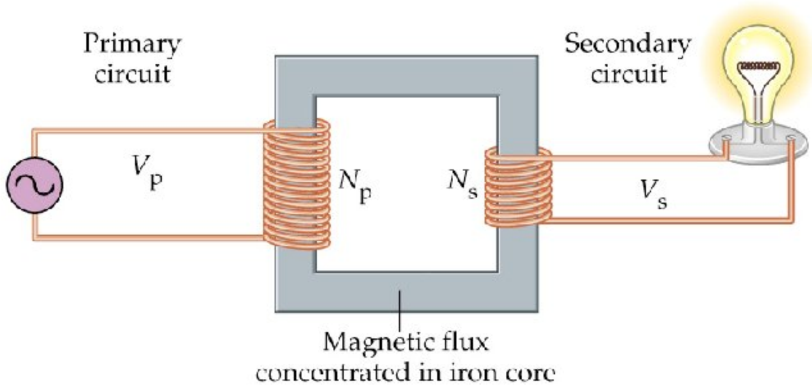
Taking the square root gives the root mean square value as $1/\sqrt{2}$ of the maximum value. Of course, it does not matter whether this is of the voltage or current:

$$\varepsilon_{rms} = \sqrt{\frac{\varepsilon_m^2}{2}} = \frac{\varepsilon_m}{\sqrt{2}} = 0.707\varepsilon_m, \text{ and } I_{rms} = \sqrt{\frac{I_m^2}{2}} = \frac{I_m}{\sqrt{2}} = 0.707 I_m.$$

Exactly the same results are obtained for cosine waves. This is what one expects since the difference between sine and cosine is only that one starts earlier than the other.

2. AC is generated by a coil rotating in a magnetic field. We know from Faraday's Law, $\varepsilon = -\frac{d\Phi_B}{dt}$, that a changing magnetic flux gives rise to an emf. Imagine a magnetic field and a coil of area *A*, rotating with frequency ω so that the flux through the coil at any instant of time is $\Phi_B = BA\cos\omega t$. Then the induced emf is $\varepsilon = BA\omega\sin\omega t$.

3. AC is particular useful because *transformers* make it possible to step up or step down voltages. The basic transformer consists of two coils - the primary and secondary - both wrapped around a core (typically iron) that enhances the magnetic field.



Suppose that the flux in the core is Φ and that its rate of change is $\frac{\partial\Phi}{\partial t}$. Call the number of turns in the primary and secondary N_p and N_s respectively. Then, from Faraday's law, the primary emf is $\varepsilon_p = -N_p\frac{\partial\Phi}{\partial t}$ and the secondary emf is $\varepsilon_s = -N_s\frac{\partial\Phi}{\partial t}$. The ratio is, $\frac{\varepsilon_p}{\varepsilon_s} = \frac{N_p}{N_s}$. So, the secondary emf is $\varepsilon_s = \frac{N_s}{N_p}\varepsilon_p$. If $\frac{N_s}{N_p}$ is less than one, then it is called a step-down transformer because the secondary voltage is less than the input voltage. Else, it is a step-up transformer. Both types are used.

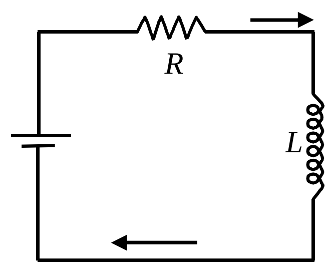
4. If this is a lossless transformer (and good transformers are 99% lossless), then the input power must equal the output power, $\varepsilon_p I_p = \varepsilon_s I_s$. From above, this shows that the ratio of currents is $I_s = \frac{N_p}{N_s}I_p$.

5. Whatever the shape or size of a current carrying loop, the magnetic flux that passes through it is proportional to the current, $\Phi \propto I$. The inductance L (called the self-inductance if

there is only one coil) is the constant of proportionality in the relation $\Phi = LI$. The unit of inductance is called Henry, 1 Henry= 1 Tesla metre^2 / Ampere. Note that inductance, like capacitance, is purely geometrical and depends only upon the shape and sizes of wires. It does not depend on the current.

6. Let us calculate the inductance of a long coil wound with n turns per unit length. As calculated earlier, the B field is $B = \mu_0 nI$, and the flux passing through N turns of the coil of length l and area A is $N\Phi_B = (nl)(BA) = \mu_0 n^2 IAl$. From the definition, we find:
   $L = \frac{N\Phi_B}{I} = \frac{\mu_0 n^2 lIA}{I} = \mu_0 n^2 lA$ (inductance of long solenoid).
   Example: Find the inductance of a coil with 3500 turns, length 10 cm, and radius 5cm.
   Solution: $L = 4\pi \cdot 10^{-7} \frac{T \cdot m}{A} \cdot 3500^2 \cdot \frac{\pi (0.05m)^2}{0.10m} = 1.21 \frac{T \cdot m^2}{A} = 1.21H.$

7. When the current changes through an inductor, by Faraday's Law it induces an emf equal to $-\frac{d\Phi_B}{dt} = -\frac{d(LI)}{dt} = -L\frac{dI}{dt}$. Let us use this to calculate the current through the circuit shown here, where a resistor is present. Then, by using Kirchoff's Law, $\varepsilon = IR + L\frac{dI}{dt}$ or $\varepsilon I = I^2 R + LI\frac{dI}{dt}$. In words: the power expended by the battery equals energy dissipated in the resistor + work done on inductor.



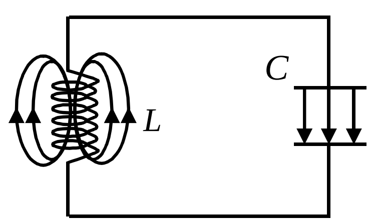
8. Before we go on to the AC case, suppose that we suddenly connect a battery to an inductor so that the voltage suddenly increases from zero to ε across the circuit. Then, $L\frac{dI}{dt} + IR = \varepsilon$ has solution: $I(t) = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right)$ where $\tau = \frac{L}{R}$. You can easily see that this true using $\frac{dI}{dt} = \frac{\varepsilon}{R}\frac{1}{\tau}e^{-t/\tau}$. The solution shows that the current increases from zero to a maximum of $\frac{\varepsilon}{R}$, i.e. that given by Ohm's Law. We need to pay a little attention to the "time constant" τ, which is the time after which the constant approaches 63% of its final value. Units: $[\tau] = \frac{[L]}{[R]} = \frac{\text{henry}}{\text{ohm}} = \frac{\text{volt.second/ampere}}{\text{ohm}} = \left(\frac{\text{volt}}{\text{ampere.ohm}}\right)\text{second} = \text{second}.$

Note that when t=τ, then $I = \frac{\varepsilon}{R}\left(1 - e^{-1}\right) = \frac{\varepsilon}{R}\left(1 - 0.37\right) = \frac{\varepsilon}{R}0.63$

9. When we pass current through an inductor a changing magnetic field is produced. This, by Faraday's Law, induces an emf across the coil. So work has to be done to force the current through. How much work? The power, or rate of doing work, is emf $\times$ current. Let U_B be the work done in passing current I. Then, $\frac{dU_B}{dt} = \left(L\frac{dI}{dt}\right)I = LI\frac{dI}{dt}$. Let us integrate $dU_B = LI\,dI$. Then, $\int_0^{U_B} dU_B = \int_0^I LIdI \quad \Rightarrow \; U_B = \frac{1}{2}LI^2$. This is an important result. It tells us that an inductor L carrying current I requires work $\frac{1}{2}LI^2$. By conservation of energy, this is also the energy stored in the inductor. Compare this result with the result $U_E = \frac{1}{2}CV^2$ for a capacitor. Notice that $C \leftrightarrow L$ and $V \leftrightarrow I$.

10. Let us use the result derived earlier for the inductance of a solenoid and the magnetic field in it, $L = \mu_0 n^2 lA$ and $B = \mu_0 nI$. Putting this into $U_B = \frac{1}{2}LI^2$ gives $U_B = \frac{1}{2}(\mu_0 n^2 lA)(B/\mu_0 n)^2$. Divide the energy by the volume of the solenoid, $\frac{U_B}{\text{volume}} = \frac{B^2}{2\mu_0}$. This directly gives the energy density (energy per unit volume) contained in a magnetic field.

11. **Electromagnetic oscillations in an LC circuit.**



We know that energy can be stored in a capacitor as well as in an inductor. What happens when we connect them up together and put some charge on the capacitor? As it discharges, it creates a current that transfers energy to the inductor. The total energy remains constant, of course.

This means that the sum $U = U_B + U_E \; = \frac{1}{2}LI^2 + \frac{1}{2}\frac{q^2}{C}$ is constant. Differentiate this:

$0 = \frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}LI^2 + \frac{1}{2}\frac{q^2}{C}\right)$. Since $I = \frac{dq}{dt}$, and $\frac{dI}{dt} = \frac{d^2q}{dt^2}$, $\therefore \quad \frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$

To make this look nicer, put $\omega^2 \equiv \frac{1}{LC} \Rightarrow \frac{d^2q}{dt^2} + \omega^2 q = 0$. We have seen this equation many times earlier. The solution is: $q = q_m \cos\omega t$. The important result here is that the charge, current, and voltage will oscillate with frequency $\omega = \sqrt{\dfrac{1}{LC}}$. This oscillation will go on for forever if there is no resistance in the circuit.

Summary of Lecture 30 – ELECTROMAGNETIC WAVES

1. Before the investigations of James Clerk Maxwell around 1865, the known laws of electromagnetism were:

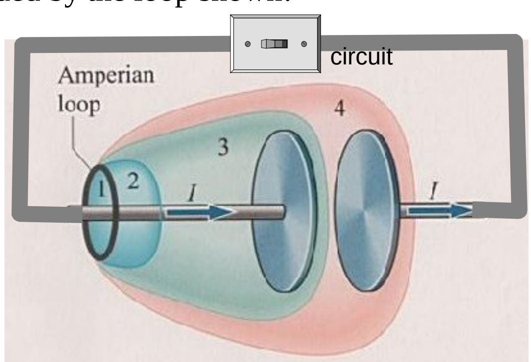
   a) Gauss' law of electricity: $\int \vec{E}\cdot d\vec{A} = \frac{q}{\varepsilon_0}$ (integral is over any closed surface)

   b) Gauss' law of magnetism: $\int \vec{B}\cdot d\vec{A} = 0$ (integral is over any closed surface)

   c) Faraday's law of induction: $\int \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt}$ (integral is over any closed loop)

   d) Ampere's law: $\int \vec{B}\cdot d\vec{s} = \mu_0 I$ (integral is over any closed loop)

2. But Maxwell realized that the above 4 laws were not consistent with the conservation of charge, which is a fundamental principle. He argued that if you take the space between two capacitors (see below) and take different surfaces 1,2,3,4 then applying Ampere's Law gives an inconsistency: $\left[\int \vec{B}\cdot d\vec{s}\right]_{(1,2,4)} \neq \left[\int \vec{B}\cdot d\vec{s}\right]_3$ because obviously charge cannot flow in the gap between plates. So Ampere's Law gives different results depending upon which surface is bounded by the loop shown!



Maxwell modified Ampere's law as follows: $\int \vec{B}\cdot d\vec{s} = \mu_0\left(I + I_d\right)$ where the "displacement current" is $I_d = \varepsilon_0 \frac{d\Phi_E}{dt}$. Let's look at the reasoning that led to Maxwell's discovery of the displacement current. The current that flows in the circuit is $I = \frac{dQ}{dt}$. But the charge on the capacitor plate is $Q = \varepsilon_0 EA$. Hence, $I = \frac{d}{dt}\left(\varepsilon_0 EA\right) = \varepsilon_0 \frac{d(EA)}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt} \equiv I_D$. In words, the changing electric field in the gap acts as source of the magnetic field in just the same way as the current in the outside wires. This is really the most important point - a magnetic field may have two separate reasons for existence - flowing charges or changing electric fields.

3. The famous Maxwell's equations are as follows:

a) $\int \vec{E}\cdot d\vec{S} = \dfrac{Q}{\varepsilon_0}$

b) $\int \vec{E}\cdot d\vec{\ell} = -\dfrac{d\Phi_B}{dt}$

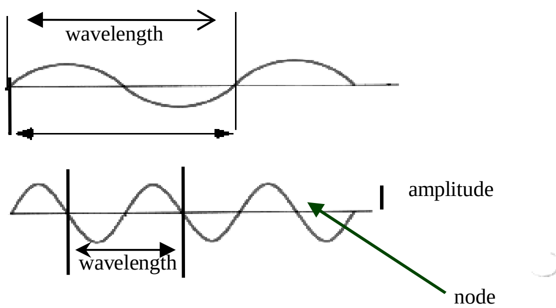
c) $\int \vec{B}\cdot d\vec{S} = 0$

d) $\int \vec{B}\cdot d\vec{\ell} = \mu_0\left(I + \varepsilon_0 \dfrac{d\Phi_E}{dt}\right)$

Together with the Lorentz Force $\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$ they provide a complete description of all electromagnetic phenomena, including waves.

4. Electromagnetic waves were predicted by Maxwell and experimentally discovered many years later by Hertz. Note that for these waves:
a) Absolutely no medium is required - they travel through vacuum.
b) The speed of propagation is c for all waves in the vacuum.
c) There is no limit to the amplitude or frequency.

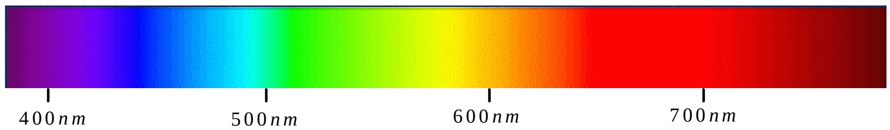
A wave is characterized by the amplitude and frequency, as illustrated below.



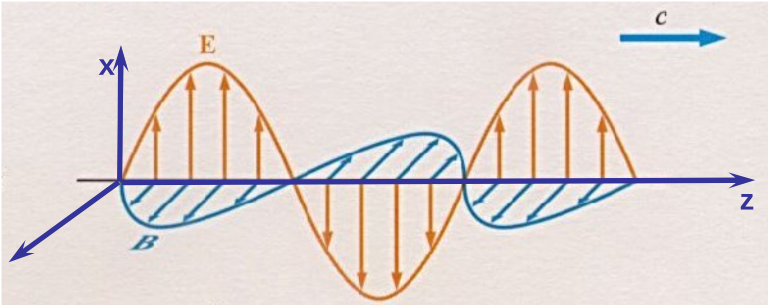
Example: Red light has λ = 700 nm. The frequency ν is calculated as follows:

$$\nu = \frac{3.0\times10^{8}\,m/\sec}{7\times10^{-7}\,m} = 4.29\times10^{14}\,Hertz$$

By comparison, the electromagnetic waves inside a microwave oven have wavelength of 6 cm, radio waves are a few metres long. For visible light, see below. On the other hand, X-rays and gamma-rays have wavelengths of the size of atoms and even much smaller.

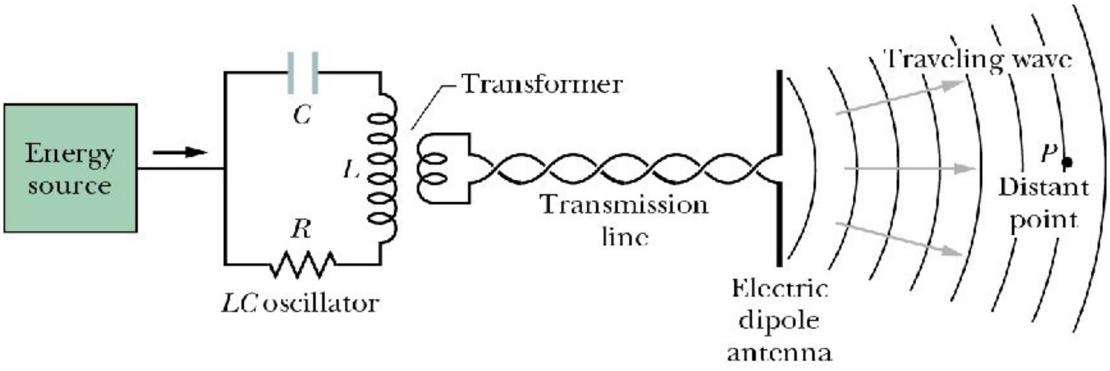


5. We now consider how electromagnetic waves can travel through empty space. Suppose an electric field (due to distant charges in an antenna) has been created. If this changes then this creates a changing electric flux $\frac{d\Phi_E}{dt}$ which, through $\int \vec{B}\cdot d\vec{s} = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$, creates a changing magnetic flux $\frac{d\Phi_B}{dt}$. This, through $\int \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt}$, creates a changing electric field. This chain of events in free space then allows a wave to propagate.



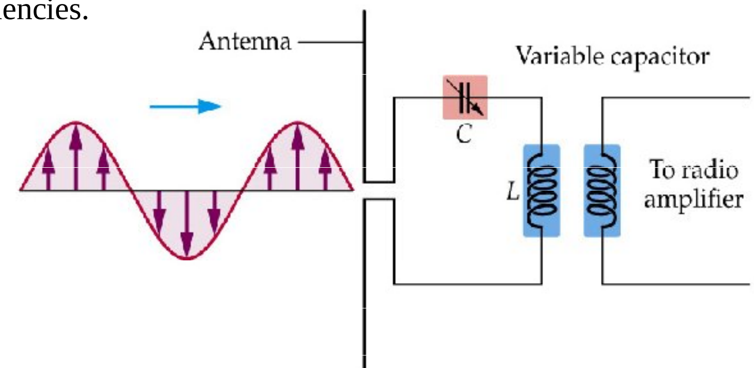
In the diagram above, an electromagnetic wave is moving in the z direction. The electric field is in the x direction, $E_x = E_0 \sin(kz - \omega t)$, and the magnetic field is perpendicular to it, $B_y = B_0 \sin(k z - \omega t)$. Here $\omega = kc$. From Maxwell's equations the amplitudes of the two fields are related by $E_0 = cB_0$. Note that the two fields are in phase with each other.

6. The production of electromagnetic waves is done by forcing current to vary rapidly in a small piece of wire. Consider your mobile phone, for example. Using the power from the battery, the circuits inside produce a high frequency current that goes into a "dipole antenna" made of two small pieces of conductor. The electric field between the two oppositely charged pieces is rapidly changing and so creates a magnetic field. Both fields propagate outwards, the amplitude falling as $1/r$.



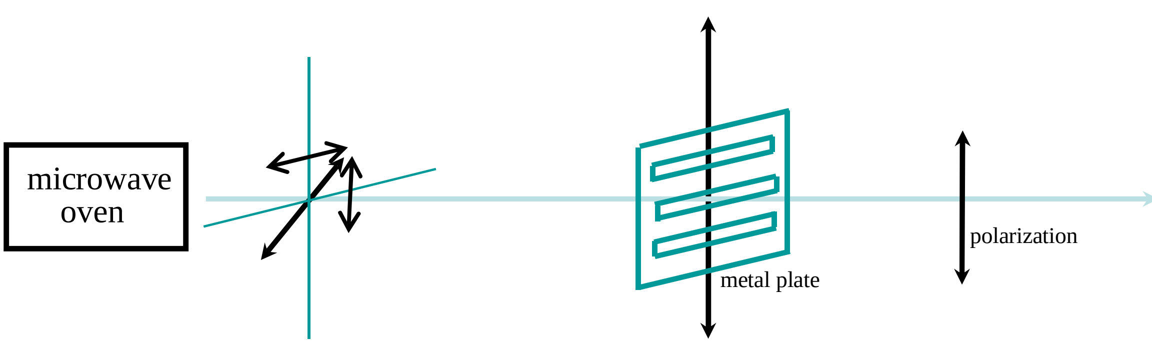
The power, which is the square of the amplitude, falls off as $I(\theta) \propto \frac{\sin^2\theta}{r^2}$. The $\sin^2\theta$ dependence shows that the power is radiated unequally as a function of direction. The maximum power is at $\theta = \pi/2$ and the least at $\theta = 0$.

7. The reception of electromagnetic waves requires an antenna. The incoming wave has an electric field that forces the electrons to run up and down the antenna wire, i.e. it produces a tiny electric current. This current is then amplified (increased in amplitude) electronically. This is schematically indicated below. Here the variable capacitor is used to tune to different frequencies.



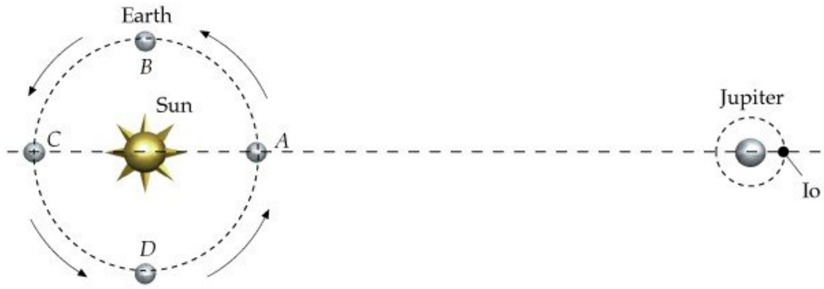
8. As we have seen, the electric field of a wave is perpendicular to the direction of its motion. If this is a fixed direction (say, $\hat{x}$), then we say that wave is polarized in the x direction. Most sources - a candle, the sun, any light bulb - produce light that is unpolarized. In this case, there is no definite direction of the electric field, no definite phase between the orthogonal components, and the atomic or molecular dipoles that emit the light are randomly oriented in the source. But for a typical linearly polarized plane electromagnetic wave polarized along $\hat{x}$, $E_x = E_0 \sin(kz - \omega t)$, $B_y = \dfrac{E_0}{c}\sin(kz - \omega t)$ with all other components zero. Of course, it may be that the wave is polarized at an angle θ relative to $\hat{x}$, in which case $E_x = E_0 \cos\theta \cdot \sin(kz - \omega t)$, $E_y = E_0 \sin\theta \cdot \sin(kz - \omega t)$, $E_z = 0$.

9. Electromagnetic waves from an unpolarized source (e.g. a burning candle or microwave oven) can be polarized by passing them through a simple polarizer of the kind below. A metal plate with slits cut into will allow only the electric field component perpendicular to the slits. Thus, it will produce linearly polarized waves from unpolarized ones.

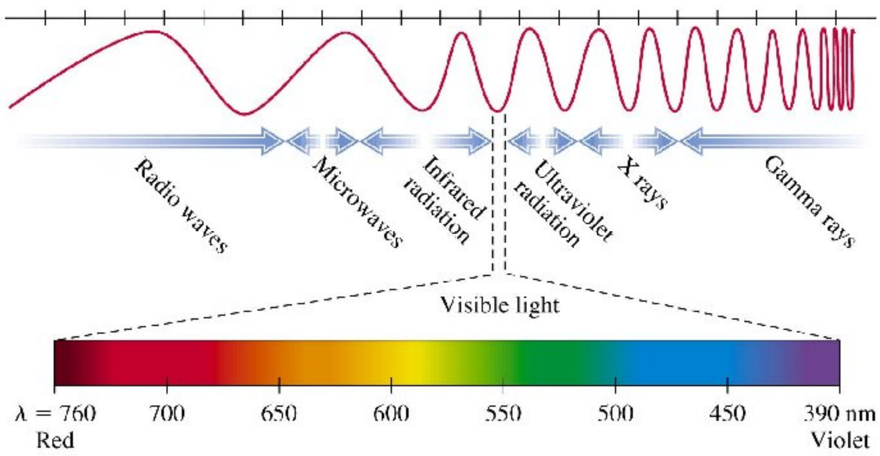


Summary of Lecture 31 – LIGHT

1. Light travels very fast but its speed is not infinite. Early attempts to measure the speed using earth based experiments failed. Then in 1675 the astronomer Roemer studied timing of the eclipse of one of Jupiter's moon called Io. In the diagram below Io is observed with Earth at A and then at C. The eclipse is 16.6 minutes late, which is the time taken for light to travel AC. Roemer estimated that $c = 3\times10^8$ metres/sec, a value that is remarkably close to the best modern measurement, $c = 299792458.6$ metres/sec.

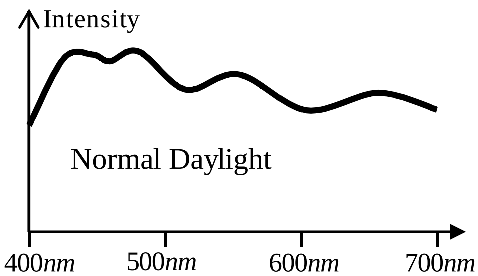


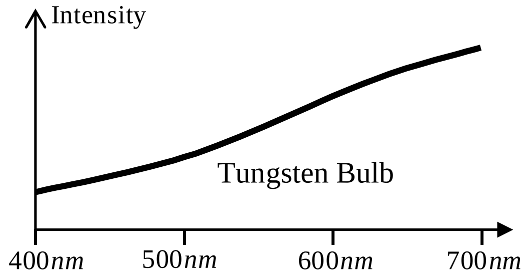
2. Light is electromagnetic waves. Different frequencies ν correspond to different colours. Equivalently, different wavelength λ correspond to different colours. Recall that the product $\nu\times\lambda = c$. In the diagram below you see that visible light is only one small part of the total electromagnetic spectrum. Here nm means nanometres or 10^{-9}metres.



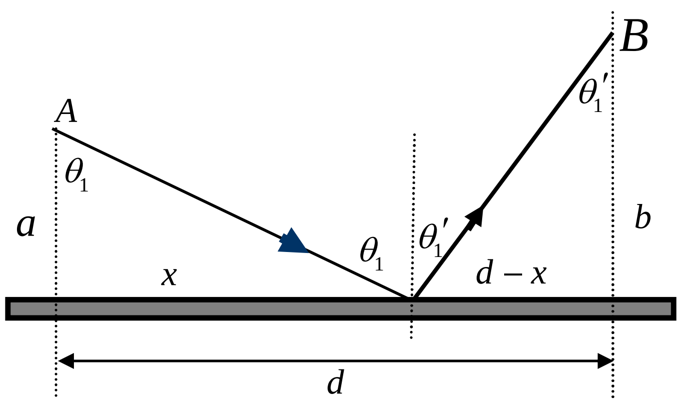
3. If light contained all frequencies with equal strength, it would appear as white to us. course, most things around us appear coloured. That is because they radiate more strongly in one range of frequency than in others. If there is more intensity in the yellow range than the green range, we will see mostly yellow. The sky appears blue to us on a clear day because tiny dust particles high above in the atmosphere reflect a lot of the blue light coming from the sun. In the figure below you can see the hump at smaller wavelengths.

Compare this with the spectrum of light emitted from a tungsten bulb. You can see that this is smoother and yellow dominates.





4. What path does light travel upon? If there is no obstruction, it obviously likes to travel on straight line which is the shortest path between any two points, say A and B. Fermat's Principle states that in all situations, light will always take that path for which it takes the least time. As an example, let us apply Fermat's Principle to the case of light reflected from a mirror, as below.



The total distance travelled by the ray is $L = \sqrt{a^2 + x^2} + \sqrt{b^2 + (d-x)^2}$. The time taken is $t = L/c$. To find the smallest time, we must differentiate and then set the derivative to zero,

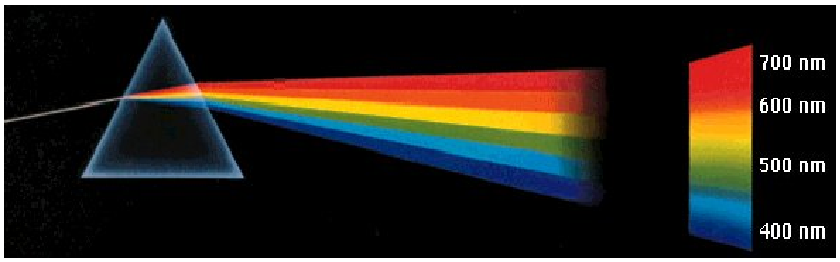
$$0 = \frac{dt}{dx} = \frac{1}{c}\frac{dL}{dx}$$

$$= \frac{1}{2c}\left(a^2 + x^2\right)^{-1/2}(2x) + \frac{1}{2c}\left[b^2 + (d-x)^2\right]^{1/2}(2)(d-x)(-1)$$

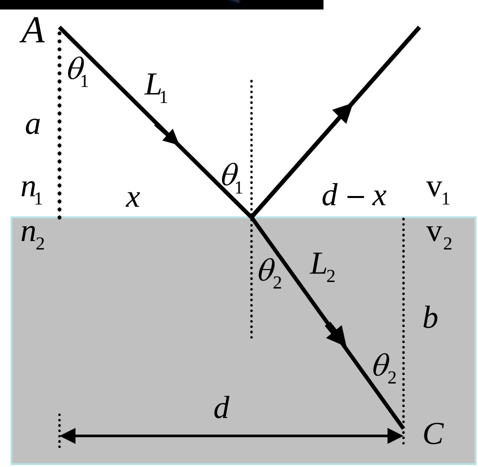
From here we immediately see that $\dfrac{x}{\sqrt{a^2+x^2}} = \dfrac{d-x}{\sqrt{b^2+(d-x)^2}}$ From the

above diagram, $\sin\theta_1 = \sin\theta_1'$. Of course, it is no surprise that the angle of reflection equals the angle of incidence. You knew this from before, and this seems like a very complicated derivation of a simple fact. But it is still nice to see that there is a deeper principle behind it.

5. The speed of light in vacuum is a fixed constant of nature which we usually call c, but in a medium light can travel slower or faster than c. We define the "refractive index" of that medium as: $\text{Refractive index} = \dfrac{\text{Speed of light in vacuum}}{\text{Speed of light in material}}$ (or $n = \dfrac{c}{\mathrm{v}}$). Usually the values of n are bigger than one (e.g. for glass it is around 1.5) but in some special media, its value can be less than one. The value of n also depends on the wavelength (or frequency) of light. This is called dispersion, and it means that different colours travel at different speeds inside a medium. This is why, as in the diagram below, white light gets separated into different colours. The fact that in a glass prism blue light travels faster than red light is responsible for the many colours we see here.



6. We can apply Fermat's Principle to find the path followed by a ray of light when it goes from one medium to another. Part of the light is reflected, and part is "refracted", i.e. it bends away or towards the normal. The total time is,



$$t = \frac{L_1}{\mathrm{v}_1} + \frac{L_2}{\mathrm{v}_2},\; n = \frac{c}{\mathrm{v}} \Rightarrow t = \frac{n_1 L_1 + n_2 L_2}{c} = \frac{L}{c}$$

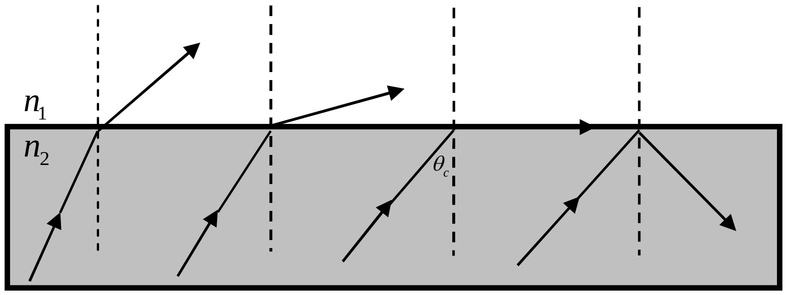
$$L = n_1 L_1 + n_2 L_2 \;= n_1\sqrt{a^2 + x^2} + n_2\sqrt{b^2 + (d - x)^2}$$

Fermat's Principle says that the time t must be minimized: $0 = \dfrac{dt}{dx} = \dfrac{1}{c}\dfrac{dL}{dx} = \dfrac{n_1}{2c}\left(a^2 + x^2\right)^{-1/2}(2x) + \dfrac{n_2}{2c}\left[b^2 + (d - x)^2\right]^{1/2}(2)(d - x)(-1)$

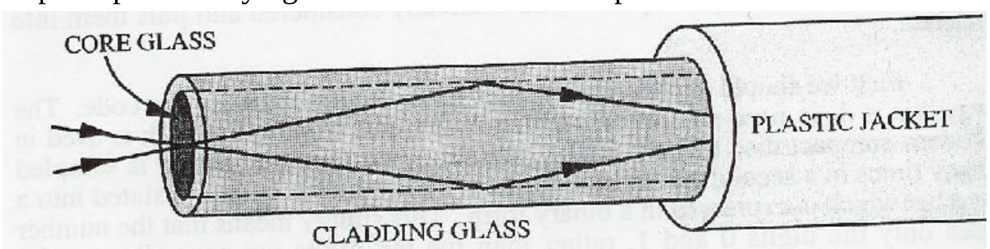
And so we get "Snell's Law", $n_1 \dfrac{x}{\sqrt{a^2 + x^2}} = n_2 \dfrac{d - x}{\sqrt{b^2 + (d - x)^2}}$ or $n_1 \sin\theta_1 = n_2 \sin\theta_2$. This required a little bit of mathematics, but you can see how powerful Fermat's Principle is!

