Lecture#1

Background
Linear $y=mx+c$
Quadratic $ax2+bx+c=0$
Cubic $ax3+bx2+cx+d=0$

Systems of Linear equations

$ax+by+c=0$
$lx+my+n=0$

Solution ?
Equation

Differential Operator

$$\frac{dy}{dx}=\frac{1}{x}$$

Taking anti derivative on both sides

$y=\ln x$

From the past

- *Algebra*
- *Trigonometry*
- *Calculus*
- *Differentiation*
- *Integration*

- *Differentiation*
  - *Algebraic Functions*
  - *Trigonometric Functions*
  - *Logarithmic Functions*
  - *Exponential Functions*
  - *Inverse Trigonometric Functions*

- More Differentiation
  - Successive Differentiation
  - Higher Order
  - Leibnitz Theorem
- Applications
  - Maxima and Minima

- Tangent and Normal
- Partial Derivatives

y=f(x)

f(x,y)=0

z=f(x,y)

Integration

- Reverse of Differentiation
- By parts
- By substitution
- By Partial Fractions
- Reduction Formula

Frequently required

- Standard Differentiation formulae
- Standard Integration Formulae

Differential Equations

- Something New
- Mostly old stuff
  - Presented differently
  - Analyzed differently
  - Applied Differently

$$\frac{dy}{dx} - 5y = 1$$

$$(y - x)dx + 4xdy = 0$$

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial y} = u$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} = 0$$

Lecture 2

Higher Order Linear Differential Equations

Preliminary theory

- A differential equation of the form

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

or $\quad a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_1(x)y'+a_0(x)y=g(x)$

where $a_0(x), a_1(x), \ldots, a_n(x), g(x)$ are functions of x and $a_n(x)\neq 0$, is called a linear differential equation with variable coefficients.

- However, we shall first study the differential equations with constant coefficients i.e. equations of the type

$$a_n\frac{d^n y}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0 y=g(x)$$

where $a_0, a_1, \ldots, a_n$ are real constants. This equation is non-homogeneous differential equation and

- If $g(x)=0$ then the differential equation becomes

$$a_n\frac{d^n y}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0 y=0$$

which is known as the **associated homogeneous differential equation**.

Initial -Value Problem

For a linear nth-order differential equation, the problem:

Solve: $\quad a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$

Subject to: $\quad y(x_0)=y_0,\quad y'(x_0)=y_0', \ldots y^{n-1}(x_0)=y_0^{n-1}$

$y_0, y_0', \ldots, y_0^{n-1}$ being arbitrary constants, is called an **initial-value problem** (IVP).

The specified values $y(x_0)=y_0$, $y'(x_0)=y_0', \ldots, y^{n-1}(x_0)=y_0^{n-1}$ are called initial-conditions.

For $n=2$ the initial-value problem reduces to

Solve: $\quad a_2(x)\frac{d^2 y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$

Subject to: $\quad y(x_0)=y_0, \ldots,\quad y'(x_0)=y_0'$

Solution of IVP

A function satisfying the differential equation on I whose graph passes through (x_0, y_0) such that the slope of the curve at the point is the number y_0' is called solution of the initial value problem.

Theorem: Existence and Uniqueness of Solutions

Let $a_n(x), a_{n-1}(x), ..., a_1(x), a_0(x)$ and $g(x)$ be c ontinuous on a n i nterval I and let $a_n(x) \neq 0, \ \forall x \in I$. If $x = x_0 \in I$, t hen a s olution $y(x)$ of th e in itial-value p roblem exist on I and is unique.

Example 1

Consider the function $y = 3e^{2x} + e^{-2x} - 3x$

This is a solution to the following initial value problem

$$y'' - 4y = 12x, \quad y(0) = 4, \ y'(0) = 1$$

Since $\dfrac{d^2y}{dx^2} = 12e^{2x} + 4e^{-2x}$

and $\dfrac{d^2y}{dx^2} - 4y = 12e^{2x} + 4e^{-2x} - 12e^{2x} - 4e^{-2x} + 12x = 12x$

Further $y(0) = 3 + 1 - 0 = 4$ and $y'(0) = 6 - 2 - 3 = 1$

Hence $y = 3e^{2x} + e^{-2x} - 3x$

is a solution of the initial value problem.