7. Light coming from air into water bends toward the normal. Conversely, light from a source in the water will bend away from the normal. What if you keep increasing the angle with respect to the normal so that the light bends and begins to just follow the surface? This phenomenon is called total internal reflection and θ_c is called the critical

angle. It obeys: $n_1 \sin\theta_c = n_2 \sin 90^0$ from which $\theta_c = \sin^{-1}\frac{n_2}{n_1}$.



8. Fibre optic cables, which are now common everywhere, make use of the total internal reflection principle to carry light. Here is what a fibre optic cable looks like from inside:

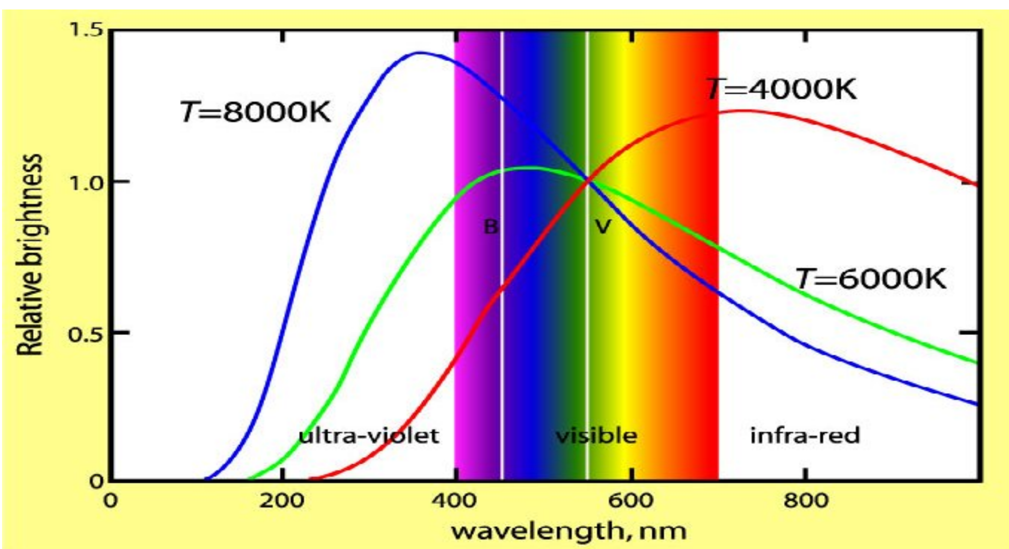


Even if the cable is bent, the light will continue to travel along it. The glass inside the cable must have exceedingly good consistency - if it thicker or thinner in any part, the refractive index will become non-uniform and a lot of light will get lost. Optical fibres now carry thousands of telephone calls in a cable whose diameter is only a little bigger than a human hair!

Summary of Lecture 32 – INTERACTION OF LIGHT WITH MATTER

1. In this lecture I shall deal with the 4 basic ways in which light interacts with matter:
   a) Emission - matter releases energy as light.
   b) Absorption - matter takes energy from light.
   c) Transmission - matter allows light to pass through it.
   d) Reflection - matter repels light in another direction.

2. When an object (for example, an iron rod or the filament of a tungsten bulb) is heated, it emits light. When the temperature is around 800°C, it is red hot. Around 2500°C it is yellowish-white. At temperatures lower than 800°C, infrared (IR) light is emitted but our eyes cannot see this. This kind of emission is called blackbody radiation. Blackbody radiation is continuous - all wavelengths are emitted. However most of the energy is radiated close to the peak. As you can see in the graph, the position of the peak goes to smaller wavelengths (or higher frequencies) as the object becomes hotter. The scale of temperature is shown in degrees Kelvin ($^{\circ}K$). To convert from °C to $^{\circ}K$, simply add 273. We shall have more to say about the Kelvin scale later.



Where exactly does the peak occur? Wien's Law states that $\lambda_{max}T = 2.90\times10^{-3}$ m K. We can derive this in an advanced physics course, but for now you must take this as given.

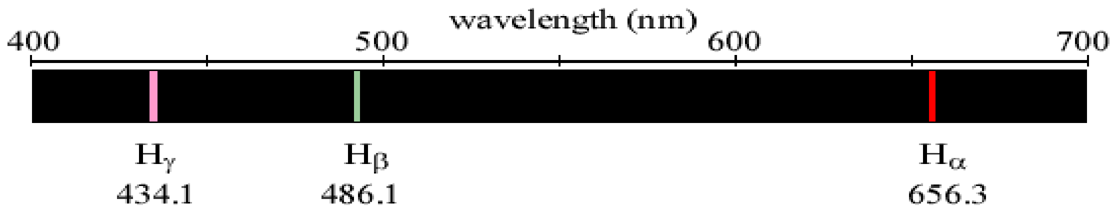
3. In the lecture on electromagnetic waves you had learnt that these waves are emitted when charges accelerate. Blackbody radiation occurs for exactly this reason as well. When a body is heated up, the electrons, atoms, and molecules which it contains undergo violent random motion. Light may emitted by electrons in one atom and absorbed in another. Even an empty box will be filled with blackbody radiation because the sides of the box are made up of material that has charged constituents that radiate energy when they undergo acceleration during their random motion.

4. When can you use Wien's Law? More generally, when can you expect a body to emit blackbody radiation? Answer: only for objects that emit light, not for those that merely reflect light (e.g. flowers). The Sun and other stars obey Wien's Law since the gases they are composed of emit radiation that is in equilibrium with the other materials. Wien's law allows astronomers to determine the temperature of a star because the wavelength at which a star is brightest is related to its temperature.

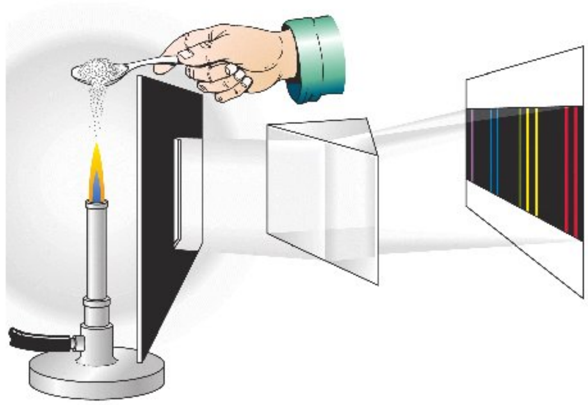
5. All heated matter radiates energy, and hotter objects radiate more energy. The famous Stefan-Boltzman Law, which we unfortunately cannot derive in this introductory course, states that the power radiated per unit area of a hot body is $P = \sigma T^4,$ where the Stefan-Boltzman constant is σ = 5.67 $\times$ 10^{-8} W m^{-2} K^{-4}.

6. Let us apply $P = \sigma T^4$ for finding the temperature of a planet that is at distance R from the sun. The sun has temperature T_{sun} and radius R_{sun}. In equilibrium, the energy received from the sun is exactly equal to the energy radiated by the planet. Now, the total energy radiated by the sun is $\sigma T_0^4 \times 4\pi R_{sun}^2$. But on a unit area of the planet, only $\frac{1}{4\pi R^2}$ of this is received. So the energy received per unit area on the planet is $\sigma T_0^4 \times 4\pi R_{sun}^2 \times \frac{1}{4\pi R^2}$. This must be equal to $\sigma T^4 \Rightarrow \quad T = T_0\sqrt{\frac{R_{sun}}{R}}$.

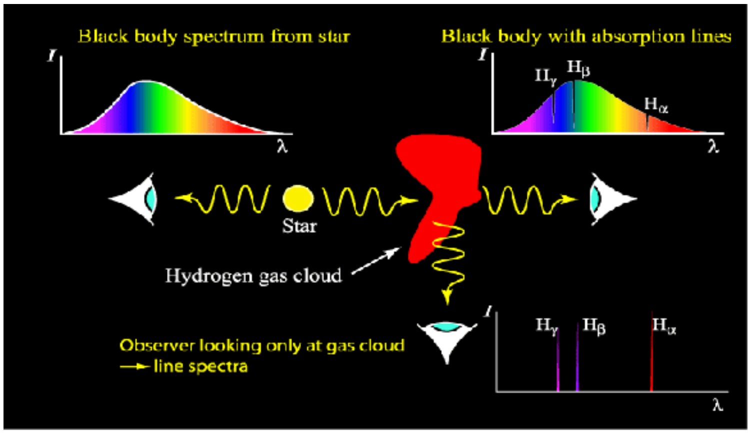
7. The above was for blackbody radiation where the emitted light has a continuous spectrum. But if a gas of identical atoms is excited by some mechanism, then only a few discrete wavelengths are emitted. Each chemical element produces a very distinct pattern of colors called an emission spectrum. So, for example, laboratory hydrogen gas lamps emit 3 lines in the visible region, as you can see below. Whenever we see 3 lines spaced apart in this way, we immediately know that hydrogen gas is present. It is as good as the thumbprint of a man!



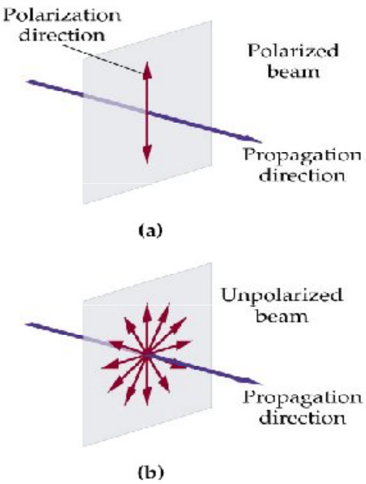
8. But how do we get atoms excited so that they can start revealing their identity? One way is to simply heat material containing those atoms. You saw in the lecture how different colours come from sprinkling different materials on a flame.



9. Everything that I have said about the emission of light applies exactly to the *absorption* of light as well. So, for example, when white light (which has all different frequencies within it) passes through hydrogen gas, you will see that all wavelengths survive except the three on the previous page. So the absorption spectrum looks exactly the same as the emission spectrum - the same lines are emitted and absorbed. This how we know that there are huge clouds of hydrogen floating in outer space. See the diagram below.



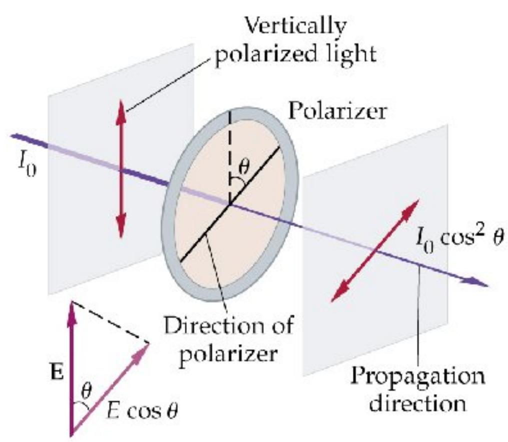
11. As you learned earlier, light is an electromagnetic wave that has an electric field vector. This vector is always perpendicular to the direction of travel, but it can be pointing anywhere in the plane. If it is pointing in a definite direction, we say that the wave is polarized, else it is unpolarized. In general, the light emitted from a source, such as a flame, will be unpolarized and it is equally possible to find any direction of the electric field in the wave. Of course, the magnetic field is perpendicular to both the direction of travel and the electric field.



10. The atmosphere contains various gases which absorb light at many different wavelengths. Molecules of oxygen, nitrogen, ozone, and water have their own absorption spectra, just as atoms have their own.

11. All the beauty of colours we see is due to the selective absorption by molecules of certain frequencies. So, for example, *carotene* is a long, complicated molecule that makes carrots orange, tomatoes red, sarson yellow, and which absorbs blue light. Similarly *chlorophyll* makes leaves green and which absorbs red and blue light.

12. From unpolarized light we can make polarized light by passing it through a polarizer as shown below.

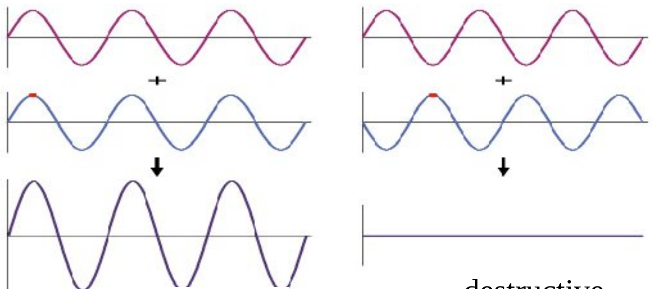


Each wave is reduced in amplitude by $\cos\theta,$ and in intensity by $\cos^2\theta.$ The wave that emerges is now polarized in the θ direction.

13. We can design materials (crystals or stressed plastics) so that they have different optical properties in the two transverse directions. These are called birefringent materials. They are used to make commonly used liquid crystal displays (LCD) in watches and mobile phones. Birefringence can occur in any material that possesses some asymmetry in its structure where the material is more springy in one direction than another.

Summary of Lecture 33 – INTERFERENCE AND DIFFRACTION

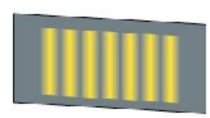
1. Two waves (of any kind) add up together, with the net result being the simple sum of the two waves. Consider two waves, both of the same frequency, shown below. If they start together (i.e. are *in phase* with each other) then the net amplitude is increased. This is called *constructive interference.* But if they start at different times (i.e. are *out of phase* with each other) then the net amplitude is decreased. This is called *destructive interference.*

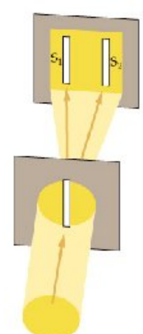


constructive destructive

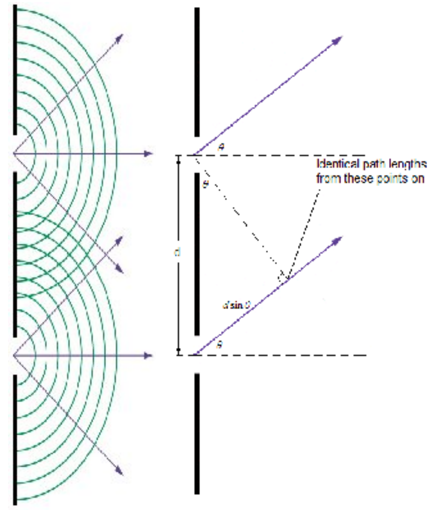
In the example above, both waves have the same frequency and amplitude, and so the resulting amplitude is doubled (constructive) or zero (destructive). But interference occurs for any two waves even when their amplitudes and frequencies are different.

2. Although any waves from different sources interfere, if one wants to observe the interference of light then it is necessary to have a *coherent* source of light. Coherent means that both waves should have a fixed phase relative to each other. Even with lasers, it is very difficult to produce coherent light from two separate sources. Observing interference usually requires taking two waves from a single source, with each going along a different path. In the figure, an incoherent light source illuminates the first slit. This creates a uniform and coherent illumination of the second screen. Then waves from the slits S_1 and S_2 meet on the third screen and create a pattern of alternating light and dark fringes.





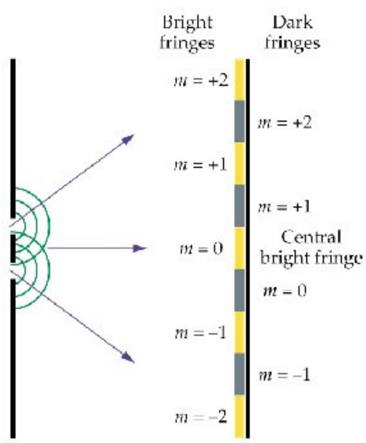
3. Wherever there is a bright fringe, constructive interference has occured, and wherever there is a dark fringe, destructive interference has occured. We shall now calculate where on the third screen the interference is constructive. Take any point on the third screen. Light reaches this point from both S_1 and S_2, but it will take different amounts of time to get there. Hence there will be a phase difference that we can calculate. Look at the diagram below. You can see that light from one of the slits has to travel an extra



distance equal to $d\sin\theta$, and so the extra amount of time it takes is $(d\sin\theta)/c$. There will be constructive interference if this is equal to $T, 2T, 3T, \cdots$ (remember that the time period is inversely related to the frequency, $T = 1/\nu$, and that $c = \lambda\nu$).

We find that $\frac{d\sin\theta}{c} = nT = \frac{n}{\nu} \quad \Rightarrow \quad d\sin\theta = n\lambda$, where $n = 1, 2, 3, \cdots$ What about for destructive interference? Here the waves will cancel each other if the extra amount of time is $\frac{T}{2}, \frac{3T}{2}, \frac{5T}{2}, \cdots$ The condition then becomes $d\sin\theta = (n + \frac{1}{2})\lambda$.

4. Example: two slits with a separation of 8.5×10^{-5}m create an interference pattern on a screen 2.3m away. If the $n = 10$ bright fringe above the central is a linear distance of 12 cm from it, what is the wavelength of light used in the experiment?

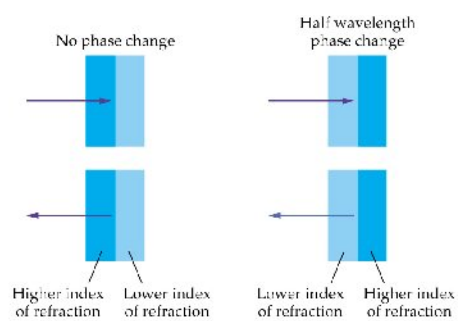


Answer: First calculate the angle to the tenth bright fringe using $y = L\tan\theta$. Solving for θ gives,

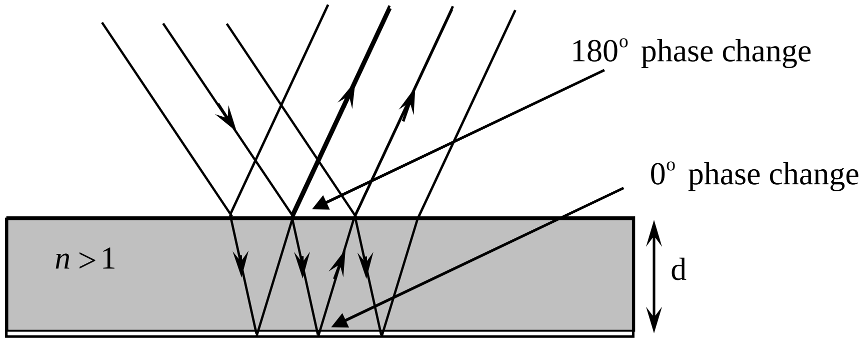
$\theta = \tan^{-1}\left(\frac{y}{L}\right) = \tan^{-1}\left(\frac{0.12m}{2.3m}\right) = 3.0^\circ$. From this,

$$\lambda = \frac{d}{n}\sin\theta = \left(\frac{8.5\times 10^{-5}}{10}\right)\sin(3.0^\circ) = 4.4\times 10^{-7}m = 440nm \text{ (nanometres)}.$$

5. When a wave is reflected at the interface of two media, the phase will not change if it goes from larger refractive index to a smaller one. But for smaller to larger, there will be phase change of a half-wavelength. One can show this using Maxwell's equations and applying the boundary conditions, but this will require some more advanced studies. Instead let's just use this fact below.



5. When light falls upon a thin film of soapy water, oil, etc. it is reflected from two surfaces. On the top surface, the reflection is with change of phase by π whereas at the lower surface there is no change of phase. This means that when waves from the two surfaces combine at the detector (your eyes), they will interfere.



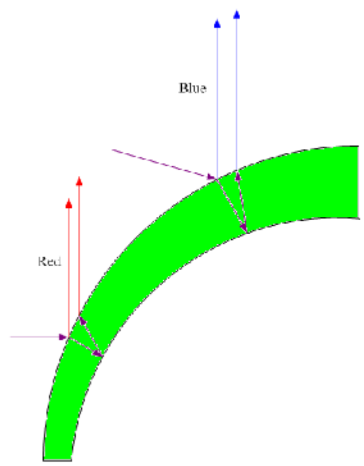
To simplify matters, suppose that you are looking at the thin film almost directly from above. Here n is the index of refraction for the medium. Then,

The condition for destructive interference is: $2nd = m\lambda \quad (m = 0,1,2,...)$

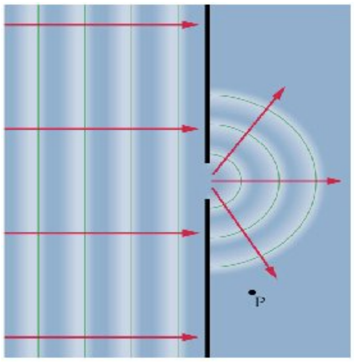
The condition for constructive interference is: $2nd = (m + \frac{1}{2})\lambda \quad (m = 0,1,2,...)$.

Prove it!

6. Interference is why thin films give rise to colours. A drop of oil floating on water spreads out until it is just a few microns thick. It will have thick and thin portions. Thick portions of any non-uniform thin film appear blue because the long-wavelength red light experiences destructive interference. Thinner portions appear red because the short-wavelength blue light interferes destructively.

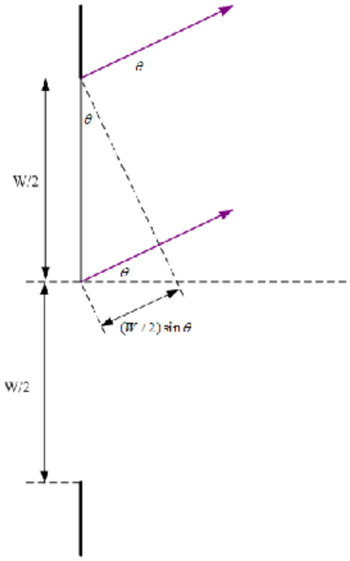


7. **Diffraction :** The bending of light around objects (into what would otherwise be a shadowed region) is known as diffraction. Diffraction occurs when light passes through very small apertures or near sharp edges. Diffraction is actually just interference, with the difference being only in the source of the interfering waves. Interference from a single slit, as in the figure here, is called diffraction.



We can get an interference pattern with a single slit provided its size is approximately equal to the wavelength of the light (neither too small nor too large).

8. Let's work out the condition necessary for diffraction of light from a single slit. With reference to the figure, imagine that a wave is incident from the left. It will cause secondary waves to be radiated from the edges of the slit. If one looks at angle θ, the extra distance that the wave emitted from the lower slit must travel is $W\sin\theta$. If this is a multiple of the wavelength λ, then constructive interference will occur. So the condition becomes $W\sin\theta = m\lambda$ with $m = \pm1, \pm2, \pm3...$ So, even from a single slit one will see a pattern of light and dark fringes when observed from the other side.

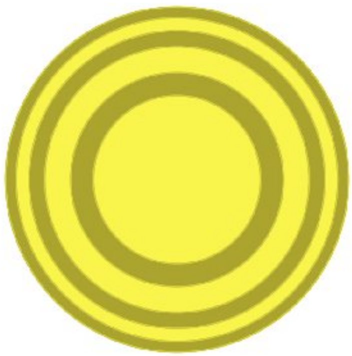


9. Light with wavelength of 511 nm forms a diffraction pattern after passing through a single slit of width 2.2×10^{-6}m. Find the angle associated with (a) the first and (b) the second bright fringe above the central bright fringe.

SOLUTION: For $m=1$, $\theta = \sin^{-1}\left(\frac{m\lambda}{W}\right) = \sin^{-1}\left(\frac{(1)(511\times10^{-9}m)}{2.20\times10^{-6}m}\right) = 13.4^\circ$

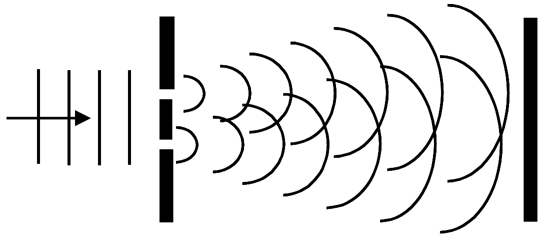
For $m=2$, $\theta = \sin^{-1}\left(\frac{m\lambda}{W}\right) = \sin^{-1}\left(\frac{(2)(511\times10^{-9}m)}{2.20\times10^{-6}m}\right) = 27.7^\circ$

10. Diffraction puts fundamental limits on the capacity of telescopes and microscopes to separate the objects being observed because light from the sides of a circular aperture interferes. One can calculate that the first dark fringe is at $\theta_{\min} = 1.22\frac{\lambda}{D}$, where D is diameter of the aperture. Two objects can be barely resolved if the diffraction maximum of one object lies in the diffraction minimum of the second object. Clearly, the larger D is, the smaller the angular diameter separation. We say that larger apertures lead to better resolution.

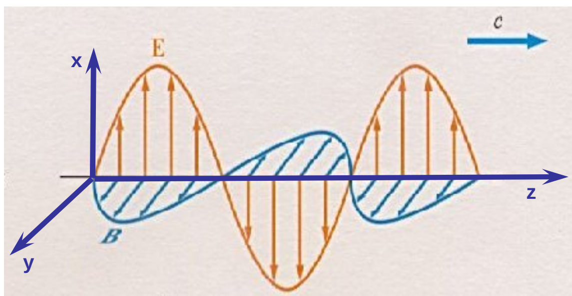


Summary of Lecture 34 – THE PARTICLE NATURE OF LIGHT

1. In the previous lecture I gave you some very strong reasons to believe that light is waves. Else, it is impossible to explain the interference and diffraction phenomena that we see in innumerable situations. Interference from two slits produces the characteristic pattern.

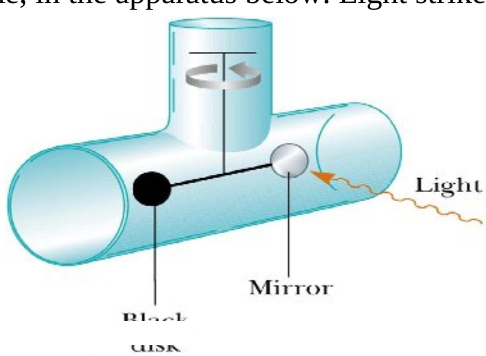


2. Light is waves, but waves in what? of what? The thought that there is some invisible medium (given the name *aether*) turned out to be wrong. Light is actually electric and magnetic waves that can travel through empty space. The electric and magnetic waves



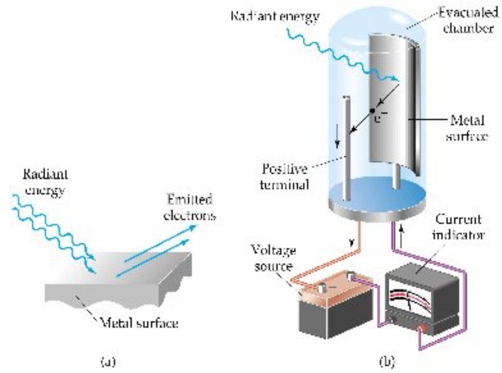
are perpendicular to each other and to the direction of travel (here the z direction).

3. Electromagnetic waves transport linear momentum and energy. If the energy per unit volume in a wave is U then it is carrying momentum p, where $p = U/c$. Waves with large amplitude carry more energy and momentum. For the sun's light on earth the momentum is rather small (although it is very large close to or inside the sun). Nevertheless, it is easily measureable as, for example, in the apparatus below. Light strikes a mirror and rebounds.



Thus the momentum of the light changes and this creates a force that rotates the mirror. The force is quite small - just 5×10^{-6} Newtons per unit area (in metre^2) of the mirror.

4. So strong was the evidence of light as waves that observation of the photoelectric effect came as a big shock to everybody. In the diagram below, light hits a metal surface and

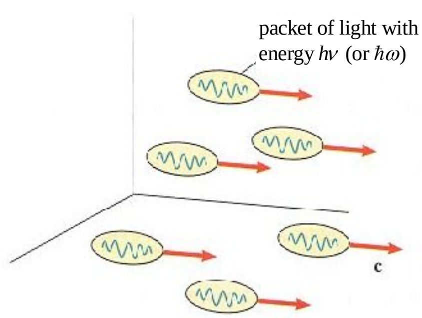


and knocks out electrons that travel to the anode. A current flows only as long as the light is shining. Above the threshold frequency, the number of electrons ejected depends on the intensity of the light. This was called the photoelectric effect. The following was observed:

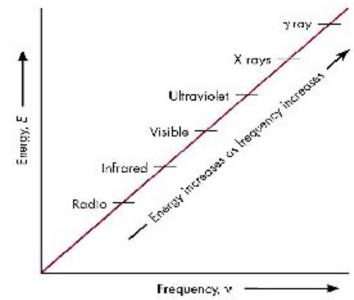
a) The photoelectric effect does not occur for all frequencies ν; it does not occur at all when ν is below a certain value. *But classically (meaning according to the Maxwell nature of light as an electromagnetic wave) electrons should be ejected at any* ν*. If an electron is shaken violently enough by the wave, it should surely be ejected!*

b) It is observed that the first photoelectrons are emitted instantaneously. *But classically the first photoelectrons should be emitted some time after the light first strikes the surface and excites its atoms enough to cause ionization of their electrons.*

5. Explanation of the photoelectric puzzle came from Einstein, for which he got the Nobel Prize in 1905. Einstein proposed that the light striking the surface was actually made of little packets (called quanta in plural, quantum in singular). Each quantum has an energy energy $\varepsilon = h\nu$ (or $\varepsilon = \hbar\omega$ where $\hbar = h/2\pi$ and $\omega = 2\pi\nu$) where h is the famous Planck's constant, 6.626×10^{-34} Joule-seconds. An electron is kicked out of the metal only when a quantum has energy (and frequency) big enough to do the job. It doesn't matter how many quanta of light - called photons - are fired at the metal. No photoelectrons will be released unless ν is large enough. Furthermore the photoelectrons are released immediately when the photon hits an electron.

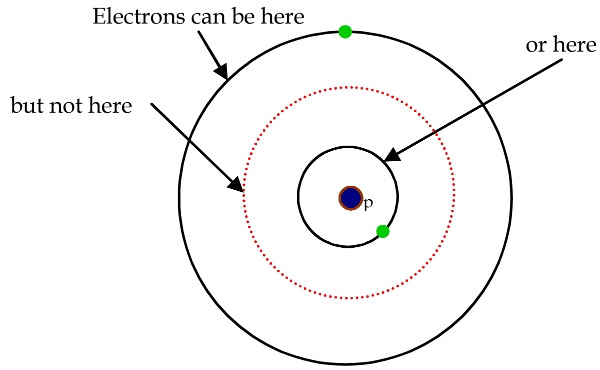


6. Microwaves, radio and TV waves, X-rays, γ-rays, etc. are all photons but of very different frequencies. Because Planck's constant h is an extremely small number, the energy of each photon is very small.

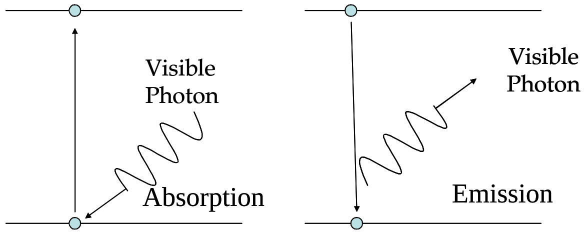


7. How many photons do we see? Here is a table that gives us some interesting numbers:
   a) Sunny day (outdoors): 10^{15} photons per second enter eye (2 mm pupil).
   b) Moonlit night (outdoors): 5×10^{10} photons/sec (6 mm pupil).
   c) Moonless night (clear, starry sky): 10^{8} photons/sec (6 mm pupil).
   d) Light from dimmest naked eye star (mag 6.5): 1000 photons/sec entering eye.

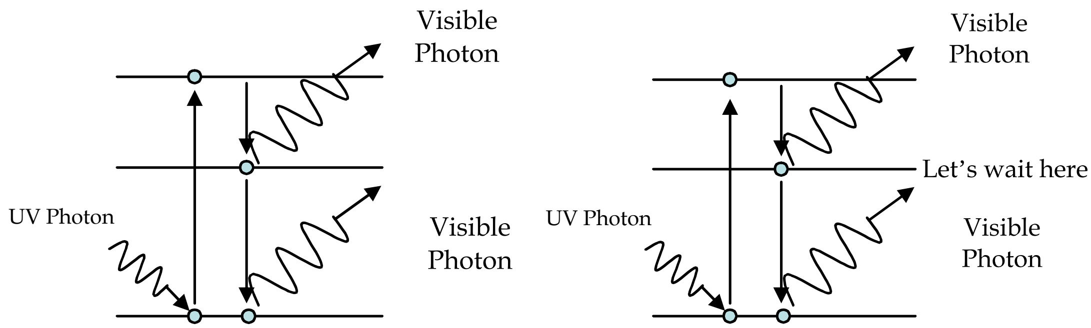
8. Where do photons come from? For this it is necessary to first understand that electrons inside an atom can only be in certain definite energy states. When an electron drops from a state with higher energy to one with lower energy, a photon is released whose energy is exactly equal to the difference of energies. Similarly a photon is absorbed when a photon of just the right energy hits an electron in the lower state and knocks it into a higher state.



The upper and lower levels can be represented differently with the vertical direction representing energy. The emission and absorption of photons is shown below.



9. Fluorescence and phosphorence are two phenomena observed in some materials. When they are exposed to a source of light of a particular colour, they continue to emit light of a different colour even after the source has been turned off. So these materials can be seen to glow even in the dark. In phosphorescence, a high-frequency photon raises an electron to an excited state. The electron jumps to an intermediate states, and then drops after a little while to the ground state. This is fluorescence. Phosphorescence is similar to fluorescence, except that phosphorescence materials continue to give off a secondary glow long (seconds to hours) after the initial illumination that excited the atoms.



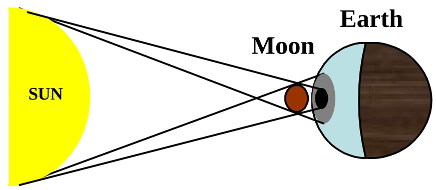
10. One of the most important inventions of the 20th century is the laser which is short for:

LASER ≡ **L**ight **A**mplification by the **S**timulated **E**mission of **R**adiation

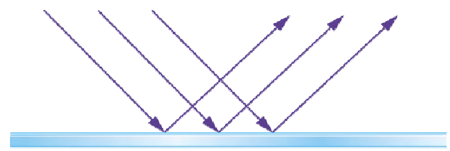
Lasers are important because they emit a very large number of photons all with one single frequency. How is this done? By some means - called optical pumping - atoms are excited to a high energy level. When one atom starts decaying to the lower state, it encourages all the others to decay as well. This is called spontaneous emission of radiation.

Summary of Lecture 35 – GEOMETRICAL OPTICS

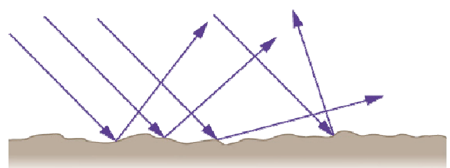
1. In the previous lecture we learned that light is waves, and that waves spread out from every point. But in many circumstances we can ignore the spreading (diffraction and interference), and light can then be assumed to travel along straight lines as rays. This is hown by the existence of sharp shadows, as for case of the eclipse illustrated here.



2. When light falls on a flat surface, the angle of incidence equals the angle of reflection. You can verify this by using a torch and a mirror, or just by sticking pins on a piece of paper in front of a mirror. But what if the surface is not perfectly flat? In that case, as shown in fig. (b), the angle of incidence and reflection are equal at every point, but the normal direction differs from point to point. This is called "diffuse reflection". Polishing a surface reduces the diffusiveness.

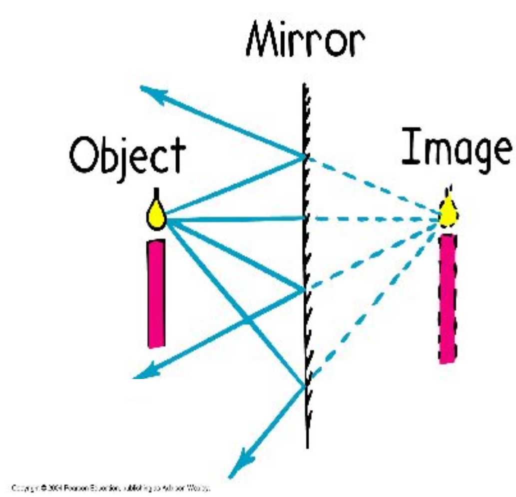


(a) Specular reflection

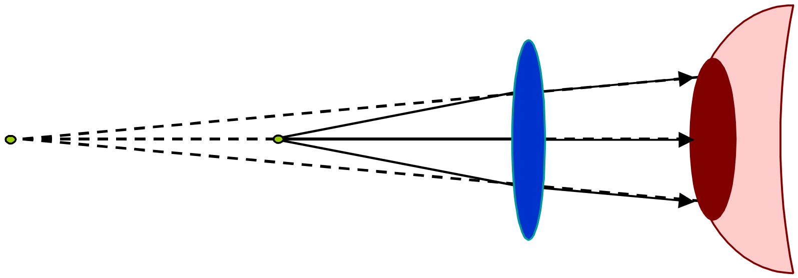


(b) Diffuse reflection

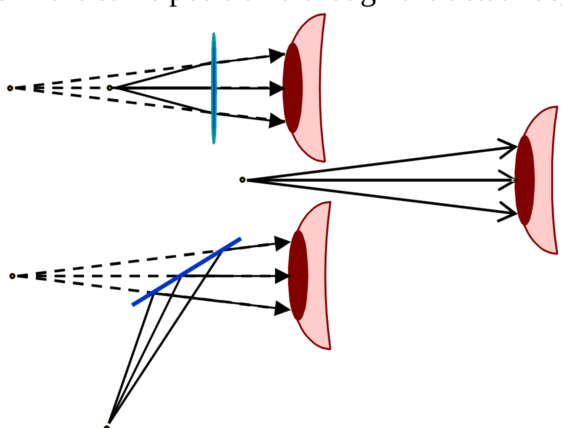
3. If you look at an object in the mirror, you will see its image. It is not the real thing, and that is why it is called a "virtual" image. You can see how the virtual image of a candle is formed in this diagram. At each point on the surface, there is an incident and reflected ray. If we extend each reflected ray backwards, it appears as if they are all coming from the same point. This point is the image of the tip of the flame. If we take other points on the candle, we will get their images in just the same way. This way we will have the image of the whole candle. The candle and its image are at equal distance from the mirror.



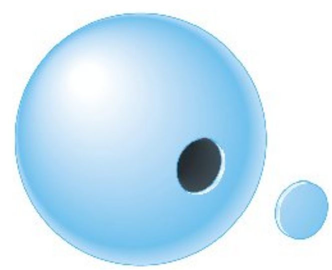
4. Here is another example of image formation. A source of light is placed in front of a bi-convex lens which bends the light as shown. The eye receives rays of light which seem to originate from a positon that is further away than the actual source.

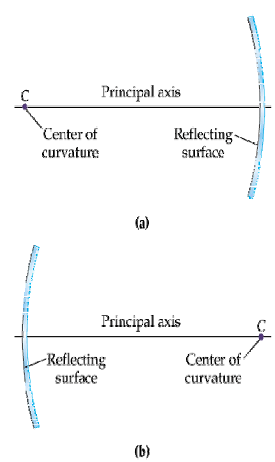


Now just to make the point even more forcefully, in all three situations below, the virtual image is in the same position although the actual object is in 3 different places.

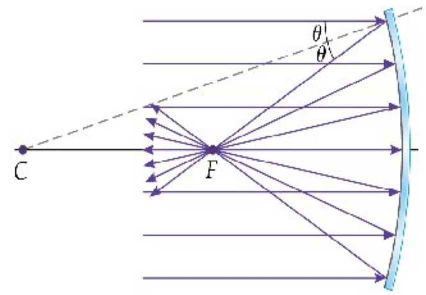


5. Imagine that you have a sphere of radius R and that you can cut out any piece you want. The outside or inside surface can be silvered, as you want. You can make spherical mirrors in this way. These can be of two kinds. In the first case, the silvering can be on the inside surface of the sphere, in which case this is is called a convex spherical mirror. The normal directed from the shiny surface to the centre of the sphere (from which it was cut out from) is called the principal axis, and the radius of curvature is *R*. The other situation is that in which the outside surface is shiny. Again, the principal axis the same, but now the radius of curvature (by definition) is - *R*. What does a negative curvature mean? It means precisely what has been illustrated - a convex surface has a positive and a concave surface has a negative curvature.

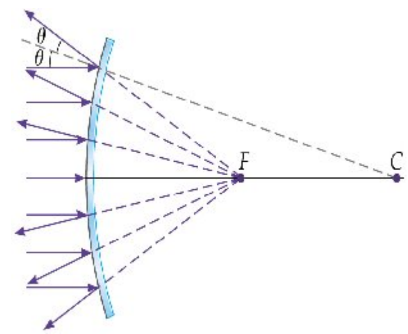




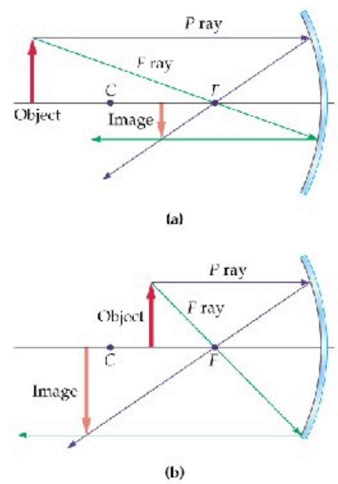
6. Now we come to the important point: a beam of parallel rays will reflect off a convex mirror and all get together (or converge) at one single point, called the focus. Now look at the figure. Since the incident and reflected angles are equal for every ray, you can see that the ray will have to pass through the focus at F, at distance $R/2$. We say that the focal length of such a mirror is $f = R/2$.



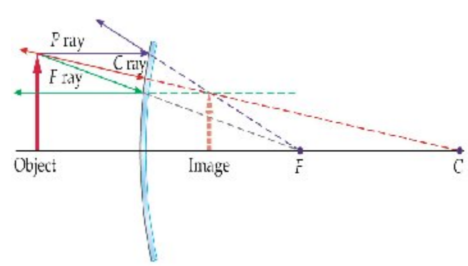
7. The concave mirror is different. A parallel beam incident from the left is reflected and the rays now spread out (diverge). However, if we were to look at any outgoing ray and extend it backward, all the rays would appear to be coming from one single point. This is the *virtual focus* because there is, in fact, no real source of light there. From the diagram you can see the virtual focus F is at distance $R/2$ *behind* the mirror. We say that the focal length of such a mirror is $f = -R/2$.



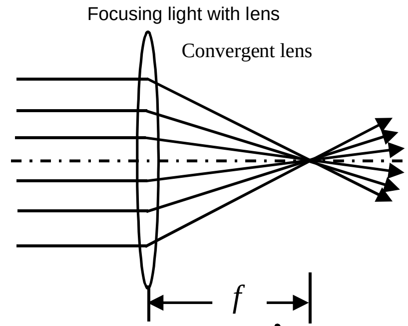
8. Now let's see how images are formed by concave mirrors. Take two rays that are emitted from the top of an object placed in front of the mirror. Call them the P and F ray, and see what happens to them after reflection. Where the two cross, that is the image point. As you can see the image is inverted and smaller than the actual object if the object is far away (i.e. lies to the left of C) and is larger if close to the mirror. The magnification of size makes this useful as a shaving mirror, among other things.



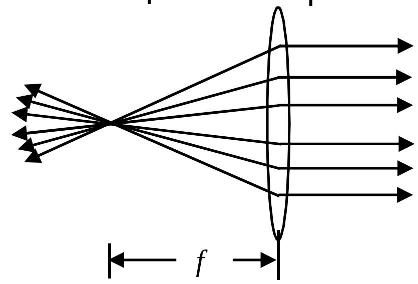
9. Now repeat for a concave mirror. Here the rays will never actually meet so we can have only a virtual object. Take 3 rays - call them P,F, and C - and see how they are reflected from the mirror's surface. Where they meet is the position of the image. The image is always smaller than the actual size of the object. This is obviously useful for driving a car because you can see a wide area.



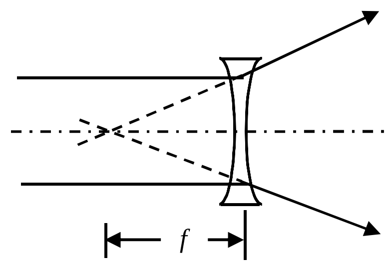
10. A lens is a piece of glass curved in a definite way. Because the refractive index of glass is bigger than one, every ray bends towards the normal. Here you see a double convex lens that focuses a beam of parallel rays. For a perfect lens, all the rays will converge to one single point that is (again) called the focus. The distance f is called the focal length.



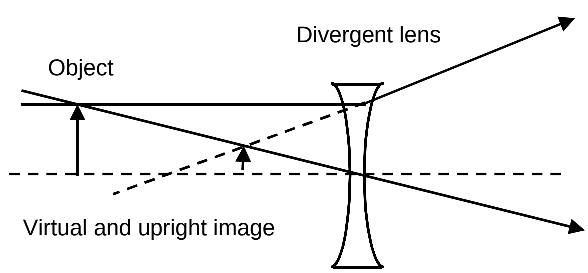
Of course, light can equally travel the other way, so if a point source is placed at the focus of a convex lens then a parallel beam of light will emerge from the other side. This is how some film projectors produce a parallel beam.



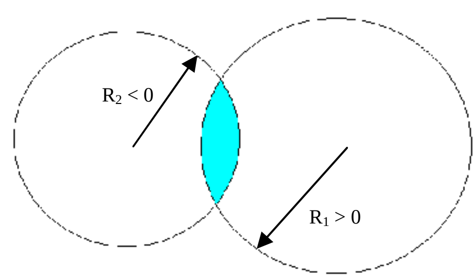
A concave lens does not cause a parallel beam to converge. On the contrary, it makes it diverge, as shown. Note, however, that if the rays are continued backwards, then they appear to come from one single point, which is here the virtual focus. The distance f is the focal length.



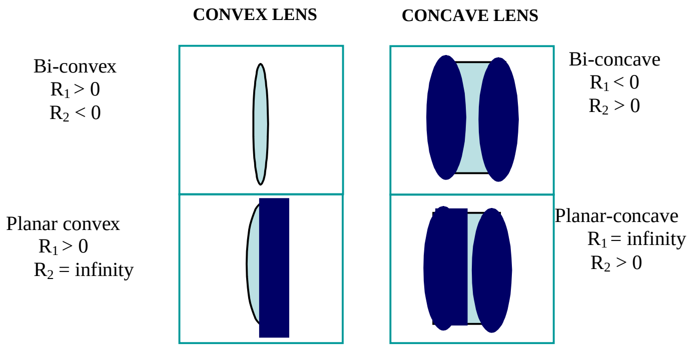
12. Here is how a concave (or divergent) lens forms an image. An observer on the side opposite to the object will see the image upright and smaller in size than the object.



13. A lens can be imagined as cut out of two spheres of glass as shown, with the spheres having radii R_1 and R_2. Note that they tend to bend light in opposite ways. By convention, R_2 is negative. It is possible to show that the focal length of the lens is , $\frac{1}{f}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$.



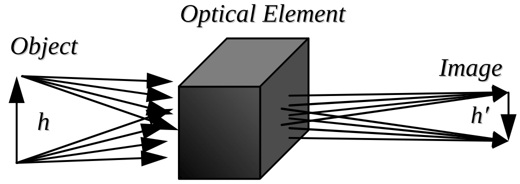
14. The following figure summarizes the shape of some common types of lenses.



Note that a flat surface has infinite radius of curvature. The focal legth of each can be calculated using the previous formula.

15. The "strength" of a lens is measured in *diopters*. If the focal length of a lens is expressed in metres, the diopter of the lens is defined as $D = 1/f$. If the refractive index of the glass in a lens is n, then the diopters due to the first interface D_1 and the second interface D_2 are, $D_1 = (n-1)/R_1$ and $D_2 = (1-n)/R_2$. The total diopter of the lens is $D = D_1 + D_2$.

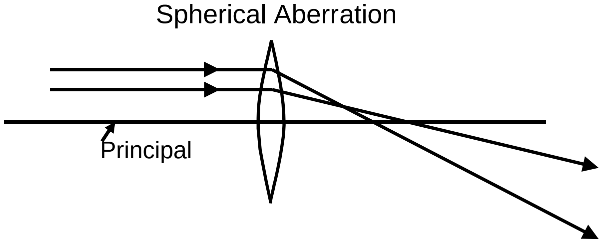
16. For any optical system - meaning a collection of lenses and mirrors - we can define a magnification factor as a ratio of sizes -- see the diagram below.



Lateral Magnification $M = h'/h$

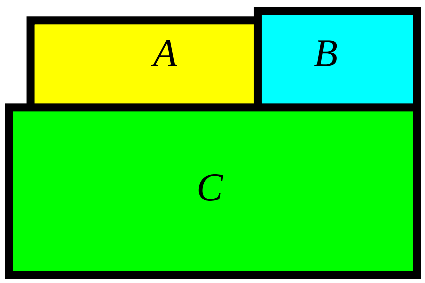
17. The perfect lens will focus a parallel beam of rays all to exactly the same focus. But no lens is perfect, and every lens suffers from aberration although this can be made quite small by following one lens with another. Below you see an example of "spherical aberration". Rays crossing different parts of the lens do not reach exactly the same focus. This distorts the image. Computers can design lens surfaces to minimize this aberration.

Spherical Aberration

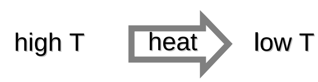


Summary of Lecture 36 – THERMAL PHYSICS I

1. The ancient view was that heat is a colourless, weightless, fluid which occupies no volume has no smell, etc. This imagined substance - called phlogiston - was supposedly stored in objects and transferred between objects. It took a long time to reject this notion. Why is it wrong? Because (as we will see) heat can be created and destroyed, whereas liquids keep their volume and cannot be created or destroyed.
2. We are all familiar with an intuitive notion of temperature. We know that hotter things have higher temperature. But let us try to define temperature more rigorously. In the diagram below, the three bodies A,B,C are in contact with each other. After sufficient time passes, one thing will be common to all three - a quantity that we call temperature.



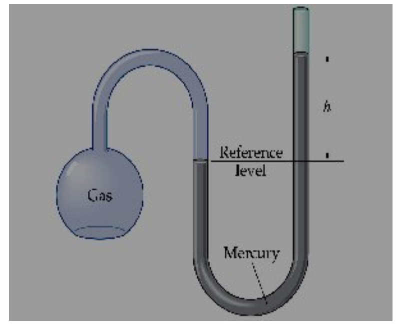
3. Now let us understand heat. Heat is energy, but it is a very special kind of energy: it is that energy which flows from a system at high temprature to a system at low temperature.



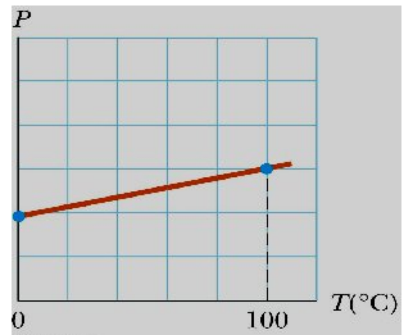
   Stated in a slightly different way: heat is the flow of internal energy due to a temperature difference. (Note that we do not have to know about atoms, molecules, and the internal composition of a body to be able to define heat - all that will come later).
4. The term "thermal equilibrium" is extremely important to our understanding of heat. If some objects (say, a glass of water placed in the open atmosphere) are put in thermal contact but there is no heat exchange, then we say that the objects are in thermal equilibrium. So, if the glass of water is hot or cold initially, after sufficient time passes it will be in thermal equilibrium and will neither receive nor lose heat to the atmosphere.

5. We have an intuitive understanding of temperature, but how do we measure it? Answer: by looking at some physical property that changes when the temperature changes. So, for example, when the temperature rises most things expand, the electrical resistance changes, some things change colour, etc. These are called "thermometric properties".

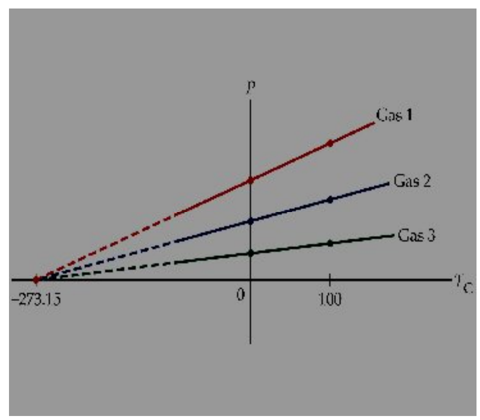
6. Let's take a practical example: the constant volume gas thermometer. Here the reference level is kept fixed by raising or lowering the tube on the right side (the tube below is made of rubber or some flexible material). So the gas volume is fixed. The gas pressure is $\rho \times g \times h$, and so the pressure is known from measuring h.



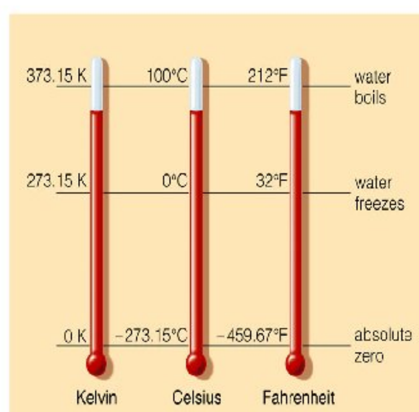
   To find the temperature of a substance, the gas flask is placed in thermal contact with the substance. When the temperature is high, the pressure is large. From the graph of pressure versus temperature, you can easily read off the temperature. Note that we are using two fixed points, which we call 0^0C and 100^0C. This is called the Centigrade scale, and the two fixed points correspond to the freezing and boiling of water.



7. There is a temperature below which it is not possible to go. In other words, if you cool and cool there comes a point after which you cannot cool any more! How do we know this? Take different gases and plot how their pressure changes as you cool them down. You can see from the graph that all the lines, when drawn backwards, meet exactly at one point. This point is the absolute zero of temperature and lies at about -273.15^0C. This is also called 0^0K, or zero degrees Kelvin.



8. Temperature Scales. If you had been born 300 years ago, and if you had discovered a reliable way to tell different levels of "hotness", then maybe today there would be a temperature scale named after you! The relation between Centigrade, Fahrenheit, and Kelvin scales is illustrated to your right. You can see that absolute zero corresponds to -460^0K, -273.15^0C, and 0^0K. The Kelvin scale is the most suited for scientific purposes, and you should be careful to use this in all heat related calculations.

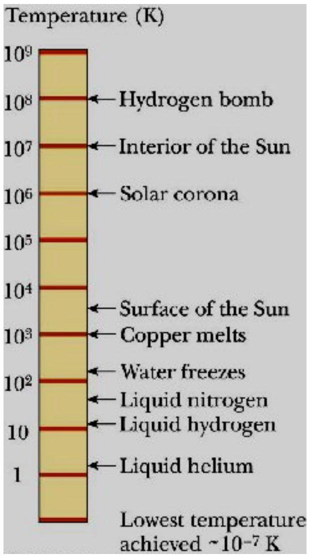


Conversion between degrees Celsius and degrees Fahrenheit: $T_F = (\frac{9}{5}F^\circ / C^\circ)T_C + 32^\circ F$

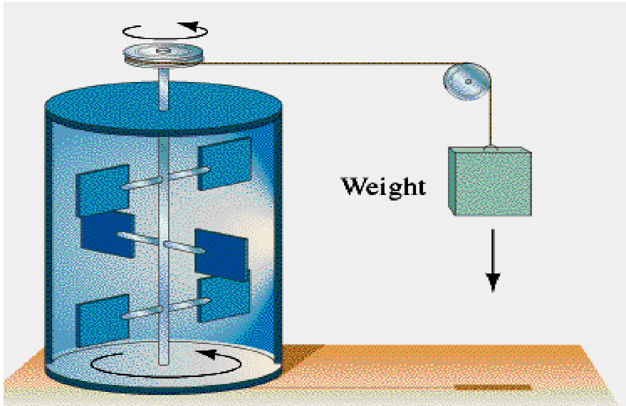
Conversion between degrees Fahrenheit and degrees Celsius: $T_C = (\frac{5}{9}C^\circ / F^\circ)(T_F - 32^\circ F)$

Conversion between Celsius and Kelvin temperatures: $T = T_C + 273.15$

9. How hot is hot, and how cold is cold? Whenever a scientist says something is large or small, it is always relative to something. The hottest thing ever was the universe when it just came into existence (more about this in the last lecture). After this comes the hydrogen bomb (10^8K). The surface of the sun is not so hot, only 5500K. Copper melts around 1000K, water turns to steam at 373K. If you cool further, then all the gases start to solidify. The lowest temperature that has ever been achieved is $10^{-7}K$, which is one tenth of one millionth of one degree! Why not still lower? We shall later why this is not possible.



10. When you rub your hands, they get hot. Mechanical work has been converted into heat. The first person to investigate this scientifically was Joule. In the experiment below, he allowed a weight to drop. This turned a paddle that stirred up the water and caused the temperature to rise. The water got hotter if the weight was released from a greater height.

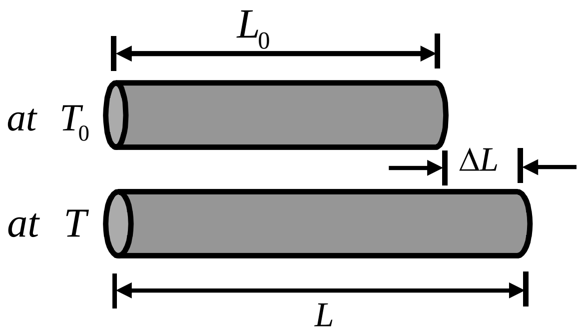


Joule established the units for the mechanical equivalent of heat. The units we use today are: 1 calorie (1 cal) raises the temperature of 1 g of water by 1 °C

1 cal = 4.186 Joule, 1 kilocalorie (1 kcal) = 1000 cal

Remember also that joule is the unit of work: when a force of one newton acts through a distance of one metre, the work done is one joule.

11. Let's now consider one important effect of heat - most things expand when heated. Of course, our world is 3-dimensional but if there is a thin long rod then the most visible effect of heating it is that the rod increases in length. By how much?



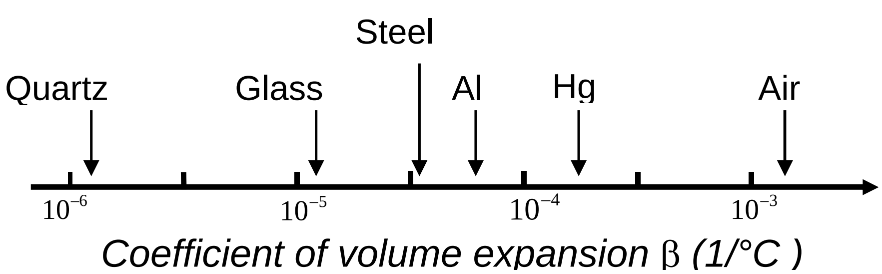
Look at the above diagram. Call the length of the rod at T_0 as L_0. When the rod is heated to T, then the length increases to $L = L_0\left[1+\alpha\left(T - T_0\right)\right]$. The difference of the lengths is, $L - L_0 = \alpha L_0\left(T - T_0\right)$, or $\Delta L = \alpha L_0 \Delta T$. Here α is just a dimensionless number that tells you how a particular material expands. It is called the coefficient of linear expansion. If α was zero, then the material would not expand at all. You can also write it as $\alpha = \dfrac{\Delta L / L_0}{\Delta T}$.

Similarly, define a coefficient of volume expansion – call it β – as $\beta \equiv \dfrac{\Delta V / V_0}{\Delta T}$.

12. There is a relation between α and β, the linear and volume coefficients. Let's look at the change in volume due to expansion: $V' = (L + \Delta L)^3 = (L + \alpha L \Delta T)^3$

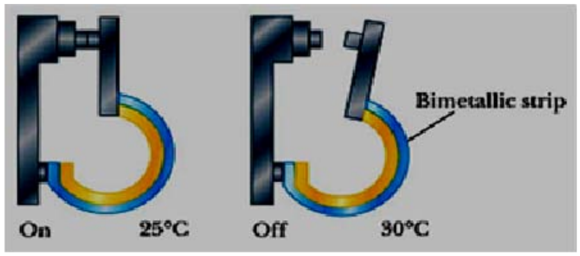
$$= L^3 + 3\alpha L^3 \Delta T + 3\alpha^2 L^3 \Delta T^2 + \alpha^3 L^3 \Delta T^3$$
$$\approx L^3 + 3\alpha L^3 \Delta T = V + 3\alpha V \Delta T$$

We are only looking for small changes, so the higher terms in ΔT can be safely dropped. Hence $\Delta V = 3\alpha V \Delta T$. From the definition, this we immediately see that $\boxed{\beta = 3\alpha}$. Just to get an idea, here is what β looks like for various different materials:

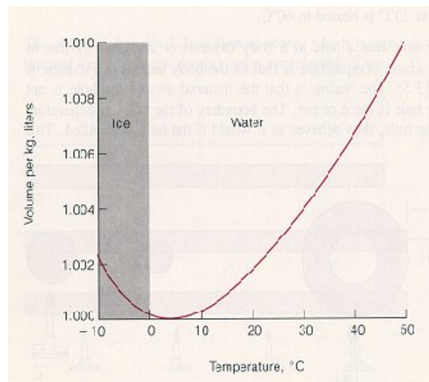


13. We can use the fact that different metals expand at different rates to make thermostats. For example, you need a thermostat to prevent an electric iron from getting too hot or a

refrigerator from getting too cold. In the diagram below, one metal is bonded to another. When they expand together, one expands less than the other and the shape is distorted. This breaks off a circuit, as shown here.



14. Water behaves strangely - for certain temperatures, it contracts when heated (instead of expanding)! Look at the graph: between 0^0C and 4^0C, it exhibits strange or *anomalous* behaviour. The reason is complicated and has to do with the molecular structure of water. If water behaved normally, it would be very bad for fish in the winter because they rely upon the bottom of the lake or sea to remain liquid even though there the surface is frozen.



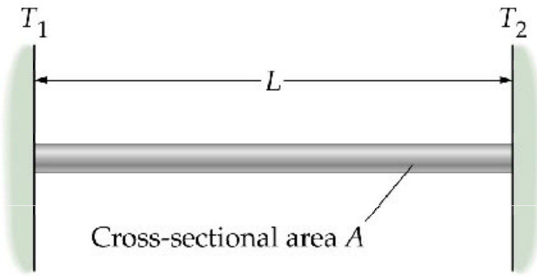
15. We define the "heat capacity" of a body as $C = \dfrac{Q}{\Delta T}$, where ΔT is the increase in temperature when an amount of heat Q is added to the body. Heat capacity is always positive; Q and ΔT have the same sign. The larger the heat capacity, the smaller is the change in the body's temperature when a fixed amount of heat is added. In general, $Q = mc\Delta T$, where $Q =$ heat added , $m =$ mass , $c =$ specific heat , and ΔT = change in temperature. Water has a very large specific heat c, $c = 1.0\ cal\ /(^oC\ g)$; this means it takes one calorie to raise the temperature of 1 gm of water by 1 degree Celsius. In joules per kilogram this is the same as 4186. In the table are the specific heats of various common materials. You can see that metals have small c, which means that it is relatively easy to raise or lower their temperatures. The opposite is true of water. Note also that steam and ice have smaller $c's$ than water. This shows that knowing the chemical composition is not enough.