We observe that

- The equation is linear differential equation.
- The coefficients being constant are continuous.
- The function $g(x) = 12x$ being polynomial is continuous.
- The leading coefficient $a_2(x) = 1 \neq 0$ for all values of x.

Hence the function $y = 3e^{2x} + e^{-2x} - 3x$ is the unique solution.

Example 2

Consider the initial-value problem

$$3y''' + 5y'' - y' + 7y = 0,$$
$$y(1) = 0, \quad y'(1) = 0, \quad y''(1) = 0$$

Clearly the problem possesses the trivial solution $y = 0$.

Since

- The equation is homogeneous linear differential equation.
- The coefficients of the equation are constants.
- Being constant the coefficient are continuous.
- The leading coefficient $a_3 = 3 \neq 0$.

Hence $y = 0$ is the only solution of the initial value problem.

Note: If $a_n = 0$?

If $a_n(x) = 0$ in the differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

for some $x \in I$ then

- Solution of initial-value problem may not be unique.
- Solution of initial-value problem may not even exist.

Example 4
Consider the function

$$y = cx^2 + x + 3$$

and the initial-value problem

$$x^2 y'' - 2xy' + 2y = 6$$

$$y(0) = 3, \qquad y'(0) = 1$$

Then $\quad y' = 2cx + 1$ and $y'' = 2c$

Therefore

$$x^2 y'' - 2xy' + 2y = x^2(2c) - 2x(2cx+1) + 2(cx^2 + x + 3)$$
$$= 2cx^2 - 4cx^2 - 2x + 2cx^2 + 2x + 6$$
$$= 6.$$

Also $\quad y(0) = 3 \Rightarrow c(0) + 0 + 3 = 3$

and $\quad y'(0) = 1 \Rightarrow 2c(0) + 1 = 1$

So that for any choice of c, the function 'y' satisfies the differential equation and the initial conditions. Hence the solution of the initial value problem is not unique.

Note that

- The equation is linear differential equation.
- The coefficients being polynomials are continuous everywhere.
- The function $g(x)$ being constant is constant everywhere.
- The leading coefficient $a_2(x) = x^2 = 0$ at $x = 0 \in (-\infty, \infty)$.

Hence $a_2(x) = 0$ brought non-uniqueness in the solution

Boundary-value problem (BVP)

For a 2nd order linear differential equation, the problem

Solve: $$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

Subject to: $y(a)=y_0, \quad y(b)=y_1$

is called a ***boundary-value problem***. The specified values $y(a)=y_0$, and $y(b)=y_1$ are called ***boundary conditions.***

Solution of BVP

A solution of the boundary value problem is a function satisfying the differential equation on some interval I, containing a and b, whose graph passes through two points (a, y_0) and (b, y_1).

Example 5

Consider the function

$$y=3x^2-6x+3$$

We can prove that this function is a solution of the boundary-value problem

$$x^2y''-2xy'+2y=6,$$
$$y(1)=0, \qquad y(2)=3$$

Since $$\frac{dy}{dx}=6x-6,\ \frac{d^2y}{dx^2}=6$$

Therefore $$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=6x^2-12x^2+12x+6x^2-12x+6=6$$

Also $$y(1)=3-6+3=0, \quad y(2)=12-12+3=3$$

Therefore, the function 'y' satisfies both the differential equation and the boundary conditions. Hence y is a solution of the boundary value problem.

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Possible Boundary Conditions

For a 2nd order linear non-homogeneous differential equation

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

all the possible pairs of boundary conditions are

$$y(a)=y_0, \qquad y(b)=y_1,$$
$$y'(a)=y'_0, \qquad y(b)=y_1,$$
$$y(a)=y_0, \qquad y'(b)=y'_1,$$
$$y'(a)=y'_0, \qquad y'(b)=y'_1$$

where y_0, y'_0, y_1 and y'_1 denote the arbitrary constants.

In General
All the four pairs of conditions mentioned above are just special cases of the general boundary conditions

$$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1$$
$$\alpha_2 y(b) + \beta_2 y'(b) = \gamma_2$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \{0,1\}$

Note that
A boundary value problem may have

- Several solutions.
- A unique solution, or
- No solution at all.