Specific Heats at Atmospheric Pressure

Substance	Specific heat, c [J / (kg·K)]
Water	4186
Ice	2090
Steam	2010
Beryllium	1820
Air	1004
Aluminum	900
Glass	837
Silicon	703
Iron (steel)	448
Copper	387
Silver	234
Gold	129
Lead	128

16. A 0.5-kg block of metal with an initial temperature of $30.0^\circ C$ is dropped into a container holding 1.12 kg of water at $20.0^\circ C$. If the final temperature of the block-water system is $20.4^\circ C$, what is the specific heat of the metal?
SOLUTION: Write an expression for the heat flow out of the block $Q_{block} = m_b c_b (T_b - T)$.
Do the same for water, $Q_{water} = m_w c_w (T - T_w)$. Now use the fact that all the energy that is lost by the block is gained by the water: $Q_{block} = Q_{water} \Rightarrow Tm_b c_b (T_b - T) - m_w c_w (T - T_w) = 0$
From this, $$c_b = \frac{m_w c_w (T - T_w)}{m_b (T_b - T)} = \frac{(1.12kg)[4186J/(kg\cdot K)](20.4^\circ C - 20.0^\circ C)}{(0.500kg)(30.0^\circ C - 20.4^\circ C)}$$
$$= 391J/(kg\cdot K)$$

17. When lifting a "daigchee" from a stove, you would be wise to use a cloth. Why? Because metals transfer, or conduct, heat easily whereas cloth does not. Scientifically we define conductivity using experiments and apparatus similar to the following:



Heat flows from the hotter to the colder plate. Let us use the following symbols:

k = thermal conductivity Q = heat transferred
A = cross sectional area t = duration of heat transfer
L = length ΔT = temperature difference

Then the heat transferred in time t is, $Q = kA\left(\frac{\Delta T}{L}\right)t$. This formula allows us to measure k if all the other quantities in it are measured.

18. Conduction is one possible way by which heat is transferred from one portion of a system to another. It does not involve physical transport of particles. However, there is another way by which heat can be transferred - convection. In convection, heat is carried by a moving fluid. So when you heat a pot of water, molecules at the bottom move up, and the ones at the top come down - the water has currents inside it that transfer heat. Another mechanism for transferring heat is through radiation. We have already talked about this while discussing blackbody radiation and the Stefan-Boltzman Law.

Summary of Lecture 37 – THERMAL PHYSICS II

1. Let us agree to call whatever we are studying the "system" (a mixture of ice and water, a hot gas, etc). The state of this system is specified by giving its pressure, volume, temperature, etc. These are called "thermodynamic variables". The relation between these variables is called the "equation of state". This equation relates P,V,T. So, for example:

   $P = f(V,T)$ (knowing V and T gives you P)

   $V = g(P,T)$ (knowing P and T gives you V)

   $T = h(P,V)$ (knowing P and V gives you T)

   For an ideal gas, the equation of state is $PV = Nk_BT$. Here N is the number of molecules in the gas, T is the temperature in degrees Kelvin, and k_B is called the Boltzman cosntant. This EOS is easy to derive, although I shall not do it here. In principle it is possible to mathematically derive the EOS from the underlying properties of the atoms and molecules in the system. In practice, however, this is a very difficult task (except for an ideal gas). However the EOS can be discovered experimentally.

2. Thermodynamics is the study of heat and how it flows. The First Law of Thermodynamics is actually just an acknowledgment that heat is a form of energy (and, of course, energy is always conserved!). Mathematically, the First Law states that:

   $$\Delta E = \Delta q + \Delta w$$

   where: E = internal energy of the system

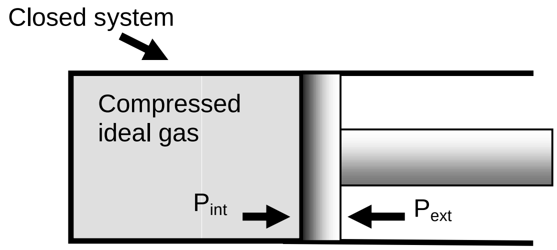
   q = heat transferred to the system from the surroundings

   w = work done on the system by the surroundings.

   In words, what the above formula says is this: if you do an amount of work Δw and also transfer an amount of heat Δq, then the sum of these two quantities will be the additional amount of energy ΔE that is stored in the system. There could be nothing simpler! Note: Δq and Δw are positive if heat is added to or work is done on a system, and negative if heat is removed from the system, or if the system does work on the environment.

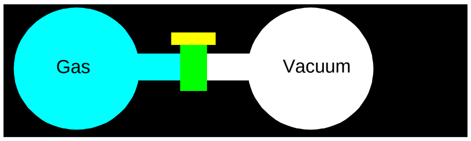
3. Work and heat are called *path variables* - their values depend on the steps leading from one state to another. To give an obvious example: suppose you drag a heavy box on the floor in a straight line from point A to point B, and then in a very roundabout way from A to B again. Obviously the work done by you will be very different. On the other hand, the internal energy does not depend upon the path. Example: a gas has internal energy proportional to its temperature. It makes no difference whether the gas had been slowly heated or rapidly; it will have the same internal energy. Since the internal energy does not depend on the path we can write $\int_{State1}^{State2} dE = E_2 - E_1 = \Delta E.$

4. Let us calculate the work done by an expanding gas. In the diagram below, suppose the piston moves out by distance dx, then the work done is $dw = Fdx$ where F is the force exerted on the gas. But $F = P \times A = -P_{ext} \times A$. (The minus sign is important to understand; remember that the force exerted by the gas on the piston is the negative of the force that the piston exerts on the gas). So $dw = Fdx = -P_{ext} \times A \times dx = -P_{ext} dV$. The work done by the expanding gas is $w = -\int_{V_1}^{V_2} P_{ext} dV$ where V_1 and V_2 are the initial and final volumes.

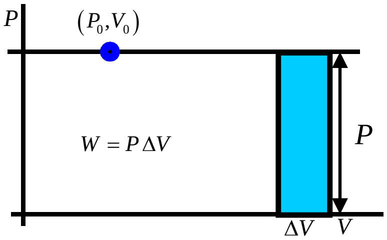


Note that w depends on the path. Let's consider 3 different paths.

a)Suppose that the gas expands into the vacuum, $P_{ext} = 0$. Then $w = 0$. So, if in the figure below, the valve is opened, then gas will flow into the vacuum without doing work.



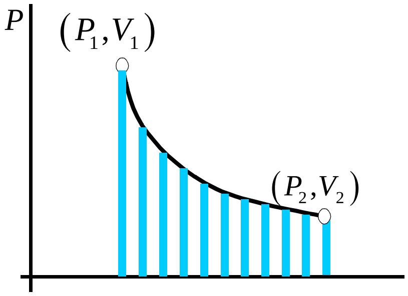
b)Suppose the pressure outside has some constant value. Then, $w = P_{ext} \times (V_2 - V_1)$ or, $w = P_{ext} \Delta V$. You can see that this is just the area of the rectangle below.



c)We can also let the gas expand so that the internal and external pressures are almost the same (this is called reversible expansion). Then the work done *on* the system is: $dw = -P_{ext} dV$. Using $PV = Nk_B T$ gives,

$$w = -\int_{V_1}^{V_2} \frac{Nk_B T}{V} dV = -Nk_B T(\ln V_2 - \ln V_1) = -Nk_B T \ln \frac{V_2}{V_1}$$

Of course, the work done *by* the gas as it expands is positive since V_2 is larger than V_1. Remember that the log function is positive if its argument is bigger than 1.



5. The internal energy of a gas depends only on the number of molecules it contains and on the temperature, $E = \frac{3}{2} Nk_B T$. In a free expansion of gas (it doesn't do any work in this case), the initial and final internal energies are equal, $E_f = E_i$.

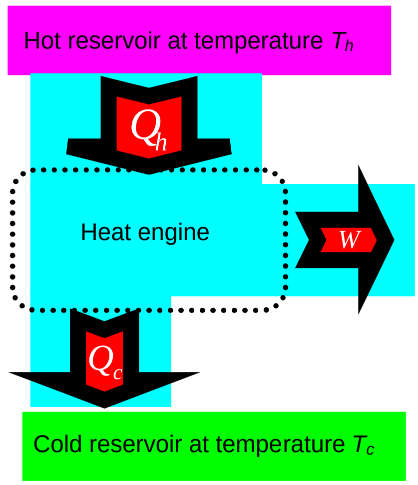
6. Imagine that a substance is heated while keeping its volume constant. Obviously, you have to supply more heat as you raise the temperature higher, $\Delta Q_V = C_V \Delta T,$ where C_V is given the name "specific heat at constant volume". No work is done since there is no expansion, $\Delta W = 0$. Hence, using the First Law, $\Delta Q_V = \Delta E + \Delta W = \Delta E,$ and so $\Delta E = C_V \Delta T$.

7. Now take the same system as above, but allow it to expand at constant pressure as it is heated. Then you will have to supply an amount of heat, $\Delta Q_P = C_P \Delta T$. Here, C_P is called the specific heat at constant pressure, $\Delta Q_P = \Delta E + \Delta w = \Delta E + P\Delta V$. Hence, $C_P \Delta T = \Delta U + P\Delta V$. If we supply only a small amount of heat then, $C_P dT = dU + PdV$. This gives, $C_P dT = C_V dT + PdV$. If we consider the special case of an ideal gas, then $PV = Nk_B T$. Hence, $PdV = Nk_B dT$ and $C_P dT = C_V dT + Nk_B dT$. Hence, $C_P = C_V + Nk_B$. Since the internal energy is $E = \frac{3}{2} Nk_B T$, it follows that $C_V = \frac{dU}{dT} = \frac{3}{2} Nk_B$. From this, the specific heat at constant pressure is: $C_P = \frac{3}{2} Nk_B + Nk_B = \frac{5}{2} Nk_B$. From the equation $C_P = C_V + Nk_B$, it follows that $\frac{Nk_B}{C_V} = \frac{C_P - C_V}{C_V}$. Let us define a new symbol, $\frac{C_P}{C_V} \equiv \gamma$.
8. We can find the EOS of an ideal gas when it expands without losing any heat (this is called adiabiatic expansion). In this case, $dQ = dE + dW = 0$. Hence, $C_V dT + PdV = 0$, or $C_V dT + Nk_B T \frac{dV}{V} = 0$. Dividing by dT gives, $\frac{dT}{T} + \frac{Nk_B}{C_V}\frac{dV}{V} = 0$ or $\frac{dT}{T} + (\gamma - 1)\frac{dV}{V} = 0$. Integrating this gives, $\ln T + (\gamma - 1)\ln V = \text{Constant}$. Equivalently, $\ln\left(TV^{\gamma-1}\right) = \text{Constant}$. A more convenient form is: $TV^{\gamma-1} = \text{Constant}$. This holds for the entire adiabatic expansion.

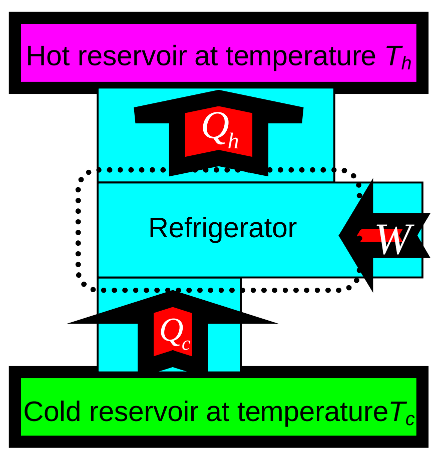
9. Almost nothing beats the importance of the Second Law Of Thermodynamics. There are many equivalent ways of stating it. The one I like is: "There can be no process whose only final result is to transfer thermal energy from a cooler object to a hotter object". This seems extremely simple (and almost useless), but in fact it tells you, among other things, that no perpetual motion machine (such as that which generates electricity without any fuel input) can ever be built. If someone could build such a machine, then it could also be used to transfer heat from a cold object to a hot object - a contradiction with the Second Law.

10. Before we consider some implications of the Second Law, I want to introduce the concept of a *thermal reservoir*. Suppose you put a thermometer in your mouth. You will transfer a small amount of heat to the thermometer, but because your body is so big this will make no measureable difference to your body temperature. Your body therefore is a thermal reservoir as far as the thermometer is concerned. More generally, any large mass of material will act as a thermal reservoir if it is at constant temperature. A *heat engine* (e.g. a car engine, refrigerator, etc) operates between reservoirs at two different temperatures.

11. A heat engine works between a high temperature T_H and a low temperature T_C. It aborbs heat $Q_{in,h}$ from the hot reservoir and rejects heat $Q_{out,c}$ into the cold reservoir. Since the internal energy of the engine does not change, $\Delta U = 0$ and the First Law gives simply, $Q = \Delta U + W = W$. The work done by the engine is $W = Q_{in,h} - Q_{out,c}$. Now define the efficiency of the machine as $\varepsilon = \frac{\text{work done by engine}}{\text{heat put in}} = \frac{W}{Q_{in,h}}$. Then $\varepsilon = \frac{Q_{in,h} - Q_{out,c}}{Q_{in,h}} = 1 - \frac{Q_{out,c}}{Q_{in,h}}$. Now, the heat that is transferred out of or into a reservoir is proportional to its absolute temperature. Hence we arrive at the important result that $\varepsilon = 1 - \frac{T_C}{T_H}$. This number is always less than one, showing that no machine can convert all the input heat into useful work. As an example, a nuclear reactor has temperature 300^0C at the core and rejects heat into a river at 30^0C. The maximum efficiency it can have is $\varepsilon = 1 - \frac{30+273}{300+273} = 0.471$

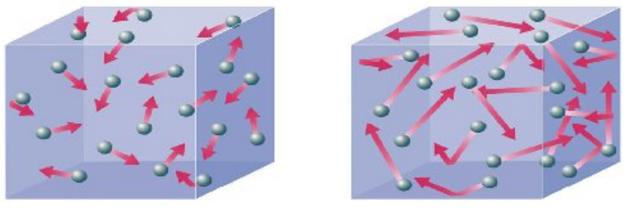


12. A refrigerator is a heat engine working in reverse. Work is done (by an electric motor) to pump heat from a cold reservoir (inside of refrigerator) to the hot exterior. The Second Law can be shown to imply the following: it is impossible for a refrigerator to produce no other effect than the transfer of thermal energy from a cold object to a hot object. Again, the efficiency of a refrigerator is always less than one.



Summary of Lecture 38 – THERMAL PHYSICS III

1. In the two previous lectures, I concentrated exclusively on heat as a form of energy that flows from a hot to a cold body. I made no mention of the fact that all matter is made of atoms and molecules; the notion of heat would exist even if we did not know this. But, the modern understanding of heat is that it is the random kinetic energy of atoms. So, for example, the difference between a cold and hot gas is illustrated below.

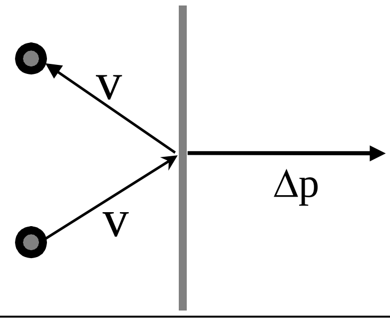


Longer arrows denote atoms that are moving faster.
The hotter gas has, on the average, faster moving atoms.

2. The study of heat, considered as arising from the random motion of the basic constituents of matter, is an area of physics called statistical mechanics. Its goal is the understanding, and prediction of macroscopic phenomena, and the calculation of macroscopic properties from the properties of individual molecules. Temperature is the average energy related to the speed of atoms in an object, and heat is the amount of energy transferred from one object to another.

3. Imagine a gas so dilute that atoms rarely collide with each other (this is also called an ideal gas). We can readily understand why the pressure is directly proportional to the temperature for a gas confined to a box: increasing the temperature T makes gas molecules move faster, striking the walls of the container harder and more often, thus giving an increase in pressure P, i.e. $P \propto T$. On the other hand, for an ideal gas at constant pressure, the volume is directly proportional to the temperature : $V \propto T$. So, as the air in a balloon is heated up, its volume will increase in direct proportion because of the impact of the atoms on the balloon walls. Note that this is where the "absolute" or Kelvin scale comes from: at 0° K an ideal gas would have zero volume because the atoms would not be moving.

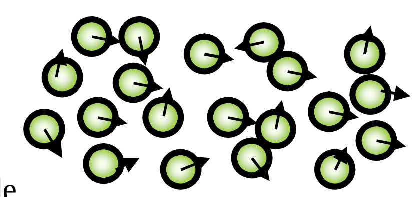
4. Pressure is related to the outward force per unit area exerted by the gas on the container wall. Newton's 2nd Law is: $F = \frac{\Delta p}{\Delta t}$. Here Δp is to be understood as the momentum destroyed in time Δt. Using this, we can do a little calculation to find the pressure in an ideal gas.



Suppose there are N atoms in a box of volume, and the average speed of an atom in the x direction is v_x. By symmetry, half are moving in the $+x$ and half in the $-x$ directions. So the number of atoms that hit the wall in time Δt is $\frac{N}{2}\frac{1}{V}\mathrm{v}_x\Delta tA$. Hence the change in momentum is $\Delta p = (2m\mathrm{v}_x)\times\left(\frac{1}{2}\frac{N}{V}\mathrm{v}_x\Delta tA\right) = \frac{N}{V}m\mathrm{v}_x^2A\Delta t$. From this, we can calculate the pressure $P = \frac{F}{A} = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{N}{V}m\mathrm{v}_x^2$. Hence $PV = Nm\mathrm{v}_x^2 = 2N\left(\frac{1}{2}m\mathrm{v}_x^2\right)_{av}$.

5. Let us return to our intuitive understanding of heat as the random energy of small particles. Now, $\left(\frac{1}{2}m\mathrm{v}_x^2\right)_{av}$ is the average kinetic energy of a particle on account of its motion in the x direction. This will be greater for higher temperatures, so $\left(\frac{1}{2}m\mathrm{v}_x^2\right)_{av} \propto T$, where T is the absolute temperature. So we write $\left(\frac{1}{2}m\mathrm{v}_x^2\right)_{av} = k_BT$, which you can think of as the definition of the Boltzmann constant, k_B. So we immediately get $PV = Nk_BT$, the equation obeyed by an ideal gas (dilute, no collisions). Now, there is nothing special about any particular direction, so $\left(\mathrm{v}_x^2\right)_{av} = \left(\mathrm{v}_y^2\right)_{av} = \left(\mathrm{v}_z^2\right)_{av}$ and therefore the average total velocity is $\left(\mathrm{v}^2\right)_{av} = \left(\mathrm{v}_x^2\right)_{av} + \left(\mathrm{v}_y^2\right)_{av} + \left(\mathrm{v}_z^2\right)_{av} = 3\left(\mathrm{v}_x^2\right)_{av}$. So the total average kinetic energy of one atom is, $K_{av} = \left(\frac{1}{2}m\mathrm{v}^2\right)_{av} = \frac{3}{2}k_BT$, and the energy of the entire gas is $K = N\left(\frac{1}{2}m\mathrm{v}^2\right)_{av} = \frac{3}{2}Nk_BT$. From this, the mean squared speed of atoms in a gas at temperature is $\left(\mathrm{v}^2\right)_{av} = \frac{3k_BT}{m}$.

6. Heat, which is random kinetic energy, causes changes of phase in matter:
   a) Water molecules attract each other, but if water is heated then they can escape and water becomes steam.
   b) If steam is heated, the water molecules break up and water becomes separated hydrogen and oxygen atoms.
   c) If a gas of atoms gets hot enough, then the atoms collide so violently that they lose electrons (i.e. get ionized).
   d) More heat (such as inside the core of a star) will break up atomic nuclei into protons and neutrons.



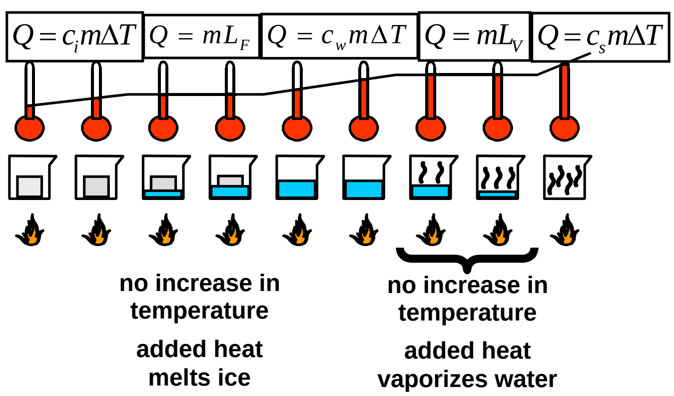
7. Evaporation: if you leave water in a glass, it is no longer there after some time. Why? Water molecules with high enough KE can break free from the water and escape. But how does a water molecule get enough energy to escape? This comes from random inter-molecular collisions, such as illustrated below. Two molecules collide, and one slows down while the other speeds up. The faster molecule might escape the water's surface.

8. Heat needs to be supplied to cause a change of phase:

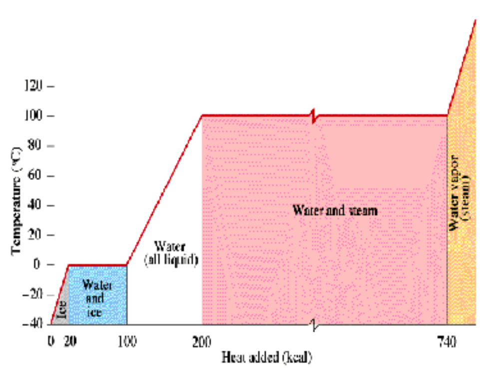
a) To melt a solid (ice, wax, iron) we must supply the "latent heat of fusion" L_F for unit mass of the solid. For mass m we must supply an amount of heat $Q = mL_F$ to overcome the attraction between molecules. No temperature change occurs in the melting.

b) To vapourize a substance (convert to gas or vapour), we must supply the "latent heat of vapourization" L_V for unit mass of the substance. For mass m we must supply an amount of heat $Q = mL_V$. No temperature change occurs in the vapourization .

c) To raise the temperature of a substance we must supply the "specific heat" C for unit mass of the substance by 1 degree Kelvin. To change the temperature of mass m by ΔT, we must supply heat $Q = mC\Delta T$.



In the above diagram you see what happens as ice is heated. The amount of heat needed is indicated as well. Note that no change of temperature happens until the phase change is completed. The same physical situation is represented to the right as well where the temperature is plotted against Q.



9. A very important concept is that of *entropy*. It was first introduced in the context of thermodynamics. Then, when the statistical nature of heat became clear a century later, it was understood in very different terms. Let us begin with thermodynamics: suppose a system is at temperature T and a small amount of heat dQ is added to it. Then, the small increase in the entropy of the system is $dS = \frac{dQ}{T}$. What if we keep adding little bits of heat? Then the temperature of the system will change by a finite amount, and the change in entropy is got by adding up all the small changes: $\Delta S = \int_{T_1}^{T_2} \frac{dQ}{T}$.

   Now let's work out how much the entropy changes when we add heat dQ to an ideal gas. From the First Law, $dQ = dU + dW = dU + PdV = C_V dT + Nk_B T\frac{dV}{V}$. The entropy change is $dS = \frac{dQ}{T} = C_V\frac{dT}{T} + Nk_B\frac{dV}{V}$, and for a finite change we simply integrate,

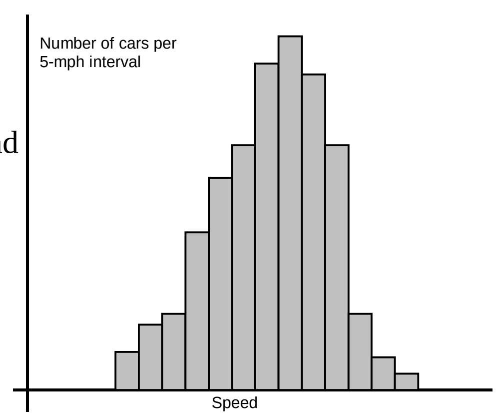
   $\Delta S = \int\frac{dQ}{T} = C_V \ln\frac{T_2}{T_1} + Nk_B \ln\frac{V_2}{V_1} = \frac{3}{2}Nk_B \ln\frac{T_2}{T_1} + Nk_B \ln\frac{V_2}{V_1}$. Note that the following:

   a)The entropy increases if we heat a gas ($T_2 > T_1$).

   b)The entropy increases if the volume increases ($V_2 > V_1$).

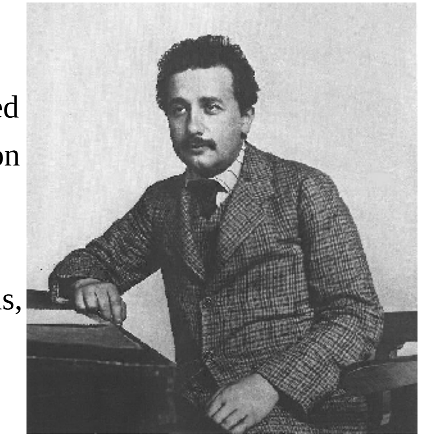
10. In statistical mechanics, we interpret entropy as the degree of disorder. A gas with all atoms at rest is considered ordered, while a hot gas having atoms buzzing around in all directions is more disordered and has greater entropy. Similarly, as in the above example, when a gas expands and occupies greater volume, it becomes even more disordered and the entropy increases.

11. We always talk about averages, but how do you define them mathematically? Take the example of cars moving along a road at different speeds. Call $n(\mathrm{v}_i)$ the number of cars with speed v_i , then $N = \sum_i n(\mathrm{v}_i)$ is the total number of cars and the average speed is defined as $\overline{\mathrm{v}} = \frac{\sum_i \mathrm{v}_i n(\mathrm{v}_i)}{\sum_i n(\mathrm{v}_i)}$.



   We define the probability of finding a car with speed v_i as $P(\mathrm{v}_i) \equiv \frac{n(\mathrm{v}_i)}{\sum_i n(\mathrm{v}_i)} = \frac{n(\mathrm{v}_i)}{N}$. We can also write the average speed as, $\overline{\mathrm{v}} = \sum_i \mathrm{v}_i P(\mathrm{v}_i)$.

Summary of Lecture 39 – SPECIAL RELATIVITY I

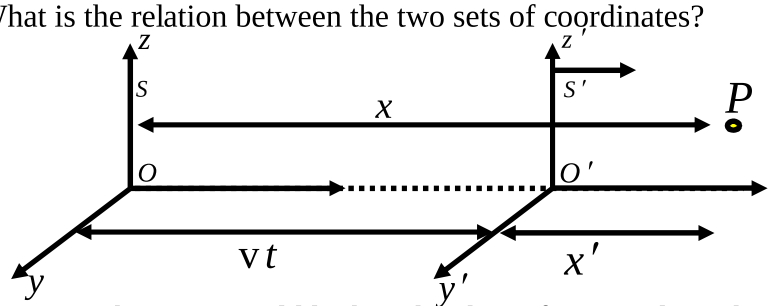


1. The Special Theory of Relativity was created by Albert Einstein, when he was a very young man in 1905. It is one of the most solid pillars of physics and has been tested thousands of times. Special Relativity was a big revolution in understanding the nature of space and time, as well as mass and energy. It is absolutely necessary for a proper understanding of fast-moving particles (electrons, photons, neutrinos..). Einstein's General Theory of Relativity - which we shall not even touch here - goes beyond this and also deals with gravity.

2. Relativity deals with time and space. So let us first get some understanding of how we measure these two fundamental quantities:
   a) *Time* **:** we measure time by looking at some phenomenon that repeats itself. There are endless examples: your heart beat, a pendulum, a vibrating quartz crystal, rotation of the earth around its axis, the revolution of the earth about the sun,... These can all be used as clocks. Of course, an atomic clock is far more accurate than using your heart beat and is accurate to one part in a trillion. Although different systems of measurement have different units it is fortunate that time is always measured in seconds.
   b) *Distance* **:** intuitively we know the difference between short and long. But to do a measurement, we first have to agree on what should be the unit of length. If you use a metre as the unit, then you can use a metre rod and measure any length you want. Of course, sometimes we may use more sophisticated means (such as finding how high a satellite is) but the basic idea is the same: the distance between point A and point B is the number of metre rods (or fractions thereof) that can be made to fit in between the two points.

3. Newton had believed that there was one single time for the entire universe. In other words, time was absolute and could be measured by one clock held somewhere in the centre of the universe. Similarly, he believed that space was absolute and that the true laws of physics could be seen in that particular frame which was fixed to the centre of the universe. (As we shall see, Einstein shocked the world by showing that time and space are not absolute quantities, but depend on the speed of your reference frame. Even more shocking was his proof that time and space are not entirely separate quantities!)

4. An *event* is something that happens at some point in space at some time. With respect to the frame S below, the event P happened at (x,y,z,t). Now imagine a girl running to the right at fixed speed v in frame S′. According to her, the same event P happened at (x′,y′,z′,t′). What is the relation between the two sets of coordinates?



If you were Newton, then you would look at the above figure and say that obviously it is the following: $x' = x - \mathrm{v}t,\ y' = y,\ z' = z,\ t' = t$. These are called *Galilean coordinate transformations*. Note that it is assumed here that the time is the same in both frames because of the Newtonian belief that there is only one true time in the world.

5. There are certain obvious consequences of using the Galilean transformations. So, for example a rod is at rest in S-frame. The length in S-frame $= x_B - x_A,$ while the length in the S′-frame $= x'_B - x'_A = x_B - x_A - \mathrm{v}(t_B - t_A)$. Since $t_B = t_A,$ the length is the same in both frames: $x'_B - x'_A = x_B - x_A$. (As we shall see later this will not be true in Einstein's Special Relativity).

6. Let us now see what the Galilean transformation of coordinates implies for transformations of velocities. Start with $x' = x - \mathrm{v}t$ and differentiate both sides. Then $\frac{dx'}{dt} = \frac{dx}{dt} - \mathrm{v}$. Since $t = t',$ it follows that $\frac{dx'}{dt} = \frac{dx'}{dt'}$ and so $\frac{dx'}{dt'} = \frac{dx}{dt} - \mathrm{v}$. Similarly, $\frac{dy'}{dt'} = \frac{dy}{dt}$ and $\frac{dz'}{dt'} = \frac{dz}{dt}$.
Now, $\frac{dx'}{dt'} = u'_x$ is the x-component of the velocity measured in S′-frame, and similarly $\frac{dx}{dt} = u_x$ is the x-component of the velocity measured in S-frame. So we have found that:

$$u'_x = u_x - \mathrm{v},\quad u'_y = u_y,\quad u'_z = u_z \ \text{(or, in vector form, } \vec{u}' = \vec{u} - \vec{\mathrm{v}})$$

Taking one further derivative, $\frac{du'_x}{dt'} = \frac{d}{dt}(u_x - \mathrm{v}) = \frac{du_x}{dt}$ (remember that $\mathrm{v} =$ constant), we find that the components of acceleration are the same in S and S′:

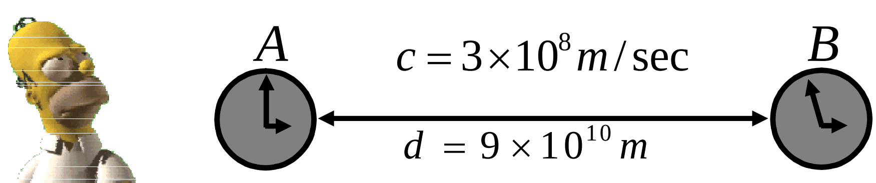
$$\frac{du'_x}{dt'} = \frac{du_x}{dt},\quad \frac{du'_y}{dt'} = \frac{du_y}{dt},\quad \frac{du'_z}{dt'} = \frac{du_z}{dt}.$$

7. We had learned in a previous chapter that light is electromagnetic wave that travels at a speed measured to be $c = 2.907925 \times 10^8 m/\sec$. Einstein, when he was 16 years old, asked himself the question: in which frame does this light travel at such a speed? If I run holding a torch, will the light coming from the torch also move faster? Yes, if we use the formula for addition of velocities derived in the previous section! So if light actually travels at $2.907925 \times 10^8 m/\sec$ then it is with respect to the frame in which the "aether" (a massless fluid which we cannot feel) is at rest. But there is no evidence for the aether!