Example 1
Consider the function

$$y = c_1 \cos 4x + c_2 \sin 4x$$

and the boundary value problem

$$y'' + 16y = 0, \quad y(0) = 0, \quad y(\pi/2) = 0$$

Then

$$y' = -4c_1 \sin 4x + 4c_2 \cos 4x$$
$$y'' = -16(c_1 \cos 4x + c_2 \sin 4x)$$
$$y'' = -16y$$
$$y'' + 16y = 0$$

Therefore, the function

$$y = c_1 \cos 4x + c_2 \sin 4x$$

satisfies the differential equation

$$y'' + 16y = 0.$$

Now apply the boundary conditions
Applying $y(0) = 0$
We obtain

$$0 = c_1 \cos 0 + c_2 \sin 0$$
$$\Rightarrow c_1 = 0$$

So that

$$y = c_2 \sin 4x.$$

But when we apply the 2nd condition $y(\pi/2) = 0$, we have

$$0 = c_2 \sin 2\pi$$

Since $\sin 2\pi = 0$, the condition is satisfied for any choice of c_2, solution of the problem is the one-parameter family of functions

$$y = c_2 \sin 4x$$

Hence, there are an ***infinite number of solutions*** of the boundary value problem.

Example 2

Solve the boundary value problem

$$y'' + 16y = 0$$

$$y(0) = 0, \; y\left(\frac{\pi}{8}\right) = 0,$$

Solution:

As verified in the previous example that the function

$$y = c_1 \cos 4x + c_2 \sin 4x$$

satisfies the differential equation

$$y'' + 16y = 0$$

We now apply the boundary conditions

$$y(0) = 0 \Rightarrow 0 = c_1 + 0$$

and $\quad y(\pi/8) = 0 \Rightarrow 0 = 0 + c_2$

So that $\quad c_1 = 0 = c_2$

Hence

$$y = 0$$

is the only solution of the boundary-value problem.

Example 3

Solve the differential equation

$$y'' + 16y = 0$$

subject to the boundary conditions

$$y(0) = 0, \quad y(\pi/2) = 1$$

Solution:

As verified in an earlier example that the function

$$y = c_1 \cos 4x + c_2 \sin 4x$$

satisfies the differential equation

$$y'' + 16y = 0$$

We now apply the boundary conditions

$$y(0) = 0 \Rightarrow 0 = c_1 + 0$$

Therefore $\quad c_1 = 0$

So that $\quad y = c_2 \sin 4x$

However $\quad y(\pi/2) = 1 \Rightarrow c_2 \sin 2\pi = 1$

or $\quad 1 = c_2.0 \Rightarrow 1 = 0$

This is a clear contradiction. Therefore, the boundary value problem has ***no solution***.

Definition: Linear Dependence

A set of functions

$$\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

is said to be ***linearly dependent*** on an interval I if $\exists$ constants $c_1, c_2, \ldots, c_n$ not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots . + c_n f_n(x) = 0, \quad \forall\, x \in I$$

Definition: Linear Independence

A set of functions

$$\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

is said to be linearly independent on an interval I if

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0, \quad \forall\, x \in I,$$

only when

$$c_1 = c_2 = \cdots = c_n = 0.$$

Case of two functions:

If $n = 2$ then the set of functions becomes

$$\{f_1(x), f_2(x)\}$$

If we suppose that

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

Also t hat t he f unctions a re l inearly d ependent on a n i nterval I then ei ther $c_1 \neq 0$ or $c_2 \neq 0$.

Let us assume that $c_1 \neq 0$, then

$$f_1(x) = -\frac{c_2}{c_1} f_2(x);$$

Hence $f_1(x)$ is the constant multiple of $f_2(x)$.

Conversely, if we suppose

$$f_1(x) = \mathrm{c}_2\ f_2(x)$$

Then $$(-1) f_1(x) + c_2 f_2(x) = 0, \ \forall x \in I$$

So that the functions are linearly dependent because $c_1 = -1$.

Hence, we conclude that:

- Any two functions $f_1(x)$ and $f_2(x)$ are linearly dependent on an interval I if and only if one is the constant multiple of the other.
- Any two functions are linearly independent when neither is a constant multiple of the other on an interval *I*.
- In general a set of n functions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ is linearly dependent if at least one of them can be expressed as a linear combination of the remaining.