8. Einstein made the two following postulates (or assumptions).
   1) The laws of physics have the same form in all inertial frames. (In other words, there is no constant-velocity frame which is preferred, or better, than any other).
   2) The speed of light in vacuum has the same value in all inertial systems, independent of the relative motion of source and observer.

   [Note: as stressed in the lecture, no postulate of physics can ever be mathematically proved. You have to work out the consequences that follow from the postulates to know whether the postulates are good ones or not.]

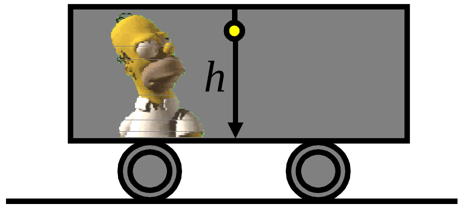
9. Let's first get one thing clear: in any one inertial frame, we can imagine that there are rulers and clocks to measure distances and times. We can synchonize all the clocks to read one time, which will be called the time in that frame S. But how do we do this? If are two clocks, then we can set them to read the same time (i.e. synchronize them) by taking account of the time light takes to travel between them.
   Example: The observer with clock A sees the time on clock B as 2:55pm. But he knows that light took 5 minutes to travel from B to A, and therefore A and B are actually reading exactly the same time.



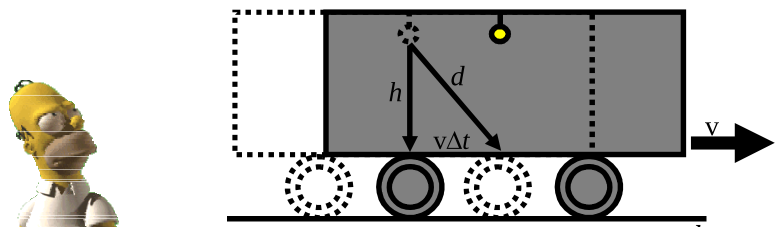
(Time taken by light in going from B to A is equal to $\frac{d}{c} = \frac{9 \times 10^{10} m}{3 \times 10^8 m/\sec} = 300\sec = 5\min$.)

10. Now I shall derive the famous formula which shows that a moving clock runs slow. This will be something completely different from the older Newtonian conception of time. Einstein derived this formula using a "gedanken" experiment, meaning an experiment which can imagine but not necessarily do. So imagine the following: a rail carriage has a

a bulb that is fixed to the ceiling. The bulb suddenly flashes, and the light reaches the floor. Time taken according to the observer inside the train is $\Delta t' = h/c$.



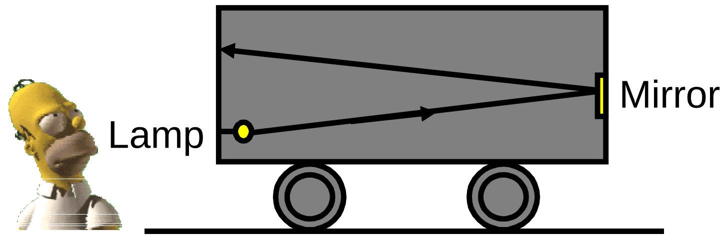
Now suppose that the same flash is observed by an observer S standing on the ground. According to S, the train is moving with speed v. Let's consider the same light ray. Clearly, the train has moved forward between the time when the light left the ceiling and when it reached the floor. According to S, the time taken is $\Delta t = \frac{d}{c} = \frac{\sqrt{h^2 + (\mathrm{v}\Delta t)^2}}{c}$. Now



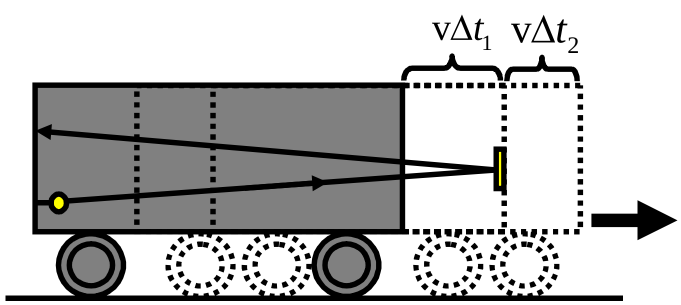
square both sides: $(c\Delta t)^2 = h^2 + (\mathrm{v}\Delta t)^2$, which gives $\Delta t = \frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\frac{h}{c} = \gamma \Delta t'$. Here γ is the relativistic factor, $\gamma = \frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}$ and is a number that is always bigger than one. As v gets closer and closer to c, the value of γ gets larger and larger. For v=4c/5, γ=5/3. So, if 1 sec elapses between the ticks of a clock in S′ (i.e. $\Delta t' = 1$), the observer in S will see 5/3 seconds between the ticks. In other words, he will think that the moving clock is slow!

11. The muon is an unstable particle. If at rest, it decays in just 10^{-18} seconds. But if traveling at 3/5 the speed of the light, it will last 25% longer because $\gamma = \frac{1}{\sqrt{1-(3/5)^2}} = \frac{5}{4}$. If it is traveling at v=0.999999c, it will last 707 times longer. We can observe these shifts due to time dilation quite easily, and they are an important confirmation of Relativity.

12. Another amazing prediction of Relativity is that objects are shortened (or contracted) along the direction of their motion. Einstein reached this astonishing conclusion on the basis of yet another gedanken experiment. Again, consider a moving railway carriage with a bulb



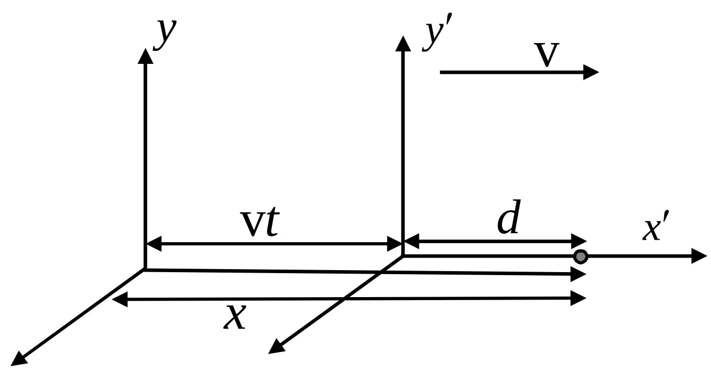
at one end that suddenly flashes. Let Δx be the length of the carriage according to ground observer S, and $\Delta x'$ be the length according to the observer S′ inside the carriage. So, $\Delta t' = \frac{2\Delta x'}{c}$ is the time taken for the light to go from one end to the other, and then return after being reflected by a mirror. Now let's look at this from the point of view of S, who is fixed to the ground. Let Δt_1 be the time for the signal to reach the front end. Then, because the mirror is moving forward, $\Delta t_1 = \frac{\Delta x + \text{v}\Delta t_1}{c}$. Call Δt_2 the return time. Then,



$\Delta t_2 = \frac{\Delta x - \text{v}\Delta t_2}{c}$. Solving for Δt_1 and Δt_2: $\Delta t_1 = \frac{\Delta x}{c - \text{v}}$ and $\Delta t_2 = \frac{\Delta x}{c + \text{v}}$. The total time is therefore $\Delta t = \Delta t_1 + \Delta t_2 = \frac{2\Delta x / c}{\left(1 - \text{v}^2/c^2\right)}$. Now, from the time dilation result derived earlier, $\Delta t' = \sqrt{1 - \text{v}^2/c^2}\,\Delta t$, and so $\Delta t' = \sqrt{1 - \text{v}^2/c^2}\left(\frac{2\Delta x / c}{\left(1 - \text{v}^2/c^2\right)}\right) = \frac{2\Delta x / c}{\sqrt{1 - \text{v}^2/c^2}}$. This gives, $\Delta x' = \frac{\Delta x}{\sqrt{1 - \text{v}^2/c^2}}$, or $\Delta x = \Delta x'/\gamma$. This an astonishing result! Suppose that there is a metre rod. Then the observer riding with it has $\Delta x' = 1$. But according to someone who sees the metre rod moving towards/away from him, the length is $1/\gamma$. This is less than 1 metre!

13. Although an object shrinks in the direction of motion (both while approaching and receding), the dimensions perpendicular to the velocity are not contracted. It is easy to conceive of a gedanken experiment that will demonstrate this. One of the exercises will guide you in this direction.

14. We shall now derive the "Lorentz Transformation", which is the relativistic version of the Galilean transformation discussed earlier. Consider an event that occurs at position x (as measured in S) at a distance d (again, as seen in S) from the origin of S′. Then, $x = d + \text{v}t$. From the Lorentz contraction formula derived earlier, $d = \frac{x'}{\gamma}$ where x' is the distance measured in S′. This gives $x' = \gamma\left(x - \text{v}t\right)$. Now, by the same logic, $x' = d' - \text{v}t'$.



Here $d' = \frac{x}{\gamma}$. This gives $x = \gamma\left(x' + \text{v}t'\right) = \gamma\left(\gamma\left(x - \text{v}t\right) + \text{v}t'\right)$. We find that the time in S′ is related to the time in S by $t' = \gamma\left(t - \frac{\text{v}}{\text{c}^2}x\right)$. Note that if we make c very large, then $t' = t$.

To summarize:

LORENTZ TRANSFORMATION

$$x' = \gamma\left(x - \text{v}t\right)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{\text{v}}{c^2}x\right).$$

(Note: in various books you will find slightly different derivations of the above Lorentz transformation. You should look at one of your choice and understand that as well.)

Summary of Lecture 40 – SPECIAL RELATIVITY II

1. Recall the Lorentz Transformation: $x' = \gamma\left(x - \text{v}t\right)$ and $t' = \gamma\left(t - \frac{\text{v}}{c^2}x\right)$. Suppose we take the space interval between two events $\Delta x = x_1 - x_2$, and the time interval $\Delta t = t_1 - t_2$.

   Then, these intervals will be seen in S′ as $\Delta t' = \gamma\left(\Delta t - \frac{\text{v}}{c^2}\Delta x\right)$ and $\Delta x' = \gamma\left(\Delta x - \text{v}\Delta t\right)$.

   Now consider two particular cases:

   a) Suppose the two events occur at the same place (so $\Delta x = 0$) but at different times (so $\Delta t \neq 0$). Note that in S′ they do not occur at the same point: $\Delta x' = \gamma\left(0 - \text{v}\Delta t\right)$!

   b) Suppose the two events occur at the same time (so $\Delta t = 0$) but at different places (so $\Delta x \neq 0$). Note that in S′ they are not simultaneous: $\Delta t' = \gamma\left(0 - \frac{\text{v}}{c^2}\Delta x\right)$.

2. As seen in the frame S, suppose a particle moves a distance dx in time dt. Its velocity u is then $u = \frac{dx}{dt}$ (in S-frame). As seen in the S′-frame, meanwhile, it has moved a distance dx' where $dx' = \gamma\left(dx - \text{v}dt\right)$ and the time that has elapsed is $dt' = \gamma\left(dt - \frac{\text{v}}{c^2}dx\right)$. The velocity in S′-frame is therefore $u' = \frac{dx'}{dt'} = \frac{\gamma\left(dx - \text{v}dt\right)}{\gamma\left(dt - \frac{\text{v}}{c^2}dx\right)} = \frac{dx/dt - \text{v}}{1 - \frac{\text{v}}{c^2}dx/dt} = \frac{u - \text{v}}{1 - \frac{u\text{v}}{c^2}}$. This is the Einstein velocity addition rule. It is an easy exercise to solve this for u in terms of u',

$$u = \frac{u' + \text{v}}{1 + \frac{u'\text{v}}{c^2}}.$$

3. Note one very interesting result of the above: suppose that a car is moving at speed v and it turns on its headlight. What will the speed of the light be according to the observer on the ground? If we use the Galilean transformation result, the answer is v+c (wrong!). But using the relativistic result we have $u' = c$ and $u = \frac{c + \text{v}}{1 + \frac{c\text{v}}{c^2}} = \frac{c + \text{v}}{1 + \frac{\text{v}}{c}} = c\frac{c + \text{v}}{c + \text{v}} = c$. In other words, the speed of the source makes no difference to the speed of light in your frame.
   Note that if either u or v is much less than $c,$ then u' reduces to the familiar result:
   $u' = \frac{u - \text{v}}{1 - \frac{u\text{v}}{c^2}} \rightarrow u - \text{v}$, which is the Galilean velocity addition rule.

4. The Lorentz transformations have an interesting property that we shall now explore. Take the time and space intervals between two events as observed in frame S, and the corresponding quantities as observed in S′. We will now prove that the quantities defined respectively as $I = (c\Delta t)^2 - (\Delta x)^2$ and $I' = (c\Delta t')^2 - (\Delta x')^2$ are equal. Let's start with I':

$$I' = (c\Delta t')^2 - (\Delta x')^2 = \gamma^2\left((c\Delta t)^2 + (\Delta x)^2 \mathrm{v}^2/c^2 - \cancel{2\mathrm{v}\Delta t\Delta x} - (\Delta x)^2 - (\mathrm{v}\Delta t)^2 + \cancel{2\mathrm{v}\Delta t\Delta x}\right)$$

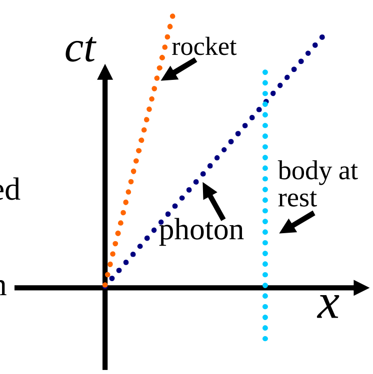
$$= \frac{1}{\left(1-\mathrm{v}^2/c^2\right)}\left(\left(c^2-\mathrm{v}^2\right)(\Delta t)^2 - (\Delta x)^2\left(1-\mathrm{v}^2/c^2\right)\right)$$

$$= (c\Delta t)^2 - (\Delta x)^2 = I$$

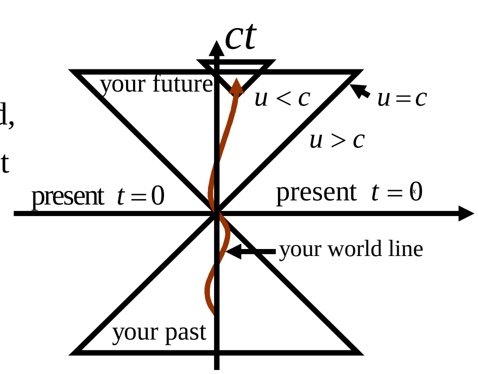
This guarantees that all inertial observers measure the same speed of light !!

5. If the time separation is large, then $I > 0$ and we call the interval *timelike.*
   If the space separation is large, then $I < 0$ and we call the interval *spacelike.*
   If $I = 0$ and we call the interval *lightlike.*
   Note : a) If an interval is timelike in one frame, it is timelike in all other frames as well.
   b) If interval between two events is timelike, their time ordering is absolute.
   c) If the interval is spacelike the ordering of two events depends on the frame from which they are observed.

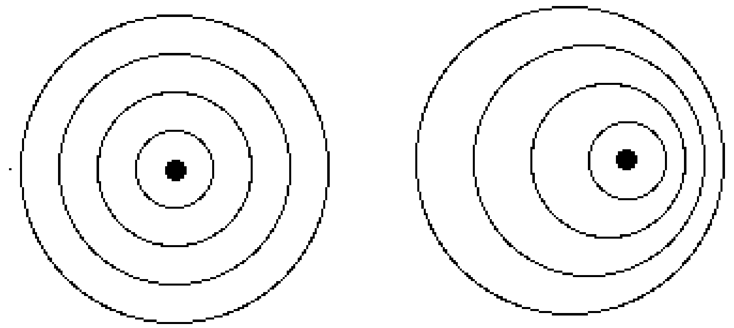
6. It is sometimes nice to look at things graphically. Here is a graph of position versus time for objects that move with different speeds along a fixed direction. First look at an object at rest. Its position is fixed even though time (plotted on the vertical axis) keeps increasing. Then look at the rocket moving at constant speed (which has to be less than c), and finally a photon (which can only move at c).



7. The trajectory of a body as it moves through space-time is called its world-line. Let's take a rocket that is at $x = 0$ at $t = 0$. It moves with non-constant speed, and that is why its world-line is wavy. A photon that moves to the right will have a world line with slope equal to +1, and that to the right with slope -1. The upper triangle (with t positive) is called the future light cone (don't forget we also have y and x). The lower light cone consists of past events.



8. Earlier on we had discussed the Doppler effect in the context of sound. Now let us do so for light. Why are they different? Because light always moves with a fixed speed while the speed of sound is different according to a moving and a fixed observer. Consider the case of a source of light at rest, and one that is moving to the right as shown below:



(a) stationary source (b) moving source

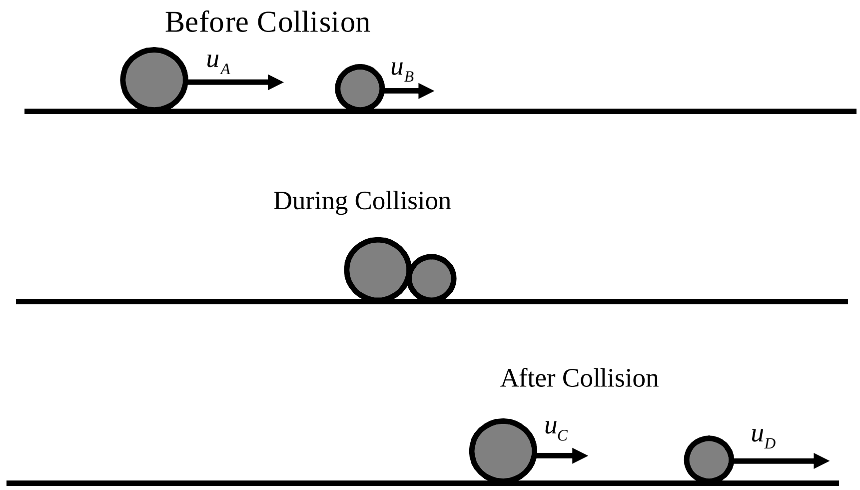
Let $\nu_0 = \dfrac{1}{T_0}$ be the frequency measured in the source's rest frame S, where T_0 is the time for one complete cycle in S. We want to calculate ν, the frequency as seen by the observer in S′ moving to the right at speed v. Call λ the distance between two successive wave crests (i.e. the wavelength according to the observer). In time T the crests ahead of the source move a distance cT, even as the source moves a shorter distance vT in the same direction. Hence $\lambda = (c - \text{v})T$ and so $\nu = \dfrac{c}{\lambda} = \dfrac{c}{(c-\text{v})T}$. Now, as discussed earlier, the time measured by observer will not be T_0 because of time dilation. Instead,

$T = \dfrac{T_0}{\sqrt{1-\text{v}^2/c^2}} = \dfrac{cT_0}{\sqrt{c^2-\text{v}^2}}$. Hence, $\nu = \dfrac{c}{(c-\text{v})T} = \left(\dfrac{c}{c-\text{v}}\right)\dfrac{\sqrt{c^2-\text{v}^2}}{c}\nu_0 = \sqrt{\dfrac{c+\text{v}}{c-\text{v}}}\nu_0.$

(If source moves away from the observer just change the sign of v: $\nu = \sqrt{\dfrac{c-\text{v}}{c+\text{v}}}\nu_0$.)

This is the famous Doppler effect formula. In the lecture I discussed some applications such as finding the speed at which stars move away from the earth, or finding the speed of cars or aircraft.

9. We now must decide how to generalize the concept of momentum in Relativity theory. The Newtonian definition of momentum is $\vec{p} = m\vec{u}$. The problem with this definition is that we are used to having momentum conserved when particles collide with each other, and this old definition will simply not work when particles move very fast. Consider the collision of two particles as in the diagrams below:



In the frame fixed to the lab (S-frame) conservation of momentum implies: $m_A u_A + m_B u_B = m_C u_C + m_D u_D$. Mass is also conserved: $m_A + m_B = m_C + m_D$. Now suppose we wish to observed the collision from a frame S′ moving at speed v. Then, from the relativistic addition of velocities formula, $u' = \dfrac{u - \text{v}}{1 - \dfrac{u\text{v}}{c^2}}$ and, from it, $u = \dfrac{u' + \text{v}}{1 + \dfrac{u'\text{v}}{c^2}}$. Insert this into the equation of conservation of momentum (using $p = m\text{v}$ as the definition):

$$m_A\left(\frac{u'_A + \text{v}}{1 + u'_A\text{v}/c^2}\right) + m_B\left(\frac{u'_B + \text{v}}{1 + u'_B\text{v}/c^2}\right) = m_C\left(\frac{u'_C + \text{v}}{1 + u'_C\text{v}/c^2}\right) + m_D\left(\frac{u'_D + \text{v}}{1 + u'_D\text{v}/c^2}\right)$$

This is clearly not the equation $m_A u'_A + m_B u'_B = m_C u'_C + m_D u'_D$ So momentum will not be conserved relativistically if we insist on using the old definition!!

10. Can we save the situation and make the conservation of momentum hold by finding some suitable new definition of momentum? The new definition must have two properties:

1) At low speeds it must reduce to the old one.

2) At all speeds momentum must be conserved.

Let's see if the definition $\vec{p} = \dfrac{m\vec{u}}{\sqrt{1 - u^2/c^2}} = \gamma m\vec{u}$ will do the job. Obviously if u<<c we get $\vec{p} = m\vec{u}$, so requirement 1 is clearly satisfied. . Let's now see if the conservation of momentum equation will hold if the new definition of momentum is used:

$$m_A\gamma_A u_A + m_B\gamma_B u_B = m_C\gamma_C u_C + m_D\gamma_D u_D.$$

After doing some algebra you find,

$$m_A\left(\frac{1}{\gamma}\gamma'_A u'_A + \gamma_A\text{v}\right) + m_B\left(\frac{1}{\gamma}\gamma'_B u'_B + \gamma_B\text{v}\right) = m_C\left(\frac{1}{\gamma}\gamma'_C u'_C + \gamma_C\text{v}\right) + m_D\left(\frac{1}{\gamma}\gamma'_D u'_D + \gamma_D\text{v}\right)$$

This gives $m_A\gamma'_A u'_A + m_B\gamma'_B u'_B = m_C\gamma'_C u'_C + m_D\gamma'_D u'_D$, which is just what we want.

11. We shall now consider how energy must be redefined relativistically. The usual expression for the kinetic energy $K = \frac{1}{2}mu^2$ is not consistent with relativistic mechanics (it does not satisfy the law of conservation of energy in relativity). To discover a new definition, let us start from the basics: the work done by a force F when it moves through distance dx adds up to an increase in knetic energy, $K = \int F dx = \int \frac{dp}{dt} dx = \int \frac{dx}{dt} dp = \int u dp$. Now use:

$$dp = md\left(\frac{u}{\sqrt{1-u^2/c^2}}\right) = \frac{mdu}{\sqrt{1-u^2/c^2}} + \frac{m\left(u^2/c^2\right)du}{\left(1-u^2/c^2\right)^{3/2}} = \frac{mudu}{\left(1-u^2/c^2\right)^{3/2}}$$

This give $K = \int_0^u u dp = m\int_0^u \frac{udu}{\left(1-u^2/c^2\right)^{3/2}} = \frac{mc^2}{\sqrt{1-u^2/c^2}} - mc^2$

Note that K is zero if there is no motion, or $K = E - E_0$ where $E = \frac{mc^2}{\sqrt{1-u^2/c^2}}$ and $E_0 = mc^2$. Now expand $\frac{1}{\sqrt{1-u^2/c^2}} = \left(1-u^2/c^2\right)^{-1/2} = 1 + \frac{u^2}{2c^2} + \cdots$

Hence, $K = mc^2\left(1 + \frac{u^2}{2c^2} + \cdots\right) - mc^2 \rightarrow \frac{1}{2}mu^2$ as $u/c \rightarrow 0$.

12. Now that we have done all the real work, let us derive some alternative expressions using our two main formulae: $\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}} = \gamma m\vec{u}$ and $E = \frac{mc^2}{\sqrt{1-u^2/c^2}} = \gamma mc^2$.

a) $\vec{u} = \frac{\vec{p}}{\gamma m} = \frac{\vec{p}c^2}{E}$ or $pc = E\left(\frac{u}{c}\right)$

b) Clearly $(pc)^2 = E^2\left(\frac{u}{c}\right)^2 = \gamma^2 m^2 c^4\left(1 - 1/\gamma^2\right) = \gamma^2 m^2 c^4 - m^2 c^4 = E^2 - m^2 c^4$. Hence,

$E^2 = p^2c^2 + m^2c^4$.

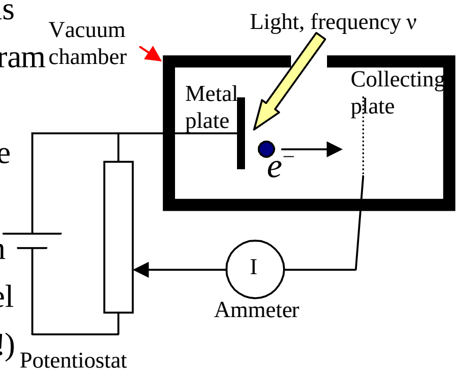
c) For a massless particle ($m = 0$), $E = pc$.

13. A particle with mass m has energy mc^2 even though it is at rest. This is called its rest energy. Since c is a very large quantity, even a small m corresponds to a very large energy. We interpret this as follows: suppose all the mass could somehow be converted into energy. Then an amount of energy equal to mc^2 would be released.

Summary of Lecture 41 – WAVES AND PARTICLES

1. We think of particles as matter highly concentrated in some volume of space, and of waves as being highly spread out. Think of a cricket ball, and of waves in the ocean. The two are completely different! And yet today we are convinced that matter takes the form of waves in some situations and behaves as particles in other situations. This is called wave-particle duality. But do not be afraid - there is no logical contradiction here! In this lecture we shall first consider the evidence that shows the particle nature of light.

2. The photoelectric effect, noted nearly 100 years ago, was crucial for understanding the nature of light. In the diagram shown, when light falls upon a metal plate connected to the cathode of a battery, electrons are knocked out of the plate. They reach a collecting plate that is connected to the battery's anode, and a current is observed. A vacuum is created in the apparatus so that the electrons can travel without hindrance. According to classical (meaning old!) physics we expect the following:

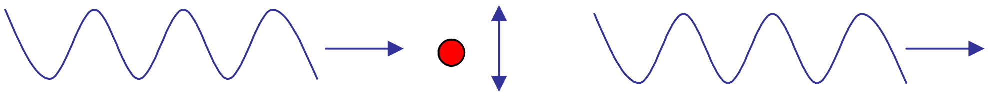


   a)As intensity of light increases, the kinetic energy of the ejected electrons should increase.
   b)Electrons should be emitted for any frequency of light ν, so long as the intensity of the light is sufficiently large.

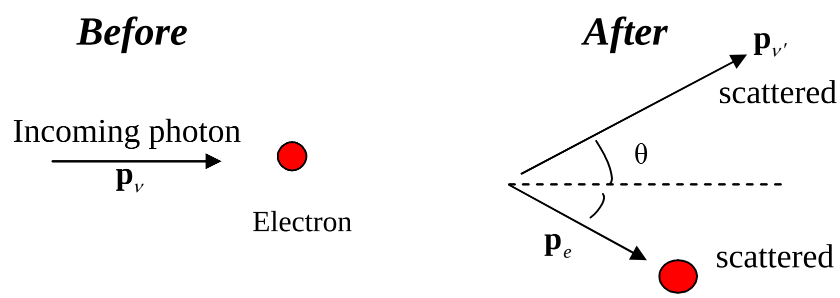
   But the actual observation was completely different and showed the following:
   a)The maximum kinetic energy of the emitted electrons was completely independent of the light intensity, but it did depend on ν.
   b)For $\nu < \nu_0$ (i.e. for frequencies below a cut-off frequency) no electrons are emitted no matter how large the light intensity was.

3. In 1905, Einstein realized that the photoelectric effect could be explained if light actually comes in little packets (or quanta) of energy with the energy of each quantum $E = h\nu$. Here h is a universal constant of nature with value $h = 6.63\times10^{-34}$ Joule-seconds, and is known as the Planck Constant. If an electron absorbs a single photon, it would be able to leave the material if the energy of that photon is larger than a certain amount W. W is called the work function and differs from material to material, with a value varying from 2-5 electron volts. The maximum KE of an emitted electron is then $K_{\max} = h\nu - W$. We visualize the photon as a particle for the purposes of this experiment. Note that this is completely different from our earlier understanding that light is a wave!

4. That light is made of photons was confirmed by yet another experiment, carried out by Arthur Compton in 1922. Suppose an electron is placed in the path of a light beam. What will happen? Because light is electromagnetic waves, we expect the electron to oscillate with the same frequency as the frequency of the incident light ν. But because a charged particle radiates em waves, we expect that the electron will also radiate light at frequency ν. So the scattered and incident light have the same ν. But this is not what is observed!

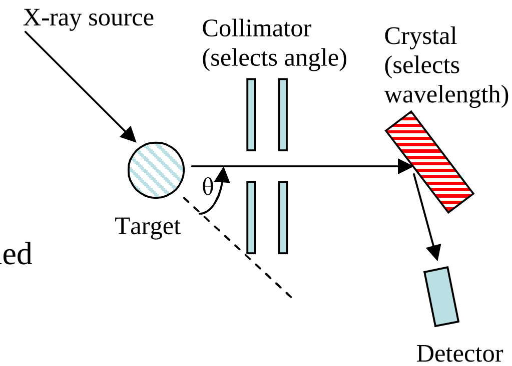


To explain the fact that the scattered light has a different frequency (or wavelength), Compton said that the scattering is a collision between particles of light and electrons. But we know that momentum and energy is conserved in scattering between particles. Specifically, from conservation of energy $h\nu + m_e c^2 = h\nu' + \left(p_e^2 c^2 + m_e^2 c^4\right)^{1/2}$. The last term is the energy of the scattered electron with mass m_e. Next, use the conservation



of momentum. The initial momentum of the photon is entirely along the $\hat{z}$ direction, $\mathbf{p}_\nu = \frac{h}{\lambda}\hat{z} = \mathbf{p}_{\nu'} + \mathbf{p}_e$. By resolving the components and doing a bit of algebra, you can get the change in wavelength $\lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta) = \lambda_c(1-\cos\theta)$, where the Compton wavelength $\lambda_c = \frac{h}{m_e c} = 2.4\times 10^{-12}$ m. Note that $\lambda' - \lambda$ is always positive because $\cos\theta$ has magnitude less than 1. In other words, the frequency of the scattered photon is always less than the frequency of the incoming one. We can understand this result because the incoming photon gives a kick to the (stationary) electron and so it loses energy. Since $E = h\nu$, it follows that the outgoing frequency is decreased. As remarked earlier, it is impossible to understand this from a classical point of view. We shall now see how the Compton effect is actually observed experimentally.

5. In the apparatus shown, X-rays are incident upon a target which contains electrons. Those X-rays which are scattered in a particular angle θ are then selected by the collimator and are incident upon a crystal. The crystal diffracts the X-rays and, as you will recall from the lecture on diffraction, $2d\sin\theta_D = n\lambda$ is the necessary condition. It was thus determined that the changed wavelength of the scattered follows $\lambda' - \lambda = \lambda_c(1-\cos\theta)$.



6. Here is a summary of photon facts:
   a)The relation between the energy and frequency of a photon is $E = h\nu$.
   b)The relativistic formula relating energy and momentum for photons is $E = pc$. Note that in general $E^2 = p^2c^2 + m^2c^4$ for a massive particle. The photon has $m = 0$.
   c)The relation between frequency, wavelength, and photon speed is $\lambda\nu = c$.
   d)From the abbove, the momentum-wavelength relation for photons is $p = \frac{h\nu}{c} = \frac{h}{\lambda}$.
   e)An alternative way of expressing the above is: $E = \hbar\omega$ and $p = \hbar k$. Here $\omega = 2\pi\nu$ and $k = \frac{2\pi}{\lambda}$, $\hbar \equiv \frac{h}{2\pi}$ ($\hbar$ is pronounced h-bar).
   f)Light is always detected as packets (photons); if we look, we never observe half a photon. The number of photons is proportional to the energy density (i.e. to square of the electromagnetic field strength).

7. So light behaves as if made of particles. But do all particles of matter behave as if they are waves? In 1923 a Frenchman, Louis de Broglie, postulated that ordinary matter can have wave-like properties with the wavelength λ related to the particle's momentum p in the same way as for light, $\lambda = \frac{h}{p}$. We shall call λ the de Broglie (pronounced as Deebrolee!) wavelength.

8. Let us estimate some typical De Broglie wavelengths:
   a) The wavelength of 0.5 kg cricket ball moving at 2 m/sec is:

$$\lambda = \frac{h}{p} = \frac{6.63\times10^{-34}}{0.5\times 2} = 6.63\times10^{-34} \text{ m}$$

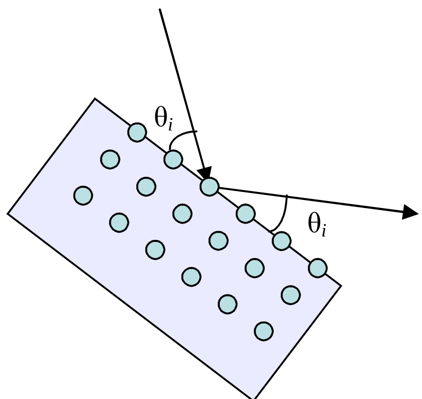
This is extremely small even in comparison to an atom, 10^{-10}m.

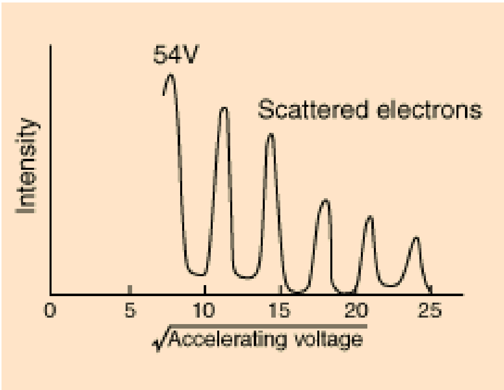
b) The wavelength of an electron with 50eV kinetic energy is calculated from:

$$K = \frac{p^2}{2m_e} = \frac{h^2}{2m_e\lambda^2} \Rightarrow \lambda = \frac{h}{\sqrt{2m_e K}} = 1.7\times 10^{-10}\,\text{m}$$

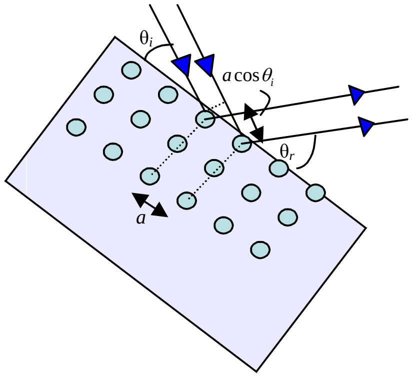
Now you see that we are close to atomic dimensions.

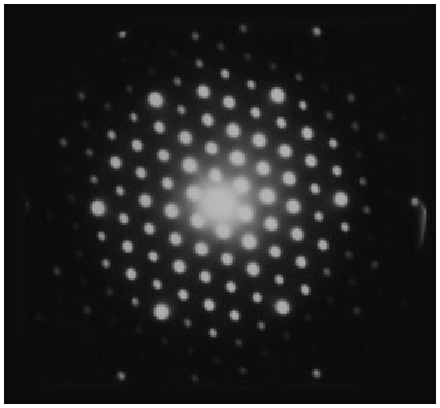
9. If De Broglie's hypothesis is correct, then we can expect that electron waves will undergo interference just like light waves. Indeed, the Davisson-Germer experiment (1927) showed that this was true. At fixed angle, one find sharp peaks in intensity as a function of electron energy. The electron waves hitting the atoms are re-emitted and reflected, and waves from different atoms interfere with each other. One therefore sees the peaks and valleys that are typical of interference (or diffraction) patterns in optical experiments.



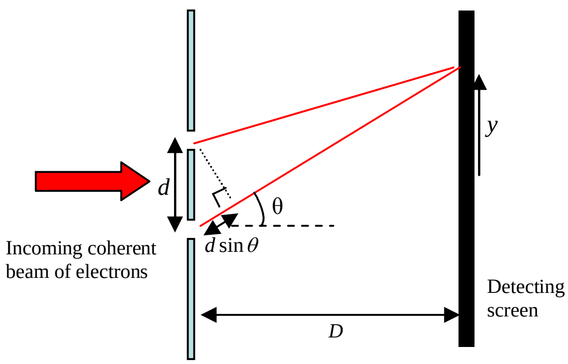


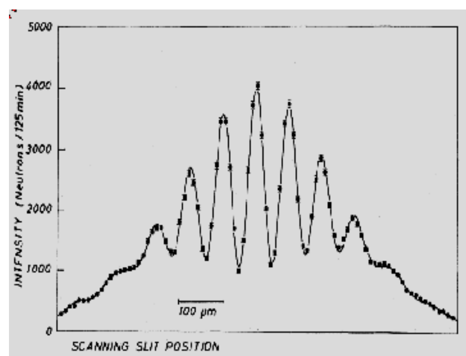
10. Let us look at the interference in some detail. When electrons fall on a crystalline surface, the electron scattering is dominated by surface layers. Note that the identical scattering planes are oriented perpendicular to the surface. Looking at the diagram, we can see that constructive interference happens when $a(\cos\theta_r - \cos\theta_i) = n\lambda$. When this condition is satified, there is a maximum intensity spot. This is actually how we find a and determine the structure of crystals.





11. Let's take a still simpler situation: electrons are incident upon a metal plate with two tiny holes punched into it. The holes - spearated by distance d - are very close together. A screen behind, at distance D, is made of material that flashes whenever it is hit by an electron. A clear interference pattern with peaks and valleys is observed. Let us analyze: there will be a maximum when $d\sin\theta = n\lambda$. If the screen is very far away, i.e. $D >> d$, then θ will be very small and $\sin\theta \approx \theta$. So we then have $\theta \approx \frac{n\lambda}{d}$, and the angular separation between two adjacent minima is $\Delta\theta \approx \frac{\lambda}{d}$. The position on the screen is y, and $y = D\tan\theta \approx D\theta$. So the separation between adjacent maxima is $\Delta y \approx D\Delta\theta$ and hence $\Delta y = \frac{\lambda D}{d}$. This is the separation between two adjacent bright spots. You can see from the experimental data that this is exactly what is observed.





12. The double-slit experiment is so important that we need to discuss it further. Note the following:

a) It doesn't matter whether we use light, electrons, or atoms - they all behave as waves. in this experiment. The wavelength of a matter wave is unconnected to any internal size of particle. Instead it is determined by the momentum, $\lambda = \frac{h}{p}$.

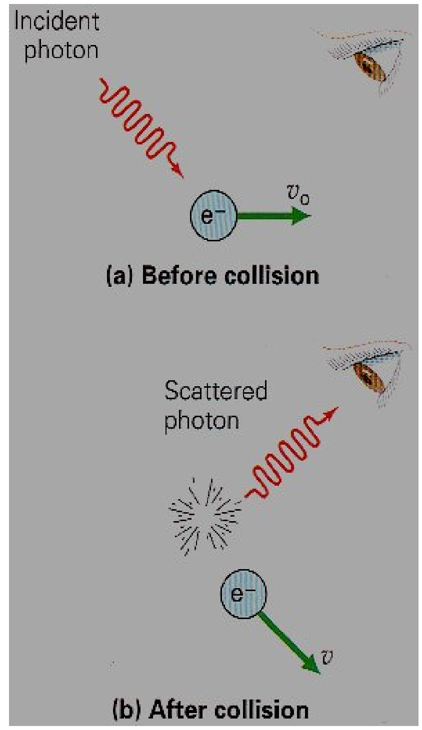
b) If one slit is closed, the interference disappears. So, in fact, each particle goes through both slits at once.

c) The flux of particles arriving at the slits can be reduced so that only one particle arrives at a time. Interference fringes are still observed !Wave-behaviour can be shown by a single atom. In other words, a matter wave can interfere with itself.

d) If we try to find out which slit the particle goes through the interference pattern vanishes! We cannot see the wave/particle nature at the same time.

All this is so mysterious and against all our expectations. But that's how Nature is!

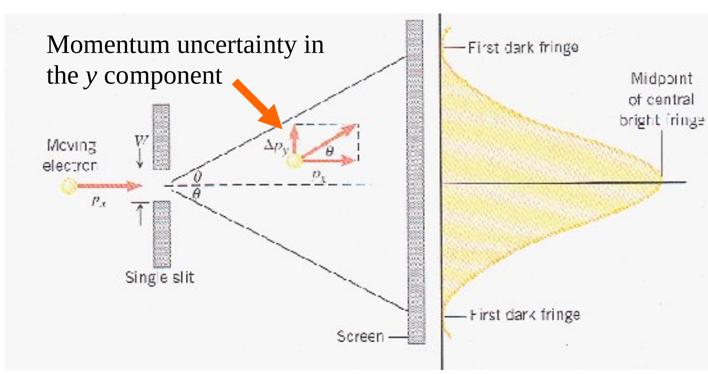
13. *Heisenberg Uncertainty Principle*. In real life we are perfectly familiar with seeing a cricket ball at rest - it has both a fixed position and fixed (zero) momentum. But in the microscopic world (atoms, nuclei, quarks, and still smaller distance scales) this is impossible. Heisenberg, the great German physicist, pointed out that if we want to see an electron, then we have to hit it with some other particle. So let's say that a photon hits an electron and then enters a detector. It will carry information to the detector of the position and velocity of the electron, but in doing so it will have changed the momentum of the electron. So if the electron was initially at rest, it will no longer be so afterwards. The act of measurement changes the state of the system! This is true no matter how you do the experiment. The statement of the principle is:



"If the position of a particle can be fixed with accuracy Δx, then the maximum accuracy with which the momentum can be fixed is Δp, where $\Delta x \Delta p \geq \hbar/2$." We call Δx the position uncertainty, and Δp the momentum uncertainty. Note that their product is fixed. Therefore, if we fix the position of the particle (make Δx very small), then the particle will move about randomly very fast (and have Δp very large).

14. By using the De Broglie hypothesis in a simple gedanken experiment, we can see how the uncertainty principle emerges. Electrons are incident upon a single slit and strike a screen far away. The first dark fringe will be when $W \sin\theta = \lambda$. Since θ is small, we can use $\sin\theta \approx \theta$, and so $\theta \approx \dfrac{\lambda}{W}$. But, on the other hand, $\tan\theta = \dfrac{\Delta p_y}{p_x}$ and $\theta \approx \dfrac{\Delta p_y}{p_x}$. So,

$\dfrac{\Delta p_y}{h/\lambda} = \dfrac{\lambda}{W}$. But W is really the uncertainty in the y position and we should call it Δy.



Thus we have found that $\Delta p_y \Delta y \approx h$. The important point here is that by localizing the y position of the electron to the width of the slit, we have forced the electron to acquire a momentum in the y direction whose uncertainty is Δp_y.

15. In a proper course in quantum mechanics, one can give a definite mathematical meaning to Δx and Δp_x etc, and derive the uncertainty relations:

$$\Delta x \Delta p_x \geq \hbar/2, \quad \Delta y \Delta p_y \geq \hbar/2, \quad \Delta z \Delta p_z \geq \hbar/2$$

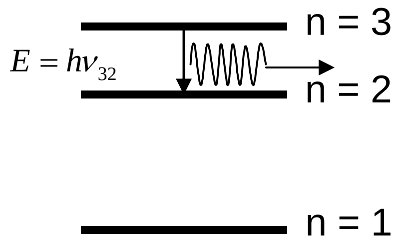
Note the following:

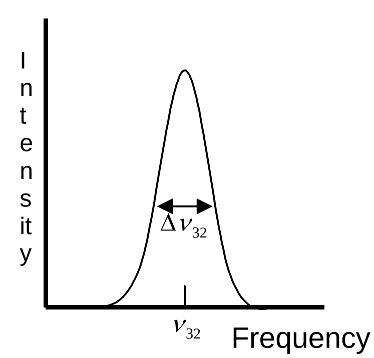
a) There no uncertainty principle for the product $\Delta x \Delta p_y$. In other words, we can know in principle the position in one direction precisely together with the momentum in another direction.

b) The thought experiment I discussed seems to imply that, while prior to experiment we have well defined values, it is the act of measurement which introduces the uncertainty by disturbing the particle's position and momentum. Nowadays it is more widely accepted that quantum uncertainty (lack of determinism) is intrinsic to the theory and does not come about just because of the act of measurement.

16. There is also an Energy-Time Uncertainty Principle which states that $\Delta E \Delta t \geq \hbar/2$. This says that the principle of energy conservation can be violated by amount ΔE, but only for a short time given by Δt. The quantity ΔE is called the uncertainty in the energy of a system.

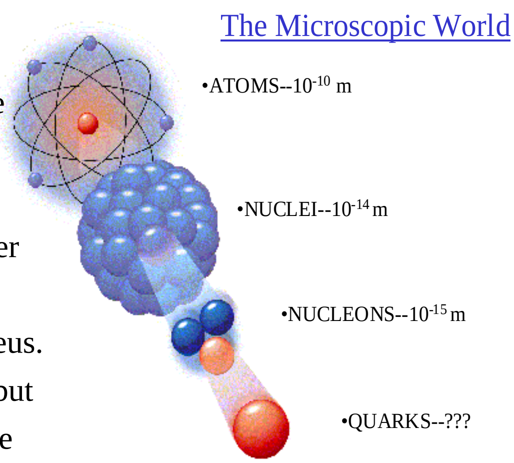
17. One consequence of $\Delta E \Delta t \geq \hbar/2$ is that the level of an atom does not have an exact value. So, transitions between energy levels of atoms are never perfectly sharp in frequency. So, for example, as shown below an electron in the $n = 3$ state will decay to a lower level after a lifetime of order $t \approx 10^{-8}$s. There is a corresponding "spread" in the emitted frequency.





Summary of Lecture 42 – QUANTUM MECHANICS

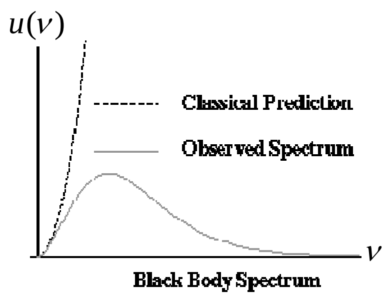
1. The word "quantum" means packet or bundle. We have already encountered the quantum of light - called photon - in an earlier lecture. Quanta (plural of quantum) are discrete steps. Walking up a flight of stairs, you can increase your height one step at a time and not, for example, by 0.371 steps. In other words your height above ground (and potential energy can take discrete values only).
2. Quantum Mechanics is the true physics of the microscopic world. To get an idea of the sizes in that world, let us start from the atom which is normally considered to be a very small object. But, as you can see, the atomic nucleus is 100,000 times smaller than the atom. The neutron and proton are yet another 10 times smaller than the nucleus. We know that nuclei are made of quarks, but as yet we do know if the quarks have a size or if they are just point-like particles.



3. It is impossible to cover quantum mechanics in a few lectures, much less in this single lecture. But here are some main ideas:
   a) Classical (Newtonian) Mechanics is extremely good for dealing with large objects (a grain of salt is to be considered large). But on the atomic level, it fails.The reason for failure is the uncertainty principle - the position and momenta of a particle cannot be determined simultaneously (this is just one example; the uncertainty principle is actually more general). Quantum Mechanics properly describes the microscopic - as well as macroscopic - world and has always been found to hold if applied correctly.
   b) Atoms or molecules can only exist in certain energy states. These are also called "allowed levels" or quantum states. Each state is described by certain "quantum numbers" that give information about that state's energy, momentum, etc.
   c) Atoms or molecules emit or absorb energy when they change their energy state. The amount of energy released or absorbed equals the difference of energies between the two quantum states.
   d) Quantum Mechanics always deals with probabilities. So, for example, in considering the outcome of two particles colliding with each other, we calculate probabilities to scatter in a certain direction, etc.

4. What brought about the Quantum Revolution? By the end of the 19th century a number of serious discrepancies had been found between experimental results and classical theory. The most serious ones were:
   A) The blackbody radiation law
   B) The photo-electric effect
   C) The stability of the atom, and puzzles of atomic spectra
   In the following, we shall briefly consider these discrepancies and the manner in which quantum mechanics resolved them.

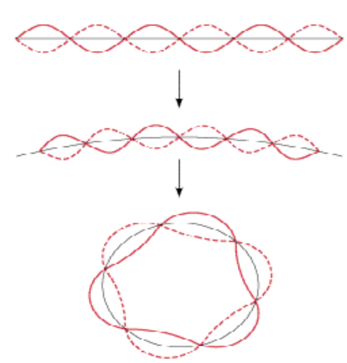
A)Classical physics gives the wrong behaviour for radiation emitted from a hot body. Although in this lecture it is not possible to do the classical calculation, it is not difficult to show that the electromagnetic energy $u(\nu)$ radiated at frequency ν increases as ν^3 (see graph). So the energy radiated over all frequencies is infinite. This is clearly wrong. The correct calculation was done by Max Planck.



Planck's result is shown above, and it leads to the sensible result that u(ν) goes to zero at large ν. He assumed that radiation of a given frequency ν could only be emitted and absorbed in quanta of energy $\varepsilon = h\nu$. If the electromagnetic field is thought of as harmonic oscillators, Planck assumed that the total energy of this large number of oscillators is made of finite energy elements $h\nu$. With this assumption, he came up with a formula that fitted well with the data. But he called his theory "an act of desperation" because he did not understand the deeper reasons.

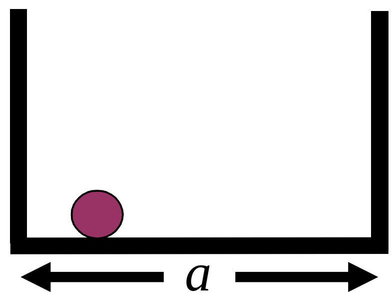
B) I have already discussed the photoelectric effect in the previous lecture. Briefly, Einstein (1905) postulated a quantum of light called photon, which had particle properties like energy and momentum. The photon is responsible for knocking electrons out of the metal - but only if it has enough energy.

C) Classical physics cannot explain the fact that atoms are stable. An accelerating charge always radiates energy if classical electromagnetism is correct. So why does the hydrogen atom not collapse? In 1921 Niels Bohr, a great Danish physicist, made the following hypothesis: if an electron moves around a nucleus so that its angular momentum is $\hbar, 2\hbar, 3\hbar, \cdots$ then it will not radiate energy. In the next lecture we explore the consequences of this hypothesis. Bohr's hypothesis called for the quantization

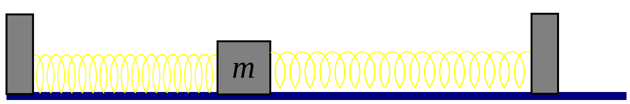


of angular momentum. If the electron is regarded as a De Broglie wave, then there must be an integer number of waves that go around the centre. Of course, this is not proper quantum mechanics and cannot be taken too seriously, but it definitely was a major step in the ultimate development of the subject. Even if one's understanding was imperfect, it was now possible to understand why atoms had certain energies only, and why the light radiated by atoms was of discrete frequencies only. In contrast, classical physics predicted that atoms could radiate at any frequency.

5. One consequence of quantum mechanics is that it explains through the uncertainty principle the stability of the atom. But before talking of that, let us consider a particle moving between two walls. Each time it hits a wall, a force pushes it in the opposite direction. There is no friction, so classically the particle just keeps moving forever between the two potential walls. In QM the uncertainty of the particle's position is $\Delta x = a$ and so, from $\Delta x \Delta p \geq \hbar/2,$ we have $\Delta p\, a \approx \frac{\hbar}{2}$. From this we learn that the kinetic energy $\frac{(\Delta p)^2}{2m} \approx \frac{\hbar^2}{8ma^2}$. This is telling us that as we squeeze the particle into a tighter and tighter space, the kinetic energy goes up and up!



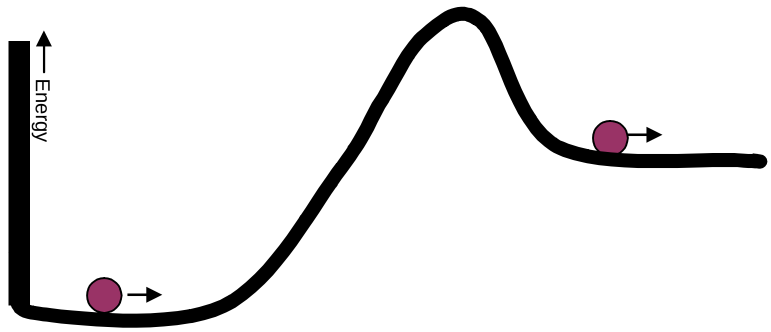
6. The story repeats for a harmonic oscillator. So imagine that a mass moves in a potential of the type shown below, and that its frequency of oscillation is ω. Classically, the lowest energy would be that in which the mass is at rest (no kinetic energy) and it is at the position where the potential is minimum. But this means that the body has both a



well-defined momentum and position. This is forbidden by the Heisenberg uncertainty principle. A proper quantum mechanical calculation shows that the minimum energy is actually $\frac{1}{2}\hbar\omega$. This is called the zero-point energy and comes about because it is not possible for the mass to be a rest. The oscillator's other energy states have energies $\frac{1}{2}\hbar\omega, \frac{3}{2}\hbar\omega, \frac{5}{2}\hbar\omega, \frac{7}{2}\hbar\omega, \cdots$ We say that the energies are quantized. This is completely impossible to understand from classical mechanics where we know that we can excite an oscillator to have any energy we want. Quantized energy levels are indeed what we observe in so many physical systems: atoms, rotating or vibrating molecules, nuclei, etc.

7. We can understand from both the previous examples why the hydrogen atom does not collapse even if the electron does not have any orbital angular momentum around the proton. The uncertainty principle essentially forces the electron to stay away from the proton - if it tried to get too close, the kinetic energy would rise enormously because $\Delta x \Delta p \geq \hbar/2$ says that if Δx becomes small then Δp must become big to compensate.

8. Quantum mechanics predicts what is called "tunneling" of particles through a potential barrier. Again, we shall use the uncertainty principle but this time the energy-time one. Imagine a particle that moves in a potential of the shape shown below. Suppose it does not have enough energy to go over the peak and on to the other side. In that case, we



know from our experience that it will just keep oscillating - moving first towards the hill and then down again, etc. But quantum mechanically, it can "steal" energy ΔE for a time Δt and this may be enough to surmount the hill. Of course, the particle must respect $\Delta E \Delta t \geq \hbar/2$ so the time is small if it needs a large amount of energy to cross over. Again, I have given only a rough argument here, but in quantum mechanics we can do proper calculations to find tunneling probabilities.

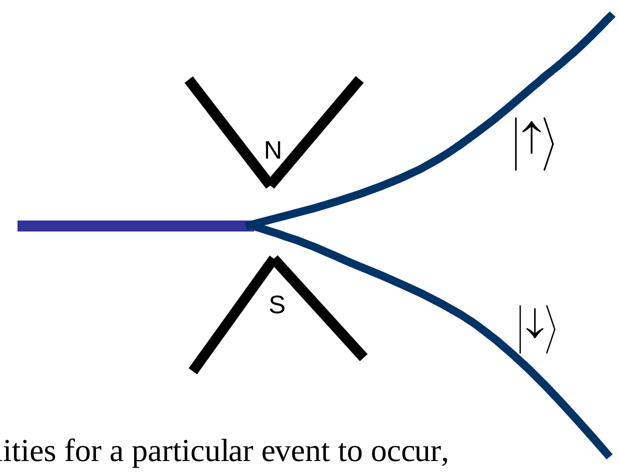
9. Without tunneling our sun would go cold. As you may know, it is powered by hydrogen fusion. The protons must somehow overcome electrostatic repulsion to get close enough so that they can feel the nuclear force and fuse with each other. But the thermal energy at the core of the sun is not high enough. It is only because the tunneling effect allows protons to sometimes get close enough that fusion happens.

10. Quantum mechanics is all about probabilities.What is probability? Probability is a measure of the likelihood that an event will occur. Probability values are assigned on a scale of zero (the event can never occur) to one (the event definitely occurs). More precisely, suppose we do an experiment (like rolling dice or flipping a coin) N times where N is large. Then if a certain outcome occurs n times, the probability is $P = n/N$.

11. The simplest system for discussing quantum mechanics is one that has only two states. Let us call these two states "up" and "down" states, $|\uparrow\rangle$ and $|\downarrow\rangle$. They could denote an electron with spin up/down, or a switch which is up/down, or an atom which can be only in one of two energy states, etc. Things like $|\uparrow\rangle$ were called kets by their inventor, Paul Dirac.

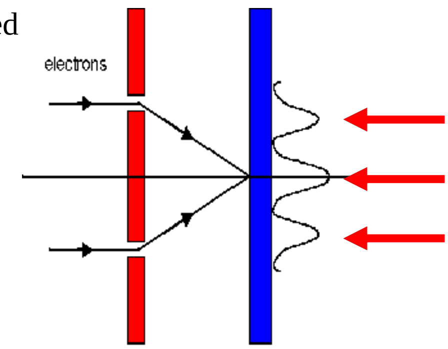
 For definiteness let us take the electron example. If the state of the electron is known to be $|\Psi\rangle = \sqrt{\frac{2}{3}}|\uparrow\rangle + \sqrt{\frac{1}{3}}|\downarrow\rangle$, then the probability of finding the electron with spin up is $P(\uparrow) = \frac{2}{3}$, and with spin down is $P(\downarrow) = \frac{1}{3}$. More generally: $|\Psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle$ denotes an electron with $P(\uparrow) = |c_1|^2$ and $P(\downarrow) = |c_2|^2$. This means that if we look at look at a large number of electrons N all of which are in state $|\Psi\rangle$, then the number with spin up is $N|c_1|^2$ and with spin down is $N|c_2|^2$. We sometimes call $|\Psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle$ a *quantum state*, and c_1 and c_2 *quantum amplitudes.*

12. The Stern-Gerlach experiment illustrated here shows an electron beam entering a magnetic field. The electrons can be pointing either up or down relative to any chosen axis. The field forces them to choose one of the two states. That the beam splits into only two parts shows that the electron has only two states. Other particle beams might split into 3,4, $\cdots$



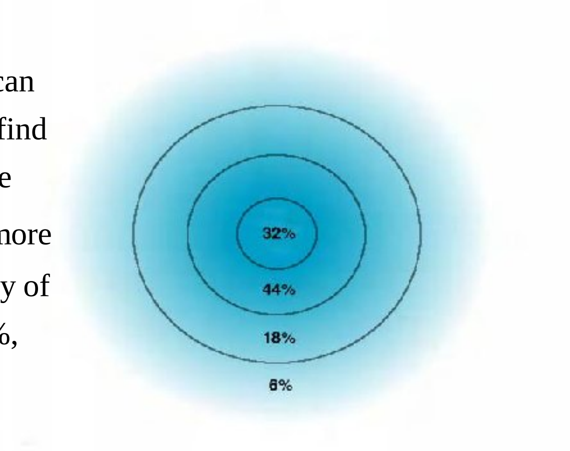
13. If a_1 and a_2 are the amplitudes of the two possibilities for a particular event to occur, then the amplitude for the total event is $A = a_1 + a_2$. Here a_1 and a_2 are complex numbers in general. But the probability for the event to occur is given by $P = |A|^2 = |a_1 + a_2|^2$. In daily experience we add probabilities, $P = P_1 + P_2$ but in quantum mechanics we add amplitudes: $P = |a_1 + a_2|^2 = a_1^* a_1 + a_2^* a_2 + a_1^* a_2 + a_2^* a_1 = P_1 + P_2 + a_1^* a_2 + a_2^* a_1$. The cross terms $a_1^* a_2 + a_2^* a_1$ are called interference terms. They are familiar to us from the lecture on light where we add amplitudes first, and then square the sum to find the intensity. Of course, if we add all possible outcomes then we will get 1. So, for example, in the electron case $P(\uparrow) + P(\downarrow) = |c_1|^2 + |c_2|^2 = 1$. Note that amplitudes can be complex but probabilities are always real.

14. Let us return to the double slit experiment discussed earlier. Here the amplitude for an electron wave coming from one slit interferes with the amplitude for an electron wave coming from the other slit. This is what causes a pattern to emerge in which electron are completely absent in certain places (destructive interference) and are present in large numbers where there is constructive interference. So what we must deal with are matter waves. But how to treat this mathematically?



15. The above brings us to the concept of a "wave function". In 1926 Schrödinger proposed a quantity that would describe electron waves (or, more generally, matter waves).
    - The wavefunction $\Psi(x,t)$ of a particle is the amplitude to be at position x at time t.
    - The probability of finding the particle at position x between x and $x+dx$ (at time t) is $|\Psi(x,t)|^2 dx$. Since the particle has to be somewhere, if we add up all possibilities then we must get one, i.e. $\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dx = 1$.
    - $\Psi(x,t)$ is determined by solving the "Schrodinger equation" which, unfortunately, I shall not be able to discuss here. This is one of the most important equations of physics. If it is solved for the atom then it tells you all that is possible to know: energies, the probability of finding an electron here or there, the momenta with which they move, etc. Of course, one usually cannot solve this equation in complicated situations (like a large molecule, for example) and this is what makes the subject both difficult and interesting.

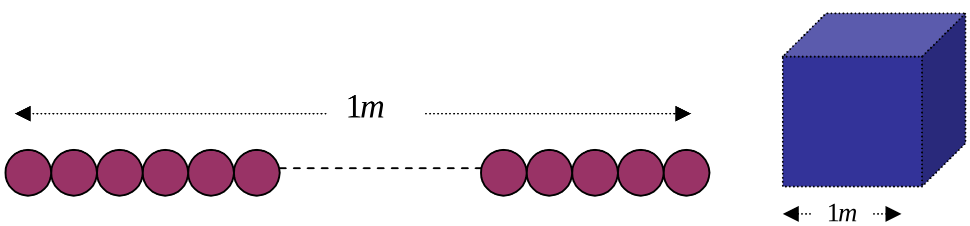
16. For an electron moving around a nucleus, one can easily solve the Schrodinger equation and thus find the wavefunction $\Psi(x,t)$. From this we compute $|\Psi(x,t)|^2$, which is large where the electron is more likely to be found. In this picture, the probability of finding the electron inside the first circle is 32%, between the second and first is 44%, etc.