Example 1

The functions

$$f_1(x) = \sin 2x, \qquad \forall x \in (-\infty, \infty)$$
$$f_2(x) = \sin x \cos x, \quad \forall x \in (-\infty, \infty)$$

If we choose $c_1 = \frac{1}{2}$ and $c_2 = -1$ then

$$c_1 \sin 2x + c_2 \sin x \cos x = \frac{1}{2}(2\sin x \cos x) - \sin x \cos x = 0$$

Hence, the two functions $f_1(x)$ and $f_2(x)$ are linearly dependent.

Example 3

Consider the functions

$$f_1(x) = \cos^2 x, \; f_2(x) = \sin^2 x, \quad \forall\, x \in (-\pi/2, \pi/2),$$
$$f_3(x) = \sec^2 x, \; f_4(x) = \tan^2 x, \quad \forall\, x \in (-\pi/2, \pi/2)$$

If we choose $c_1 = c_2 = 1, c_3 = -1, c_4 = 1$, then

$$\begin{aligned} & c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + c_4 f_4(x) \\ &= c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x \\ &= \cos^2 x + \sin^2 x + -1 - \tan^2 x + \tan^2 x \\ &= 1 - 1 + 0 = 0 \end{aligned}$$

Therefore, the given functions are linearly dependent.

Note that

The function $f_3(x)$ can be written as a linear combination of other three functions $f_1(x), f_2(x)$ and $f_4(x)$ because $\sec^2 x = \cos^2 x + \sin^2 x + \tan^2 x$.

Example 3
Consider the functions

$$f_1(x) = 1 + x, \ \forall\, x \in (-\infty, \infty)$$
$$f_2(x) = x, \quad \forall\, x \in (-\infty, \infty)$$
$$f_3(x) = x^2, \ \forall\, x \in (-\infty, \infty)$$

Then

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

means that

$$c_1(1+x) + c_2 x + c_3 x^2 = 0$$

or $$c_1 + (c_1 + c_2)x + c_3 x^2 = 0$$

Equating coefficients of x and x^2 constant terms we obtain

$$c_1 = 0 = c_3$$
$$c_1 + c_2 = 0$$

Therefore $c_1 = c_2 = c_3 = 0$

Hence, the three functions $f_1(x), f_2(x)$ and $f_3(x)$ are linearly independent.

Definition: Wronskian

Suppose that the function $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n-1$ derivatives then the determinant

$$\begin{vmatrix} f_1 & f_2 & \cdots\cdots\cdots & f_n \\ f_1^{/} & f_2^{/} & \cdots\cdots\cdots & f_n^{/} \\ \vdots & \vdots & & \vdots \\ f_1^{n-1} & f_2^{n-1} & \cdots\cdots\cdots & f_n^{n-1} \end{vmatrix}$$

is called Wronskian of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ and is denoted by $W\left(f_1(x), f_1(x), \ldots, f_1(x)\right)$.

Theorem: Criterion for Linearly Independent Functions

Suppose the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possess at least *n-1* derivatives on an interval I. If

$$W(f_1(x), f_2(x), \ldots, f_n(x)) \neq 0$$

for at least one point in I, then functions $f_1(x), f_2(x), \ldots, f_n(x)$ are linearly independent on the interval I.

Note that

This is only a sufficient condition for linear independence of a set of functions.

In other words

If $f_1(x), f_2(x), \ldots, f_n(x)$ possesses a t l east $n-1$ derivatives on an i nterval an d are linearly dependent on I , then

$$W(f_1(x), f_2(x), \ldots, f_n(x)) = 0, \quad \forall x \in I$$

However, the converse is not true. i.e. a Vanishing Wronskian does not guarantee linear dependence of functions.

Example 1

The functions

$$f_1(x) = \sin^2 x$$
$$f_2(x) = 1 - \cos 2x$$

are linearly dependent because

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

We observe that for all $x \in (-\infty, \infty)$

$$W(f_1(x), f_2(x)) = \begin{vmatrix} \sin^2 x & 1 - \cos 2x \\ 2\sin x \cos x & 2\sin 2x \end{vmatrix}$$

$$= 2\sin^2 x \sin 2x - 2\sin x \cos x$$
$$+ 2\sin x \cos x \cos 2x$$
$$= \sin 2x [2\sin^2 x - 1 + \cos 2x]$$
$$= \sin 2x [2\sin^2 x - 1 + \cos^2 x - \sin^2 x]$$
$$= \sin 2x [\sin^2 x + \cos^2 x - 1]$$
$$= 0$$

Example 2

Consider the functions

$$f_1(x) = e^{m_1 x}, \; f_2(x) = e^{m_2 x}, \quad m_1 \neq m_2$$

The functions are linearly independent because

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

if and only if $c_1 = 0 = c_2$ as $m_1 \neq m_2$

Now for all $x \in R$

$$W\left(e^{m_1x}, e^{m_2x}\right) = \begin{vmatrix} e^{m_1x} & e^{m_2x} \\ m_1e^{m_1x} & m_2e^{m_2x} \end{vmatrix}$$
$$= (m_2 - m_1)e^{(m_1+m_2)x}$$
$$\neq 0$$

Thus f_1 and f_2 are linearly independent of any interval on *x*-axis.

Example 3

If α and β are real numbers, $\beta \neq 0$, then the functions

$$y_1 = e^{\alpha x}\cos\beta x \text{ and } y_2 = e^{\alpha x}\sin\beta x$$

are linearly independent on any interval of the *x*-axis because

$$W\left(e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x\right)$$

$$= \begin{vmatrix} e^{\alpha x}\cos\beta x & e^{\alpha x}\sin\beta x \\ -\beta e^{\alpha x}\sin\beta x + \alpha e^{\alpha x}\cos\beta x & \beta e^{\alpha x}\cos\beta x + \alpha e^{\alpha x}\sin\beta x \end{vmatrix}$$
$$= \beta e^{2\alpha x}\left(\cos^2\beta x + \sin^2\beta x\right) = \beta e^{2\alpha x} \neq 0.$$

Example 4

The functions

$$f_1(x) = e^x,\ f_2(x) = xe^x,\ \text{and}\ f_3(x) = x^2e^x$$

are linearly independent on any interval of the *x*-axis because for all $x \in R$, we have

$$W\left(e^x, xe^x, x^2e^x\right) = \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & xe^x + e^x & x^2e^x + 2xe^x \\ e^x & xe^x + 2e^x & x^2e^x + 4xe^x + 2e^x \end{vmatrix}$$

$$= 2e^{3x} \neq 0$$

Exercise

1. Given that

   $$y = c_1 e^{x} + c_2 e^{-x}$$

   is a two-parameter family of solutions of the differential equation

   $$y'' - y = 0$$

   on $(-\infty, \infty)$, find a member of the family satisfying the boundary conditions

   $$y(0) = 0, \quad y'(1) = 1.$$

2. Given that

   $$y = c_1 + c_2 \cos x + c_3 \sin x$$

   is a three-parameter family of solutions of the differential equation

   $$y''' + y' = 0$$

   on the interval $(-\infty, \infty)$, find a member of the family satisfying the initial conditions $y(\pi) = 0,\ y'(\pi) = 2, y''(\pi) = -1$.

3. Given that

   $$y = c_1 x + c_2 x \ln x$$

   is a two-parameter family of solutions of the differential equation $x^2 y'' - xy' + y = 0$ on $(-\infty, \infty)$. Find a member of the family satisfying the initial conditions

   $$y(1) = 3, \quad y'(1) = -1.$$

   Determine whether the functions in problems 4-7 are linearly independent or dependent on $(-\infty, \infty)$.

4. $f_1(x) = x, \quad f_2(x) = x^2, \quad f_3(x) = 4x - 3x^2$
5. $f_1(x) = 0, \quad f_2(x) = x, \quad f_3(x) = e^{x}$
6. $f_1(x) = \cos 2x, \quad f_2(x) = 1, \quad f_3(x) = \cos^2 x$
7. $f_1(x) = e^{x}, \quad f_2(x) = e^{-x}, \quad f_3(x) = \sinh x$

Show by computing the Wronskian that the given functions are linearly independent on the indicated interval.

8. $\tan x,\ \cot x; \qquad (-\infty, \infty)$
9. $e^{x}, e^{-x}, e^{4x}; \quad (-\infty, \infty)$
10. $x, x\ln x, x^2 \ln x; \quad (0, \infty)$

Lecture 3
Solutions of Higher Order Linear Equations

Preliminary Theory

- In order to solve an nth order non-homogeneous linear differential equation
$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$
we first solve the associated homogeneous differential equation
$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

- Therefore, we first concentrate upon the preliminary theory and the methods of solving the homogeneous linear differential equation.