Summary of Lecture 43 – INTRODUCTION TO ATOMIC PHYSICS

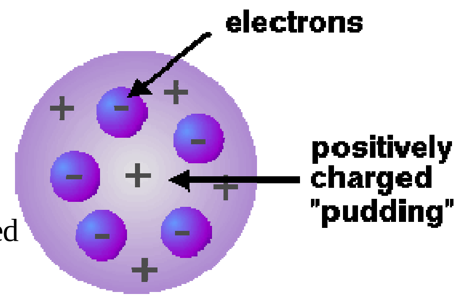
1. • About 2500 years ago, the ancient Greek philosopher Democritus asked the question: what is the world made of? He conjectured that it is mostly empty, and that the remainder is made of tiny "atoms". By definition these atoms are indivisible.
   • Then 300 years ago, it was noted by the French chemist Lavoisier that in all chemical reactions the total mass of reactants before and after a chemical reaction is the same. He demonstrated that burning wood caused no change in mass. This is the Law of Conservation of Matter.
   • A major increase in understanding came with Dalton (1803) who showed that:
     1) Atoms are building blocks of the elements.
     2) All atoms of the same element have the same mass.
     3) Atoms of different elements are different.
     4) Two or more different atoms bond in simple ratios to form compounds.
2. Avogadro made the following hypothesis : "Equal volumes of all gases, under the same conditions of temperature and pressure, contain equal numbers of molecules". Why? Because we know that pressure is caused by molecules hitting the sides of the containing vessel. If the temperature of two gases is the same, then their molecules move with the same speeds, and so Avogadro's hypothesis follows for ideal gases. The famous number $N_0 = 6.023\times 10^{26}$ per kilogram-mole is called Avogadro's Number.

3. Let's get an idea of the size of atoms. Amazingly, we do not need high-powered particle accelerators to do so. Consider a cube of $1m\times 1m\times 1m$. If the radius of an atom is r, then we have $(1/2r)^3$ atoms in the cube. Now in 1kg.atom we have $N_0 = 6\times 10^{26}$ atoms and each atom occupies a volume (A/ρ) m^3, where $A =$ atomic weight and $\rho =$ density.

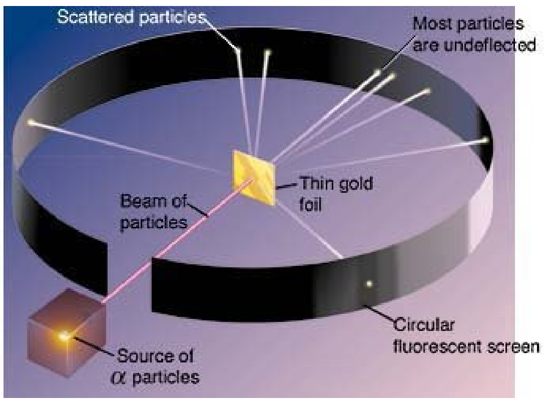


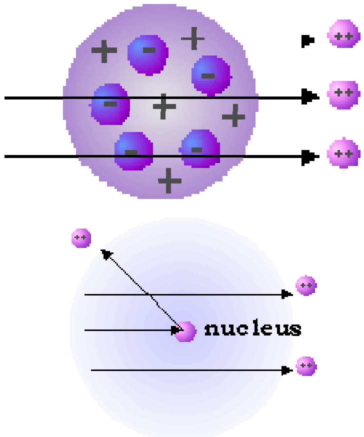
Hence $N_0 = (1/2r)^3\times A/\rho$. This give $r = \frac{1}{2}\left(\frac{A}{\rho N_0}\right)^{1/3}$. Putting in some typical densities, we find that $r_{Ag}\approx r_{Be}\approx 10^{-10}m$. This shows that atoms are mostly of the same size. This is quite amazing because one expects a Be atom to be much smaller than an Ag atom.

4. Even if you know how big an atom is, this does not mean that its internal structure is known. In 1895 J.J.Thomson proposed the "plum pudding" model of an atom. Here the atom is considered as made of a positively charged material with the negatively charged electrons scattered through it.



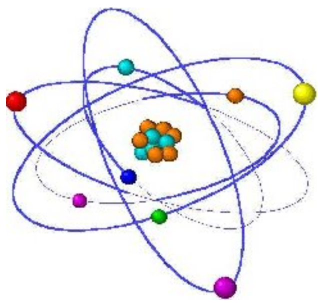
5. But the plum-pudding model was wrong. In 1911, Rutherford carried out his famous experiment that showed the existence of a small but very heavy core of the atom. He arranged for a beam of α particles to strike gold atoms in a thin foil of gold.





If the positive and negative charges in the atom were randomly distributed, all α's would go through without any deflection. But a lot of backscattering was seen, and some α's were even deflected back in the direction of the incident beam. This was possible only if they were colliding with a very heavy object inside the atom. Rutherford had discovered the atomic nucleus.

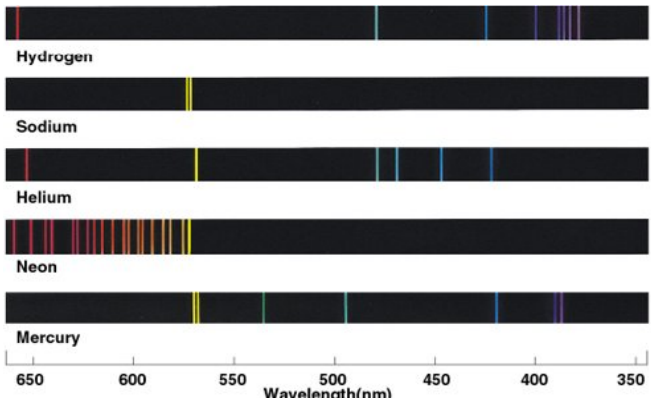
6. The picture that emerged after Rutherford's discovery was like that of the solar system - the atom was now thought of as mostly empty space with a small, positive nucleus that contained protons. Negative electrons moved around the outside in orbits hat resembled those of planets, attracted towards the centre by a coulomb force.



7. This sounds fine, but there is a serious problem: we know that a charge that accelerates radiates energy. In fact the power radiated is $P \propto e^2 a^2$, where e is the charge and a is the acceleration. Now, a particle moving in a circular orbit has an acceleration even if it is moving at constant speed because it is changing its direction all the time. So this means

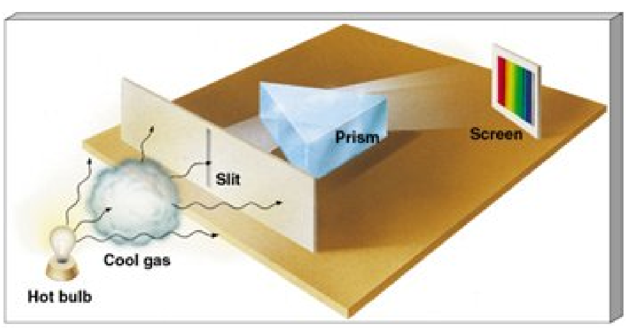
that the electron will be constantly radiating power and thus will slow down, collapsing eventually into the nucleus.

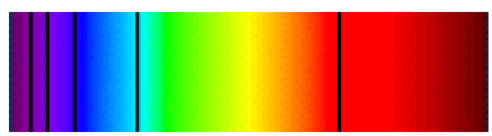
8. This is not the only thing wrong with the solar system model. If you look at the light emitted by any atom, you do not see a continuous distribution of colours (frequencies). Instead, a spectroscope will easily show that light is emitted at only certain discrete frequencies.



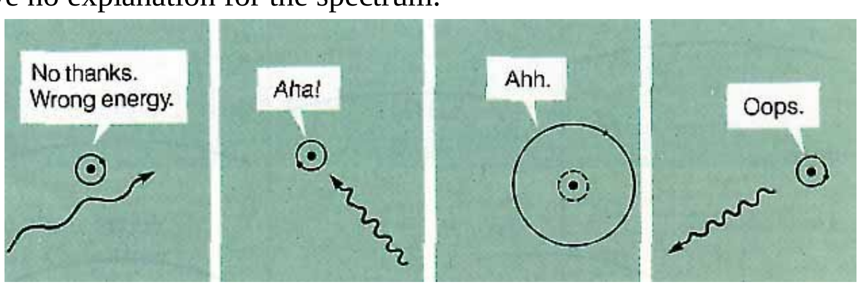
In the above you see the emission spectrum of different atoms.

9. Similarly, if white light is passed through a gas of atoms of a certain type, only certain colours are absorbed, and the others pass through without a hindrance.

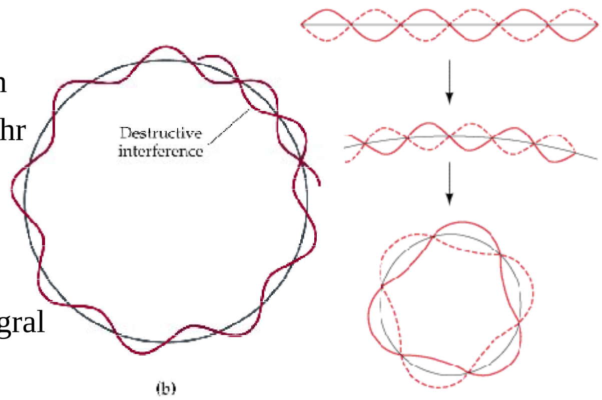




The above shows the absorption spectrum of a certain atom. The wavelengths for both emission and absorption lines are exactly equal. Classical physics and the Rutherford model have no explanation for the spectrum.

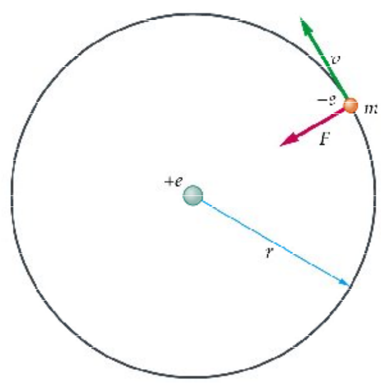


10. Then came Niels Bohr. By this time it was known that electrons had a dual character as waves (De Broglie relation and Davisson-Germer experiment). Bohr said: suppose I bend a standing wave into a circle. If the wavelength is not exactly correct, wave interference will make the wave disappear. So only integral numbers of wavelengths can interfere constructively.



11. Let us pursue this idea further. The electron has a wavelength and forms standing waves in its orbit around the nucleus. An integral number of electron wavelengths must fit into the circumference of the circular orbit. Hence $n\lambda = 2\pi r$ with $n = 1, 2, 3\cdots$ The momentum is $p = m\text{v} = \dfrac{h}{\lambda} = \dfrac{h}{(2\pi r/n)} = \dfrac{n\hbar}{r}$. The angular momentum $L = rm\text{v} = n\hbar$ is therefore quantized in units of $\hbar$.

12. Now let us suppose that the electron moves in an orbit of radius r when it has $L_n = n\hbar$. Equilibrium demands that the centrifugal force be equal to the coulomb attraction: $\dfrac{m\text{v}_n^{\,2}}{r} = k\dfrac{e^2}{r^2}$. From $\text{v}_n = \dfrac{n\hbar}{mr_n}$ we find that the radius $r_n = \left(\dfrac{\hbar^2}{mke^2}\right)n^2$.



- For $n = 1$ the electron orbit which is closest to the nucleus, $r_n = a_0 \equiv 0.53\times 10^{-8}\,cm$ (this is called the Bohr radius).
- For higher n, $r_n = a_0 n^2$. The atom becomes huge for n ≈ 100, the so-called Rydberg atom. Such atoms are of experimental interest these days.
- Note that the speed of the electron is smaller in orbits farther from the nucleus, $\text{v}_n = \dfrac{ke^2}{n\hbar}$. As n becomes very large, the electron is very far out and very slow.
- In the above $n = 0$ is strictly not allowed. As you can see, none of the formulae make any sense for this case. The minimum angular momentum that the electron can have is $\hbar$. (In proper quantum mechanics the minimum is $0\hbar$ and this is a big difference with the Bohr model of the atom).

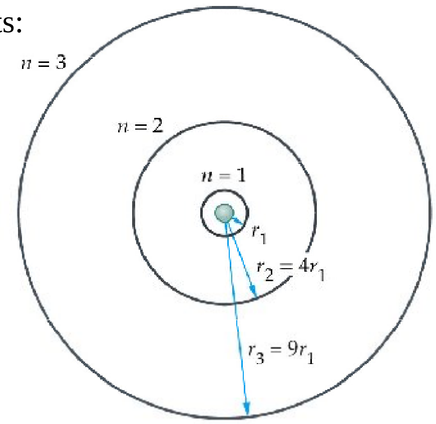
13. We can compute the energies of the various orbits:

$$E = K + U = \tfrac{1}{2}mv^2 + U$$

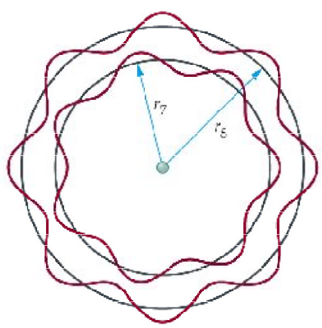
$$= \tfrac{1}{2}\left(\frac{ke^2}{r}\right) - \frac{ke^2}{r} = -\frac{ke^2}{2r}$$

Hence, $E_n = -\left(\frac{ke^2}{2}\right)\left(\frac{mke^2}{\hbar^2}\right)\frac{1}{n^2}$

$$= -\left(\frac{mk^2e^4}{2\hbar^2}\right)\frac{1}{n^2} = -\frac{13.6\ \text{eV}}{n^2}$$



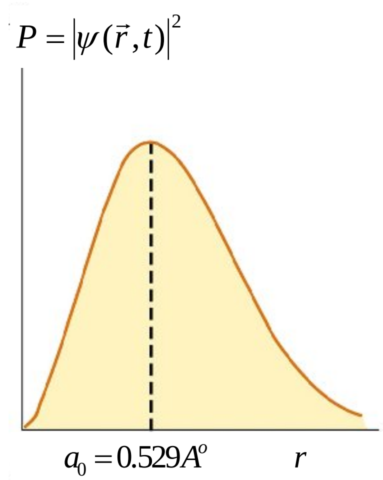
14. In the Bohr model, electrons can jump between different orbits due to the absorption or emission of photons. Dark lines in the absorption spectra are due to photons being absorbed, and bright lines in the emission spectra are due to photons being emitted. The energy of the emitted or absorbed photon is equal to the difference of the initial and final energy levels, $h\nu = E_f - E_i$. The picture below shows the electron in the $n = 7$ and $n = 8$ levels. The photon emitted has $h\nu = E_8 - E_7 = -13.6\left(\frac{1}{8^2} - \frac{1}{7^2}\right)$ ev.



(a)

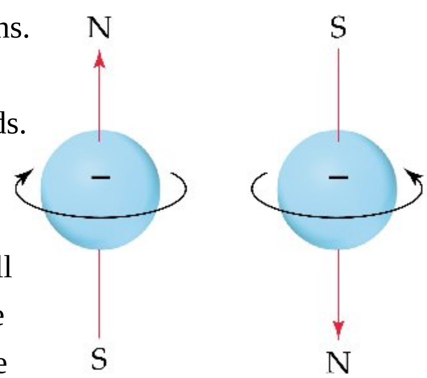
15. The Bohr model gave wonderful results when compared against the hydrogen spectrum. It was the among the first indications that some "new physics" was needed at the atomic level. But this model is not to be taken too seriously - it fails to explain many atomic properties, and fails to explain why the H atom can exist even when the electron has no orbital angular momentum (and hence no centrifugal force to balance against the Coulomb attraction). It cannot predict all the lines observed for H, much less for multi-electron atoms such as Oxygen. The real value of this model was that it showed the way forward towards developing quantum mechanics, which is the true physics of the world, both microscopic and macroscopic. I have discussed some elements of QM in the last lecture, in particular the wavefunction $\psi(\vec{r},t)$ of the electron.

16. Quantum mechanics gives a picture that is quite different from the solar-system model. We solve the "Schrodinger Equation" to find the wavefunction $\psi(\vec{r},t)$, whose square gives the probability of finding the electron at the point $\vec{r}$. In the lowest energy state, the electron can be viewed as a spherical cloud surrounding the nucleus. The densest regions of the cloud represent the highest probability for finding the electron.



- The principal quantum number $n = 1,2,3\cdots$ determines the allowed energy levels. An electron can only have energy $E_n = -\frac{13.6}{n^2}$ eV. Miraculously this is the same result as in the Bohr model.
- The orbital angular momentum is determined by the number l, and $L = \sqrt{l(l+1)}\hbar$. Allowed values of l are, $l = 0,1,2,3\cdots,n-1$.
- The magnetic quantum number m_l determines the projection (or component) of the vector angular momentum $\vec{L}$ on to any fixed axis, $L_z = m_l\hbar$. Allowed values are, $m_l = -l,\cdots -2,-1,0,1,2,\cdots,l$.

17. Electrons can be thought of as little spinning balls of charge. All electrons spin at the same speed (more accurately, their spin angular momentum is the same and equals $\hbar/2$). An electron can spin in only one of two possible directions. When charges move around in a circle, that constitutes a current. As you know, currents give rise to magnetic fields. This is why electrons are also little magnets that interact with other magnets. Now go back to the Stern-Gerlach experiment described in the previous lecture, and you will understand better why the electrons were deflected by the applied magnetic field. Now that we have learned that the electron has spin, we can describe the two spin states by giving the "magnetic" quantum number m_s where $m_s = -\frac{1}{2},\frac{1}{2}$. These two states have the same energy except when there is some magnetic field present.

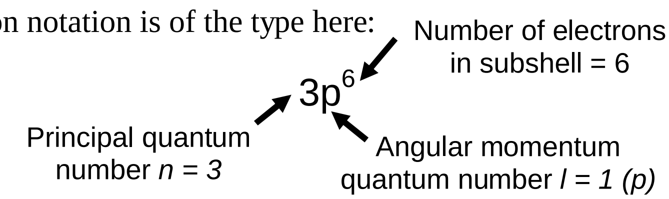


18. States (or orbitals) having $l = 0,1,2,\cdots$ are called s,p,d,$\cdots$. The s-states are spherical. As n increases, the s-orbitals get larger and the wavefunction is larger away from the nucleus.

19. In the world we are used to, we can always tell apart identical particles (same mass, charge, spin,...) by simply watching them. But in QM, their idetities can get confused and identical particles are indistinguishable. Suppose that A and B are particles that are identical in every possible way, and we exchange them. Of course, the probability of finding one or the other must remain unchanged. In other words, $|\Psi(1,2)|^2 = |\Psi(2,1)|^2$, where 1 and 2 denote the positions of the first and second particles. But something very interesting happens now because either one of two possibilities can be true: $\Psi(1,2) = +\Psi(2,1)$ or $\Psi(1,2) = -\Psi(2,1)$. Particles obeying the first are called *bosons*, while those obeying the second are called *fermions*. What if we bring two fermions to same point in space? Then: $\Psi(1,1) = -\Psi(1,1)$. This means that $\Psi(1,1) = 0$! In other words, two identical fermions will never be at the same point or in the same quantum state. This is the famous Pauli Exclusion Principle.

20. Let us apply the Pauli Exclusion Principle to the multi-electron atom where each electron has the quantum numbers $\{n, l, m_l, m_s\}$. Only one electron at a time may have a particular set of quantum numbers. Now for some definitions:
    - Shell - electrons with the same value of n
    - Subshell - electrons with the same values of n and l
    - Orbital - electrons with the same values of n, l, and m_l

    Once a particular state is occupied, other electrons are excluded from that state. The electron configuration is how the electrons are distributed among the various atomic orbitals in an atom. A common notation is of the type here:

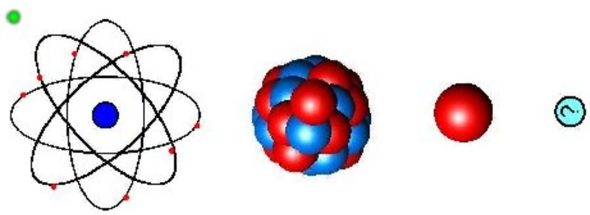


21. Building the shell structure of multi-electron atoms through $n = 4$ using the Pauli Principle.

n	Possible Values of l	Subshell Designation	Possible Values of m_l	Number of Orbitals in Subshell	Total Number of Orbitals in Shell
1	0	1*s*	0	1	1
2	0	2*s*	0	1	
	1	2*p*	1, 0, −1	3	4
3	0	3*s*	0	1	
	1	3*p*	1, 0, −1	3	
	2	3*d*	2, 1, 0, −1, −2	5	9
4	0	4*s*	0	1	
	1	4*p*	1, 0, −1	3	
	2	4*d*	2, 1, 0, −1, −2	5	
	3	4*f*	3, 2, 1, 0, −1, −2, −3	7	16

Summary of Lecture 44 – INTRODUCTION TO NUCLEAR PHYSICS

Q.1 In the previous lecture you learned how it was discovered that the atom is mostly empty space with a cloud of electrons. At the centre is a small but very heavy nucleus that has protons and neutrons. The word "nucleon" refers to both of these. So you can think of



the neutron or proton as being two different varieties of the nucleon. The masses of the two are very similar, and they are roughly 2000 times heavier than the electron.

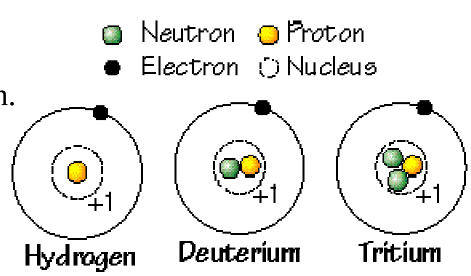
$$\text{proton mass} = M_p = 1.672\times10^{-27}\,kg$$
$$\text{neutron mass} = M_n = 1.675\times10^{-27}\,kg$$
$$\text{electron mass} = M_e = 9.109\times10^{-31}\,kg$$

The neutron is neutral, of course, but the charge on the proton is $1.6\times10^{-19}C$ while the charge on the electron is the negative of this, $-1.6\times10^{-19}C$.

2. Using kilograms is very awkward if you are dealing with such small particles. Instead we we use $E = mc^2$ to write the mass of a particle in terms of its rest energy, $m = E/c^2$. So mass is measured in units of MeV/c^2.

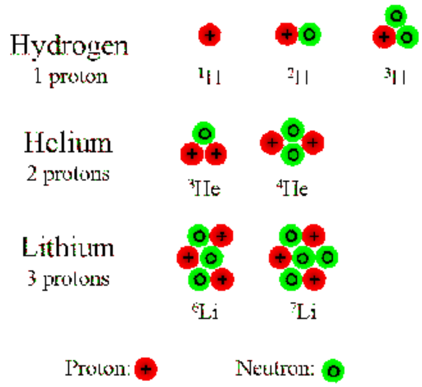
$$\text{proton mass} = M_p = 938\ MeV/c^2$$
$$\text{neutron mass} = M_n = 940\ MeV/c^2$$
$$\text{electron mass} = M_e = 0.5\ MeV/c^2$$

3. a) A hydrogen nucleus is just one proton.
   b) A deuteron has a proton plus one neutron.
   c) A triton (or tritium nucleus) has a proton plus two neutrons.
   All three atoms have one electron only, and thus completely identical chemical properties.

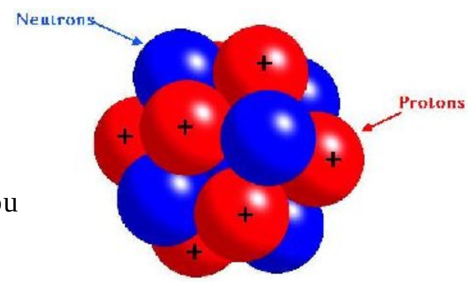


4. Hydrogen, deuterium, and tritium are called isotopes. If a nucleus with Z protons has N neutrons then its isotopes will have fewer, or more, neutrons. Since the number of electrons is also Z, the chemical properties of all isotopes are exactly the same. But for any given element, at most there is only one stable isotope.

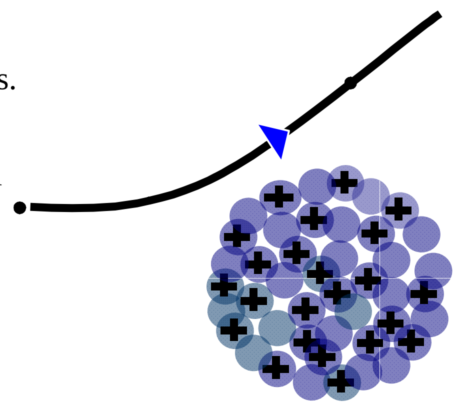
4. A commonly used notation is $^{A}_{Z}X$ where X is the element, and $A = Z + N$. Of the elements that you see on the right, the most stable ones are ^{1}H, ^{3}He, and ^{6}Li. Now consider oxygen. The most stable isotope is ^{16}O. When you breathe in oxygen from the atmosphere, 99.8% is ^{16}O, 0.037% is ^{17}O, and 0.163% is ^{18}O. We shall see later what unstable means and how nuclei decay.



5. The diameter of the nucleus is about 10 million times smaller that the overall diameter of the atom. Nuclei follow an approximate rule for the radius, $r \approx r_0 A^{1/3}$ where $r_0 = 1.2\ fm$ (remember, 1 fermi $= 10^{-13}$ cm) and $A = Z + N$. Now, $A^{1/3}$ increases very slowly with A. You can check that $16^{1/3} = 2.52$ while $208^{1/3} = 5.93$. This means that a very heavy lead nucleus ^{208}Pb is only about 2.4 times the size of the much lighter ^{16}O nucleus.

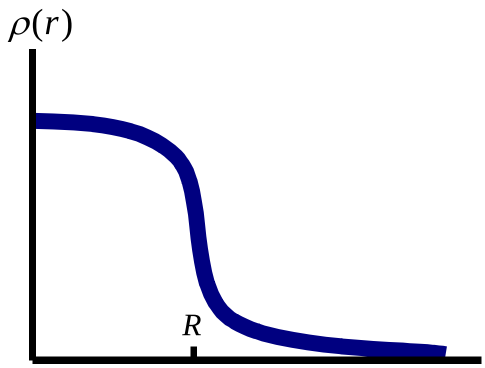


6. To learn about how protons are distributed inside a nucleus, we send a beam of electrons at a nucleus and observe how they scatter in different directions. The negatively charged electrons interact with the positively charged protons, but they obviously will not see the neutrons. The scattered electrons are captured in a detector which can be moved around to different angles. In this way one can reconstruct the charge distribution which caused the electrons to be scattered in that particular way.



7. What energy should electrons have in order to see a nucleus? We know that electrons are waves with $\lambda = h/p$ (the De Broglie relation). To see something as small as $1\,fm$ requires a wave with wavelength at least $\lambda \approx 1\,fm$. A wave with longer wavelength would simply pass over the nucleus without being disturbed. So the minimum electron energy is,
$E = \dfrac{p^2}{2m} = \dfrac{h^2}{2m\lambda^2}$. Evaluation gives this to be a few MeV, requiring an electron accelerator of more than this minimum energy.

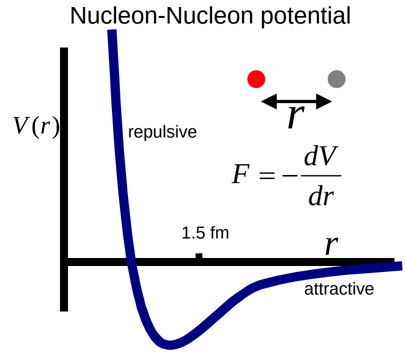
8. From electron scattering we see that the proton density is almost constant throughout the nucleus and falls sharply at the surface. So nuclei should be thought of as rather fuzzy balls. A typical plot of density versus distance from the centre looks like this. Here the distance $R \approx r_0 A^{1/3}$ can be called the nuclear radius. The distribution of neutrons is very similar to this plot, but requires other techniques.



9. Let us consider the implication of the approximate formula for the nuclear radius, $r \approx r_0 A^{1/3}$ where $r_0 = 1.2\ fm.$ The volume of the nucleus is: $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi r_0{}^3 A.$ From this $\frac{A}{V} = \frac{3}{4\pi r_0^3} \approx 0.14$ nucleons/fm^3. This is the number of nucleons per cubic fermi, and is independent of the nucleus considered. This is the density you would find at the centre of any nucleus. Of course, this is approximately true only but it is quite remarkable.

10. Protons repel protons through the electrostatic force. So why does the nucleus not blow apart. Obviously there must be some attractive force that is stronger than this repulsion. It is, in fact, called the strong force. From what we have learned so far, we can guess some of its important features:
    a) Since neutron and proton distributions are almost the same, the N-P force cannot be very different from the N-N or P-P force.
    b) Since the density of nucleons in large nuclei is the same as in lighter nuclei, this means that a given nucleon feels only the force due to its immediate neighbours, and does not interact much with nucleons on the other side of the nucleus. In other words, the range of the nucleon-nucleon force is very short and of the order of 1-2 fm only.

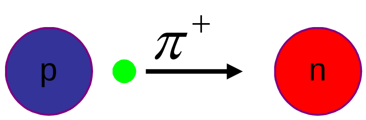
11. The force between two charges is always of one sign - repulsive if the signs are the same, and attractive if they are opposite. In the early years of quantum theory, people realized that this force comes about because of the exchange of photons between charges. The nucleon-nucleon force is different. It has to be attractive to keep the nucleus together, and has to short range (as discussed above). But, to prevent nucleons from sticking to each other, it must be repulsive at short distances. Now here, "short" and long means distances on the scale of fermis. Typically the distances between nucleons is on this scale as well.



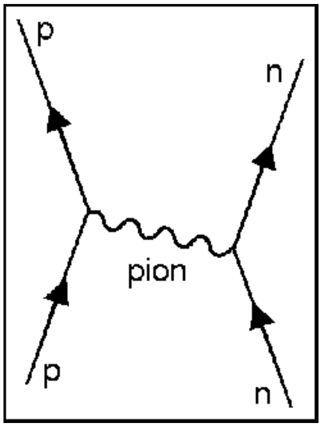
Here is roughly what the N-N potential looks like. Since the force is the negative of the slope, you can see that for large value of r, the force is attractive (negative means directed towards smaller values of r). At roughly 1.4 fm the potential reaches its most attractive point. For values of r smaller than this, the force is repulsive. In fact there is a very strong core that almost completely forbids the nucleons from getting closer than about 0.5 fm.

12. In 1935, the Japanese physicist Hideki Yukawa made an astonishing breakthrough in understanding the basis for the attractive N-N force. He assumed that, just as between charges, there must be a particle that is emitted by one nucleon and then captured by the other. He called this a "pi-meson" or "pion", and made a good guess for what its mass should be. His argument uses the time-energy uncertainty principle discussed earlier:

- Let Δt be the time that the pion takes between emission and capture. In this time it will have travelled a distance approximately equal to $c\Delta t$ because light particles can travel no faster than light.
- Creating a pion from "nothing" means that energy conservation has been violated. The amount of violation is $\Delta E =$ minimum energy of pion $\approx mc^2$ and this must obey $\Delta E\,\Delta t \approx \hbar/2$. Hence, $\Delta t \approx \frac{\hbar}{2mc^2}$ and the distance travelled by the pion $\approx$ c $\Delta t \approx \frac{\hbar}{2mc}$.



- Put $\frac{\hbar}{2mc} \approx 1.2\ fm$ (range of nuclear force). This gives the mass of the pion as close to $mc^2 \approx 124\ MeV$. This was an amazing prediction - the first time a particle had been predicted to exist on the basis of a theoretical argument. When experimentalists searched for it in 1947, they indeed found a particle of mass rather close to it, with $mc^2 \approx 138\ MeV$. It was a very dramatic confirmation for which Yukawa got the Nobel prize.