- We recall that a function $y=f(x)$ that satisfies the associated homogeneous equation
$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$
is called solution of the differential equation.

Superposition Principle

Suppose that $y_1, y_2, \ldots, y_n$ are solutions on an interval I of the homogeneous linear differential equation

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

Then

$$y=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x),$$

$c_1, c_2, \ldots, c_n$ being arbitrary constants is also a solution of the differential equation.

Note that

- A constant multiple $y=c_1y_1(x)$ of a solution $y_1(x)$ of the homogeneous linear differential equation is also a solution of the equation.
- The homogeneous linear differential equations always possess the trivial solution $y=0$.

- The superposition principle is a property of linear differential equations and it does not hold in case of non-linear differential equations.

Example 1

The functions

$$y_1 = e^x, y_2 = c^{2x}, \text{ and } y_3 = e^{3x}$$

all satisfy the homogeneous differential equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$$

on $(-\infty, \infty)$. Thus y_1, y_2 and y_3 are all solutions of the differential equation

Now suppose that

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}.$$

Then

$$\frac{dy}{dx} = c_1e^x + 2c_2e^{2x} + 3c_3e^{3x}.$$

$$\frac{d^2y}{dx^2} = c_1e^x + 4c_2e^{2x} + 9c_3e^{3x}.$$

$$\frac{d^3y}{dx^3} = c_1e^x + 8c_2e^{2x} + 27c_3e^{3x}.$$

Therefore

$$\begin{aligned}
&\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y \\
&= c_1\left(e^x - 6e^x + 11e^x - 6e^x\right) + c_2\left(8e^{2x} - 24e^{2x} + 22e^{2x} - 6e^{2x}\right) \\
&\quad + c_3\left(27e^{3x} - 54e^{3x} + 33e^{3x} - 6e^{3x}\right) \\
&= c_1(12-12)e^x + c_2(30-30)e^{2x} + c_3(60-60)e^{3x} \\
&= 0
\end{aligned}$$

Thus

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}.$$

is also a solution of the differential equation.

Example 2

The function

$$y = x^2$$

is a solution of the homogeneous linear equation

$$x^2 y'' - 3xy' + 4y = 0$$

on $(0,\infty)$.

Now consider

$$y = cx^2$$

Then $\quad y' = 2cx \quad$ and $\quad y'' = 2c$

So that $\quad x^2 y'' - 3xy' + 4y = 2cx^2 - 6cx^2 + 4cx^2 = 0$

Hence the function

$$y = cx^2$$

is also a solution of the given differential equation.

The Wronskian

Suppose that y_1, y_2 are 2 solutions, on an interval I, of the second order homogeneous linear differential equation

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

Then either $\quad W(y_1, y_2) = 0, \quad \forall\, x \in I$

or $\quad W(y_1, y_2) \neq 0, \quad \forall\, x \in I$

To verify this we write the equation as

$$\frac{d^2 y}{dx^2} + \frac{Pdy}{dx} + Qy = 0$$

Now $\quad W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$

Differentiating $w.r.to\ x$, we have

$$\frac{dW}{dx} = y_1 y_2'' - y_1'' y_2$$

Since y_1 and y_2 are solutions of the differential equation

$$\frac{d^2y}{dx^2} + \frac{Pdy}{dx} + Qy = 0$$

Therefore

$$y_1'' + Py_1' + Qy_1 = 0$$

$$y_2'' + Py_2' + Qy_2 = 0$$

Multiplying 1st equation by y_2 and 2nd by y_1 the have

$$y_1''y_2 + Py_1'y_2 + Qy_1y_2 = 0$$

$$y_1y_2'' + Py_1y_2' + Qy_1y_2 = 0$$

Subtracting the two equations we have:

$$\left(y_1y_2'' - y_2y_1''\right) + P\left(y_1y_2' - y_1'y_2\right) = 0$$

or

$$\frac{dW}{dx} + PW = 0$$

This is a linear 1st order differential equation in W, whose solution is

$$W = ce^{-\int Pdx}$$

Therefore

- If $c \neq 0$ then $W\left(y_1, y_2\right) \neq 0, \quad \forall\, x \in I$
- If $c = 0$ then $W\left(y_1, y_2\right) = 0, \quad \forall\, x \in I$

Hence Wronskian of y_1 and y_2 is either identically zero or is never zero on I.