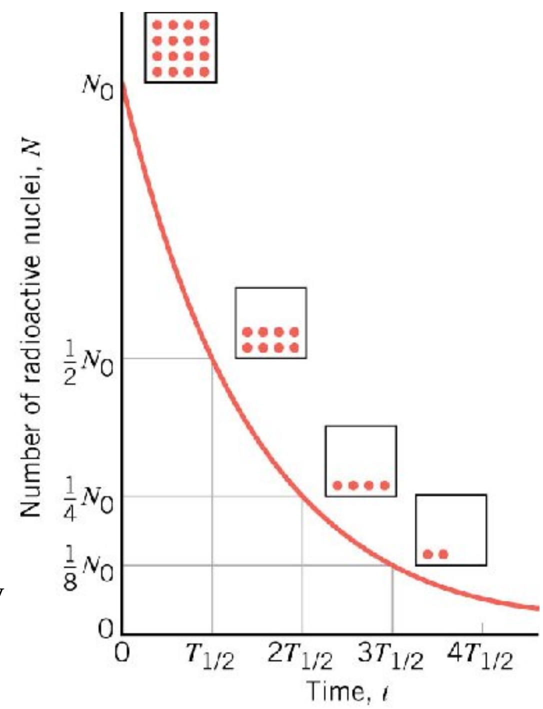
- Pions can rightfully be called the carriers of the strong nuclear force. They have 3 possible charge states: π^+, π^0, π^-. They belong to a larger family of particles called mesons. Other family members are rho-mesons, omega-mesons, K-mesons,... Today we can produce mesons in huge amounts by smashing nucleons against each other in an accelerator.

13. A few nuclei are stable, most decay. The decay law is simply derived: if the number of nuclei decreases by dn in time dt, then dn must be proportional to both the the number of nuclei n that are decaying and dt, so $dn \propto -ndt$ (minus sign for decrease). with the proportionality constant λ, we have $dn = -\lambda ndt$, or $\frac{dn}{dt} = -\lambda n$. We have encountered the solution of this type of equation before, $n = n_0 e^{-\lambda t}$. You can see that at $t = 0$, $n = n_0$. Taking the log, we have $\ln\frac{n}{n_0} = -\lambda t$. We define the half-life $T_{\frac{1}{2}}$ as the time it takes for half the original sample to decay. If $n = n_0/2$ then $\log\frac{n_0}{2n_0} = -\lambda t$, from which the half life is related to λ by, $T_{\frac{1}{2}} = \frac{\log 2}{\lambda} = \frac{0.693}{\lambda}$. The larger λ, the more radioactively unstable a nucleus is. Some typical half-lives are:

Polonium	$^{214}_{84}P$	$1.64 \times 10^{-4} s$
Krypton	$^{89}_{36}$K	3.16 minutes
Strontium	$^{90}_{38}$Sr	28.5 years
Radium	$^{226}_{88}$Ra	1600 years
Carbon	$^{14}_{6}$C	5730 years
Uranium	$^{238}_{92}$U	4.5×109 years

You can see how hugely different the lifetimes of different nuclei are!

14. Here is a plot of the number of unstable nuclei left as a function of time. After each half-life, the number of nuclei decreases in number by half of the previous. Eventually there is only one nucleus left, and that too will eventually decay. So how can the derivation for the decay law be correct? Strictly speaking, we are not allowed to write down, or solve, a differential equation like $\frac{dn}{dt} = -\lambda n$ because this assumes that $n(t)$ is a continuous function. But this is almost true because in real life we deal with very large numbers of nuclei and so it makes a lot of sense to think of $n(t)$ as continuous.



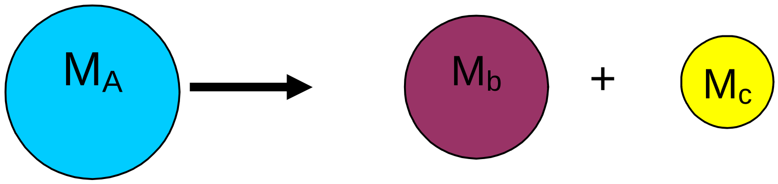
15. Just to get an idea, consider the decay of $^{222}_{86}\text{Rn}$ (Radon, a very dangerous gas that is found underground) into $^{218}_{84}\text{Po}$ (Polonium, another terrible poison) and $^{4}_{2}\text{He}$ (harmless, fortunately!). The half life is 3.8 days. So, if we started with 20,000 atoms of $^{222}_{86}\text{Rn}$, then in 3.8 days we would have 10,000 atoms of $^{222}_{86}\text{Rn}$ and 10,000 atoms of $^{218}_{84}\text{Po}$
In 7.6 days we would have 5000 atoms of $^{222}_{86}\text{Rn}$, in 11.4 days, 2500 , $^{222}_{86}\text{Rn}$ etc.

16. The decay law can be used to see how old things are. This is called radioactive dating. Carbon dating is widely used for living things that died a few hundred or few thousand years ago. How does it work?This uses the decay of the unstable isotope, $^{14}_{6}C$. Of course, the stable isotope of carbon is $^{12}_{6}C$.
   - When a living organism dies, CO_2 is no longer absorbed. Thus the ratio of carbon 14:12 decreases by half every 5730 years. We can measure the rate of decrease through $N = N_o e^{-\lambda t}$ or the "activity" $\text{A} = A_o \ e^{-\lambda t}$ with $A_o = 0.23\ Bq/g$. (The becquerel Bq is the unit of radioactivity, defined as the activity of a quantity of radioactive material in which one nucleus decays per second.)
   - The amount of isotopes in the atmosphere is approximately constant, despite a half-life of 5730 y because there is a constant replenishment of $^{14}_{6}C$ through the reaction, $^{14}_{7}N + n \rightarrow {}^{14}_{6}C + p$

17. Let us use the above idea to find the time when this man died. His body was found a few years ago buried under deep snow in a mountain pass, so it it did not decay as usual. By looking at the radioactivity in his body, it was found that that the activity of $^{14}_{6}C$ was $0.121\ Bq/g$ of body tissue. This is less than the normal activity $0.23\ Bq/g$. because $^{14}_{6}C$ has been decaying away. First find λ, $\lambda = \dfrac{0.693}{T_{1/2}} = \dfrac{0.693}{5730} = 1.21\times10^{-4}\ \ y^{-1}$
Then use, $0.121 = 0.23\ \ e^{-1.21\times10^{-4}\times t}$ which gives,
$\ln\dfrac{0.121}{0.23} = -1.21\times10^{-4}\times t$ and so $t\ =\ 5300$ years is when this poor man was killed (or died somehow)!

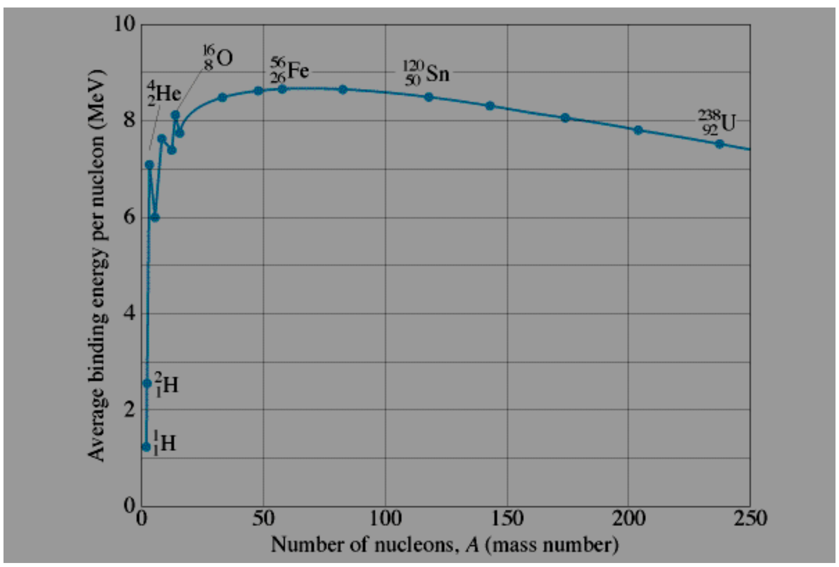
18. The most famous formula of physics, $E = mc^2$, is the basis for nuclear energy. In 1935, it was discovered by two German physicists, Otto Robert Frisch and Lise Meitner, that a heavy nucleus can fission (or break up) into two or more smaller nuclei. The total energy is, of course, conserved but the mass is not. This is completely different from the

usual situation. In the picture below you see an example of fission.



The masses of the two nuclei add up to less than the mass of the parent nucleus, and the energy released is $Q=(M_A-M_b-M_c)c^2$. This goes into kinetic energy and sends the two daughter nuclei flying apart at a large velocity. There happen to be NO completely stable nuclei above $Z=82,$ and no naturally occurring nuclei above $Z=92$. Above these limits the nuclei decay or fall apart in some fashion to get below these limits.

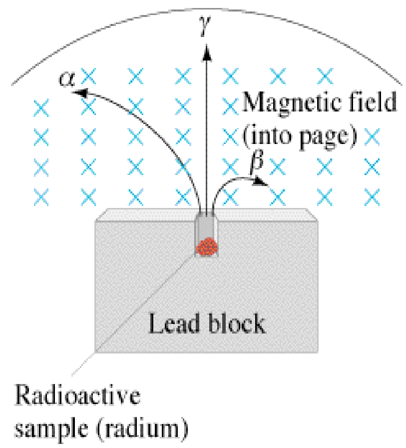
19. A very useful concept is "binding energy". Suppose you want to take a nucleon out of a nucleus. The binding energy is the amount of energy that you would have to provide to pull it on the average. Nuclei with the largest BE per nucleon are the most stable. As you



can see from the graph below, the most stable element is iron, ^{56}Fe with a BE per nucleon of about 8.6 MeV. This is why iron is the heavy element found in the largest quantity on earth and inside stars. The curve is not smmoth and you see that a ^{4}He nucleus (i.e. an α parrticle, has a relatively high binding energy and so is relatively stable. In contrast, Uranium ^{235}U or deuterium ^{2}H are much less bound and they decay.

20. Nuclei can be unstable in different ways. A nucleus can emit $\alpha,\beta,$ and γ radiations. Usually a nucleus will emit one of these three, but it is possible to emit two, or even all three of these. (In addition, as we have discussed above, a nucleus can break up into

two or more smaller fragments through the process of fission.) What is the nature of α, β, γ radiations? In the experiment shown here, you see that a piece of radium has been put in a lead block (as shielding). A magnetic field bends the charged particles emitted during the decay. We find that they are of three types:

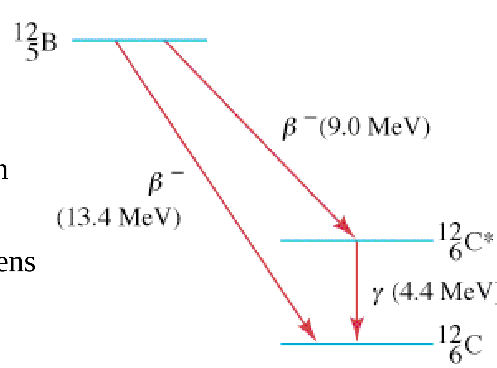


a)Heavy positively charged particles are bent to one side by the magnetic field. The amount of bending shows that they are α particles.

b)Other particles are bent much more, and in the other direction. They are electrons (β particles).

c)Some particles are not bent at all, hence must be neutral. They are γ particles (or rays, or γ photons, same thing!).

21. Let's first consider alpha decay, ${}^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}X' + {}^{4}_{2}He$. As you see A changes by 4 and Z by 2. You can think of α particles as a gang of 4 particles that always stays together in a big nucleus. Unstable nuclei simply cannot overcome the proton repulsion and an α particle ultimately succeeds in esaping the nucleus. indefinitely.
22. The simplest beta decay reaction is when a neutron decays, $n \rightarrow p + e^{-} + \bar{\nu}$. Other than a proton and electron, an anti-neutrino is also emitted. As discussed in the lecture, the (anti) neutrino is a neutral particle with a very tiny mass that interacts very weakly with matter. A nucleus can undergo beta decay with either an electron being produced, i.e ${}^{A}_{Z}X \rightarrow {}_{Z+1}{}^{A}X' + e^{-} + \bar{\nu}$ (called β^{-} decay) or with an anti-electron (positron, or positive electron), ${}^{A}_{Z}X \rightarrow {}_{Z-1}{}^{A}X' + e^{+} + \nu$, (called β^{+} decay). Beta decay involves the weak nuclear force. This is one of the four fundamental forces in the world, and its small strength means that the decay happens much more slowly than most other reactions.
23. Just as for an atom, a nucleus can only exist in certain definite energy states. When a nucleus goes from one state to the other, it can emit a photon (γ ray). Because the spacing between nuclear levels is of the order of one MeV (i.e. a million times more than in atoms), the photon is much more energetic than an optical photon. In going between levels, β emission also happens as can be seen in this diagram.



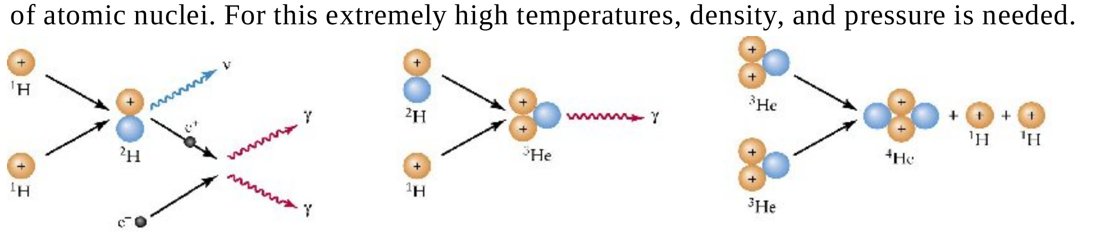
Summary of Lecture 45 – PHYSICS OF THE SUN

1. In this lecture I shall pull together the different parts of physics that you have learned in this course and apply them to understanding the source of all life on earth - our sun. We will learn that the sun operates according to principles that we can understand, and on the basis of this we can even predict the manner in which it will eventually die.

2. Basic solar facts:
   a) Mass of sun $= 2\times10^{30}$ kg = 333,000 Earth's
   b) Diameter of sun $=$ 1,392,000 km = 10^9 Earth's
   c) Age of sun = 4.6 billion years
   d) Rotation Period = 25 days at equator, 36 at poles (surface)
   e) Temperature = 15 million ^{0}K at core, 5770 ^{0}K at surface
   f) Density = $8\times$ gold at the core, average is ~ 1.5 water
   g) Composition: 72% H, 25% He, rest is metals

3. The sun puts out a huge amount of energy. In quantitative terms we measure this in by its luminosity, 3.83×10^{27} joules per second. The power output is 3.83×10^{24} kilowatts. This is equal to 8×10^{16} of the largest power plants on Earth, meaning those which produce ~5000 MW of power. Another way of expressing this: every second the sun puts out as much energy as 2.5×10^9 (2 billion) such power plants would put out every year.

4. What powers the sun? The earth is very old (billions of years). If there was a chemical fuel (say, coal or oil) at most that would last a few million years. But the sun is many thousands of tons older than that. Only after the discovery of $E = mc^2$ did we know the real secret. The sun gets its energy from the fusion (the coming together and combining of atomic nuclei. For this extremely high temperatures, density, and pressure is needed.

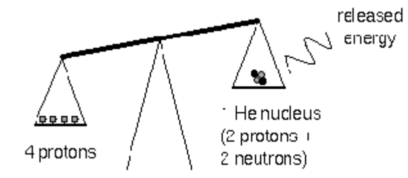


a Step 1:
- Two protons (hydrogen nuclei, ^{1}H) collide.
- One of the protons changes into a neutron (shown in blue), a neutral, nearly massless neutrino (ν), and a positron (e^+), a positively charged electron.
- The proton and neutron form a hydrogen isotope (^{2}H).
- The positron encounters an ordinary electron (e^-), annihilating both particles and converting them into gamma-ray photons (γ).

b Step 2:
- The ^{2}H nucleus from the first step collides with a third proton.
- A helium isotope (^{3}He) is formed and another gamma-ray photon is released.

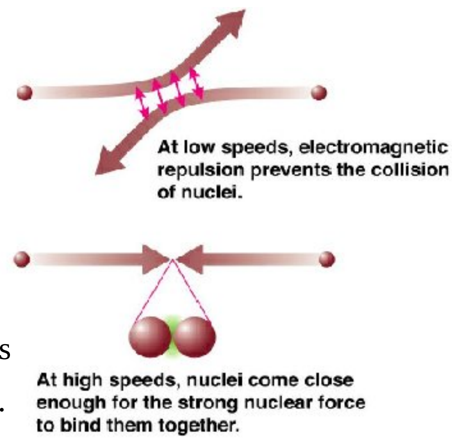
c Step 3:
- Two ^{3}He nuclei collide.
- A different helium isotope with two protons and two neutrons (^{4}He) is formed and two protons are released.

4. The basic point is that, as you can see in the picture, the combined mass of 4 protons is higher than than that of the helium nuclei into which they convert through the process shown earlier. The difference then appears in the form of kinetic energy of the released particles, which in random form is heat.

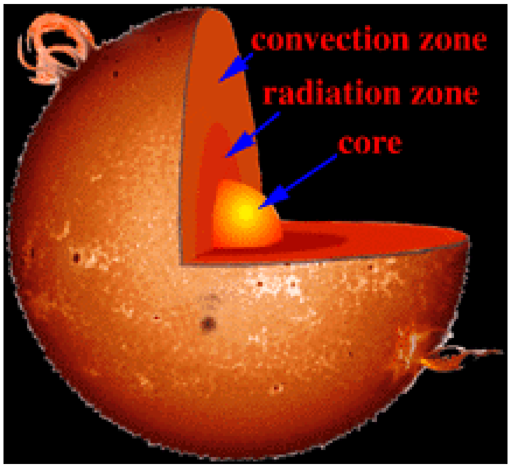


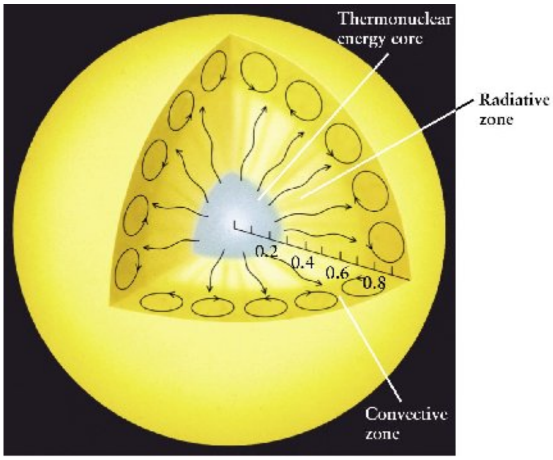
Some mass is converted into energy ($E=mc^2$)

5. Protons repel protons, but the only way in which they will participate in a fusion reaction is when they can come sufficiently close. This requires that they smash into each other at sufficiently high speeds, and hence nuclear fusion in the sun requires core temperatures greater than about 8 million ^{0}K. He nuclei can also fuse with each other to release energy, but because they are heavier much higher temperatures & densities (about 100 million ^{0}K for helium fusion) are required. Thus, stars fuse hydrogen first. Each second, the sun turns 4 million tons of hydrogen into energy.



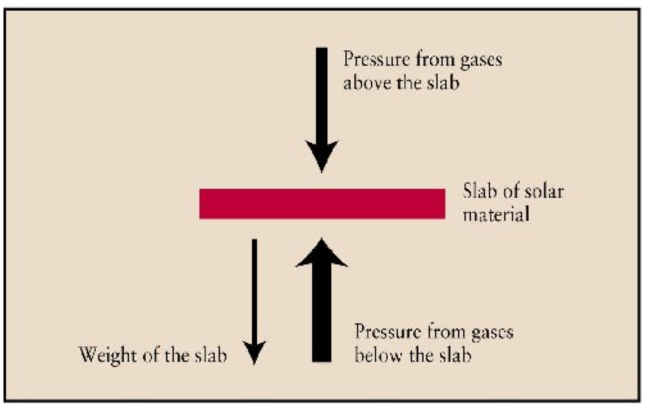
6. Fusion takes place only in the core of the sun (see diagrams below) because it is only there that the hydrogen gas is hot enough. From the core, the heat gets out by the emission of photons (radiation zone). The hot gas then exchanges heat with the sun's relatively cold exterior through convective currents. Huge columns of hot gas move from the inside towards the surface. After giving up most of the heat, the "cold" gas sinks towards the centre and the cycle goes on.

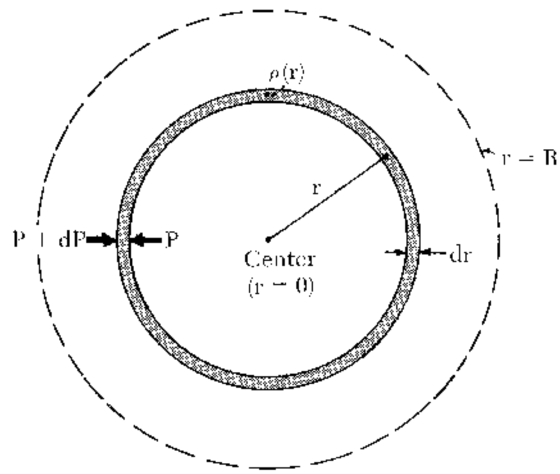




7. For over 4 billion years the sun has been nearly steady in maintaining its size. It is in a state of equilibrium between two big forces acting oppositely to each other:
   a) The hot hydrogen gas seeks to expand outwards because the thermal velocity of H atoms leads to a pressure directed outwards.
   b) Gravity tries to squeeze the star inwards because every piece of matter attracts every other piece. So the gravitational pressure is inward, increasing toward the core.

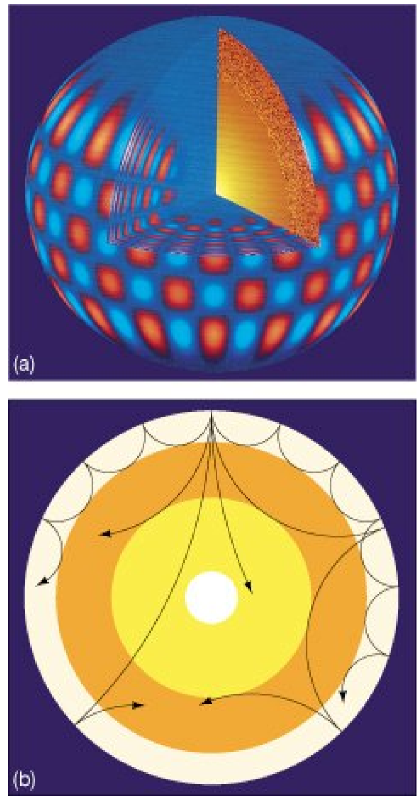
   As in the diagrams below, imagine a piece of solar material at some distance from the centre of the sun. The two forces acting upon it must exactly balance in equilibrium. If for some reason the sun cools down, the pressure of gas will decrease and the sun will contract to a new equilibrium point. Conversely, the sun will expand if the rate of fusion were to increase.



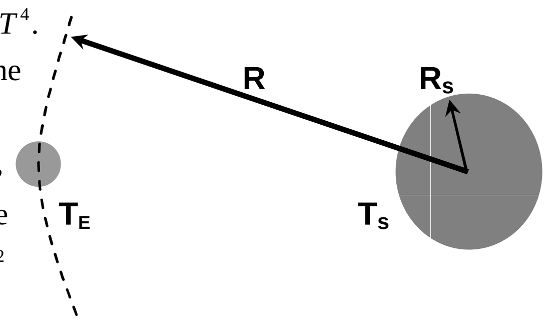


Look at the second diagram. Let $M(r)$ be the mass contained within radius r. We will not assume that the density is constant in r. First find the inward directed gravitational force on a shell of matter at radius r and thickness dr, $dF = -\dfrac{GM(r)\times\rho(r)4\pi r^2 dr}{r^2}$. There is a net pressure as shown, and we will call dP the difference of pressures. Then obviously $dF = dP\times 4\pi r^2$. Hence, $dP = -\dfrac{GM(r)\rho(r)dr}{r^2}$ or, $\dfrac{dP}{dr} = -\dfrac{GM(r)\rho(r)}{r^2}$. The total mass upto radius r is, $M(r) = \int_0^r \rho(r')4\pi r'^2 dr'$. If we knew $\rho(r)$ then $M(r)$ would also be known. By solving the differential equation, we would also then know $P(r)$. To give us a better understanding, suppose for simplicity that the sun is approximately uniform. Then the density is, $\rho \approx \dfrac{M}{(4\pi/3)R^3} \approx 1.4\text{gm cm}^{-3}$. Hence the pressure at the centre of the sun can be computed, $P_{centre} \approx \dfrac{GM\rho}{R} \approx 3\times 10^9$ atmospheres.

8. The equilibrium of forces is a nearly perfect one, but there are small disturbances and these cause "earthquakes" (actually sun-quakes) to occur on the surface. The study of the surface of the sun and sun-quakes is an area known as "helioseismology" (helio=sun, seismology is the study of vibrations and earthquakes). The pulsating motions of the sun can be seen by measuring the Doppler shifts of hydrogen lines across the face of the sun. Some parts are expanding towards the earth while adjacent regions contract away. This is like the modes of a ringing bell. Vibrations propagate inside the sun, and the waves travel through a hot dense gas. They experience reflection and refraction since the speed at which a wave travels changes when it goes from a region of high to low density or vice-versa. All this is being studied by astrophysicists today as they map out the sun's interior.

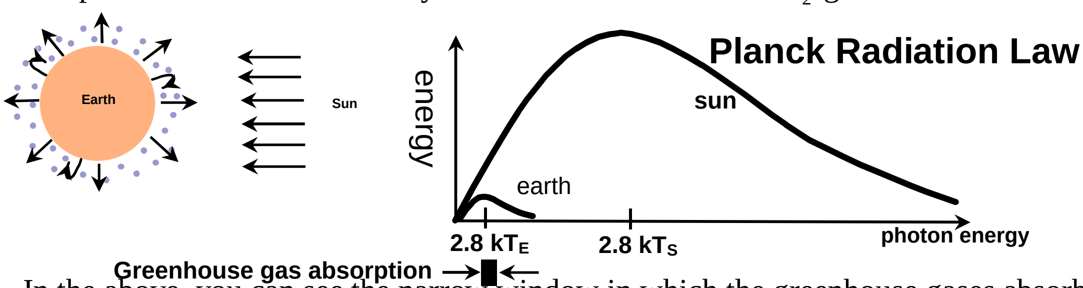


9. Let us calculate the surface temperature of a planet circulating the sun. We shall use the Stefan Boltzman law that you studied in the lecture of heat: the power radiated by a black body per unit surface area at temperature T is σT^4. For thermal equilibrium, we must have that all the power absorbed from the sun is re-radiated by the planet. Let P_S be the sun's flux at its surface, Then, $P_S = \sigma T_S^4$. Since radiation decreases by the distance squared, $P_R =$ flux at earth $= \sigma T_S^4 \times \left(\frac{R_S}{R}\right)^2$



This must be multiplied by πR_E^2, which is the crossectional area of the planet. On the other hand, for emission from the planet, $P_E =$ flux at its surface $= \sigma T_E^4$. We must multiply this by $4\pi R_E^2$, the area of planet, to get the total radiated power. Now impose the equilibrium condition: $P_R \times \pi R_E^2 = P_E \times 4\pi R_E^2$. This gives $\sigma T_S^4 \times \left(\frac{R_S}{R}\right)^2 = 4\sigma T_E^4$, and hence $T_E = \left(\frac{R_S}{2R}\right)^{1/2} T_S$. This is true for any planet, so let's see what this gives for the earth using $R_S = 7\times 10^8$ metres, $R = 1.5\times 10^{11}$ metres, and $T_S = 5800^o K$. This gives $T_E = 280K$, which is very sensible! Of course, we have assumed that the earth absorbs and emits as a black body. True?

10. Actually, the black body assumption is only approximately true. About 30% of the sun's radiation reflects off our atmosphere! In fact the average surface temperature of the earth is about 290K, or about 13 ^{0}C. The extra warming is mainly due to the "greenhouse effect". Certain gases trap the sun's radiation and it is not able to get out, thus leading to greater absorption. The main greenhouse gases are CO_2, CH_4, H_2O. (N_2, O_2 and Ar are transparent to solar and earth radiation.) CO_2 is now 360 ppm (parts per million) of the atmosphere, up from 227 ppm in 1750, before the industrial revolution. Plants turn H_2O + CO_2 into O_2 plus organics. Some estimates predict a 3 to 10 ^{0}C rise in the earth's surface temperature over the next 100 years due to the increased CO_2 greenhouse effect.

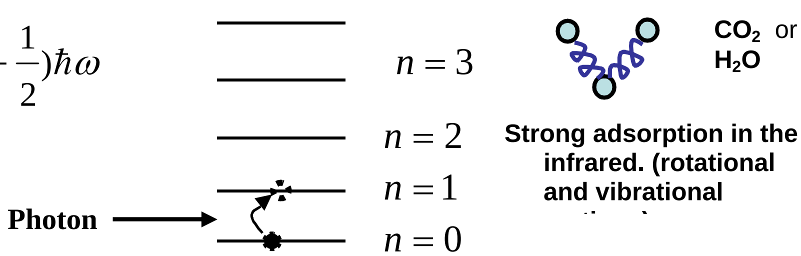


In the above, you can see the narrow window in which the greenhouse gases absorb the sun's radiation. But why are only some gases responsible, and not others?

11. The answer lies in quantum mechanics. In a previous lecture you learned that molecules can exist only in certain states that have very specific values of energy. For example, a molecule of CO_2 can oscillate and have equally spaced energy levels as shown below:

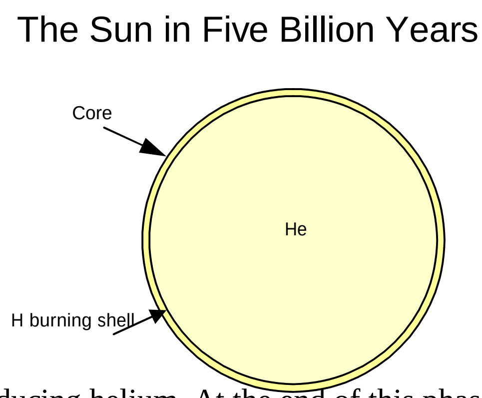
Energy levels of an oscillator:

$$\varepsilon_n = (n + \frac{1}{2})\hbar\omega$$



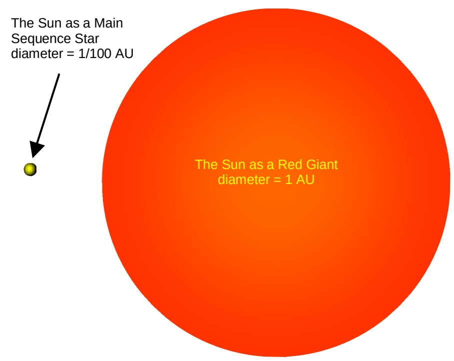
The energy at which these molecules can absorb radiation happen to lie in the spectrum of the re-radiated energy from the earth's surface. Thus, they effectively trap the out-going radiation, leading to enhanced earth temperatures. For any molecule, we can both calculate (using quantum mechanics) the frequencies at which radiation is absorbed or emitted. We can also experimentally measure the frequencies. Both of these are in very good agreement, and this is one of the reasons why we have such confidence in the correctness of quantum mechanics.

12. Finally, I have some bad news: the sun is going to die because the hydrogen supply is eventually going to run out. There is, of course, no immediate danger - the Sun can last another 5 billion years on core hydrogen fusion. During this time it will be consuming



hydrogen and producing helium. At the end of this phase, the core of the sun will have mostly helium with the little bit of hydrogen left almost entirely outside in a thin layer. The reaction rate will fall, and the core will no longer be able to balance the pull due to gravity. This will cause a shrinkage of the core. As a consequence the temperature will increase. A new phase of the sun is about to start.

13. At this point a fusion reaction in next core zone begins. This lifts the envelopes and the sun will brighten up significantly. It is now a Red Giant star, and as you can see below, it is huge! The earth will be swallowed up by the expanded sun.



14. As remarked earlier, helium can also "burn" to produce carbon, but it needs a much higher temperature. Eventually even the He will be depleted, and the reaction rate will fall. Again the small reaction rate means that the core will shrink, and heat up to the point that it crosses the carbon fusion threshold of 600 million K. Low mass stars cannot reach this temperature. Envelope expands to a supergiant, many times larger than even a Red Giant.