In general

If $y_1, y_2, \ldots, y_n$ are n solutions, on an interval I, of the homogeneous nth order linear differential equation with constants coefficients

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$$

Then

Either $\qquad W\left(y_1, y_2, \ldots, y_n\right) = 0, \quad \forall x \in I$

or $\qquad W\left(y_1, y_2, \ldots, y_n\right) \neq 0, \quad \forall x \in I$

Linear Independence of Solutions:

Suppose that

$$y_1, y_2, \ldots, y_n$$

are n solutions, on an interval I, of the homogeneous linear nth-order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Then the set of solutions is linearly independent on *I* if and only if

$$W(y_{1,} y_2, \ldots, y_n) \neq 0$$

In other words

The solutions

$$y_1, y_2, \ldots, y_n$$

are linearly dependent if and only if

$$W(y_{1,} y_2, \ldots, y_n) = 0, \quad \forall x \in I$$

Fundamental Set of Solutions

A set

$$\{y_1, y_2, \ldots, y_n\}$$

of n linearly independent solutions, on interval I, of the homogeneous linear nth-order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

is said to be a fundamental set of solutions on the interval I.

Existence of a Fundamental Set

There always exists a fundamental set of solutions for a linear nth-order homogeneous differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on an interval *I*.

General Solution-Homogeneous Equations

Suppose that

$$\{y_1, y_2, \ldots, y_n\}$$

is a fundamental set of solutions, on an interval *I*, of the homogeneous linear *n*th-order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Then the general solution of the equation on the interval *I* is defined to be

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

Here $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example 1

The functions

$$y_1 = e^{3x} \text{ and } y_2 = e^{-3x}$$

are solutions of the differential equation

$$y'' - 9y = 0$$

Since $$W\left(e^{3x}, e^{-3x}\right) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0, \quad \forall x \in I$$

Therefore y_1 and y_2 from a fundamental set of solutions on $(-\infty, \infty)$. **Hence** general solution of the differential equation on the $(-\infty, \infty)$ is

$$y = c_1 e^{3x} + c_2 e^{-3x}$$

Example 2

Consider the function $y = 4\sinh 3x - 5e^{-3x}$

Then $y' = 12\cosh 3x + 15e^{-3x}$, $y'' = 36\sinh 3x - 45e^{-3x}$

$\Rightarrow$ $y'' = 9\left(4\sinh 3x - 5e^{-3x}\right)$ or $y'' = 9y$,

Therefore $y'' - 9y = 0$

Hence $$y = 4\sinh 3x - 5e^{-3x}$$

is a particular solution of differential equation.

$$y'' - 9y = 0$$

The general solution of the differential equation is

$$y = c_1 e^{3x} + c_2 e^{-3x}$$

Choosing $c_1 = 2, c_2 = -7$

We obtain $y = 2e^{3x} - 7e^{-3x}$

$$y = 2e^{3x} - 2e^{-3x} - 5e^{-3x}$$

$$y = 4\left(\frac{e^{3x} - e^{-3x}}{2}\right) - 5e^{-3x}$$

$$y = 4\sinh 3x - 5e^{-3x}$$

Hence, the particular solution has been obtained from the general solution.

Example 3

Consider the differential equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$$

and suppose that $y_1 = e^x,\ y_2 = e^{2x}$ and $y_3 = e^{3x}$

Then $\dfrac{dy_1}{dx} = e^x = \dfrac{d^2y_1}{dx^2} = \dfrac{d^3y_1}{dx^3}$

Therefore $\dfrac{d^3y_1}{dx^3} - 6\dfrac{d^2y_1}{dx^2} + 11\dfrac{dy_1}{dx} - 6y_1 = e^x - 6e^x + 11e^x - 6e^x$

or $\dfrac{d^3y_1}{dx^3} - 6\dfrac{d^2y_1}{dx^2} + 11\dfrac{dy_1}{dx} - 6y_1 = 12e^x - 12e^x = 0$

Thus the function y_1 is a solution of the differential equation. Similarly, we can verify that the other two functions i.e. y_2 and y_3 also satisfy the differential equation.

Now for all $x \in R$

$$W\left(e^x, e^{2x}, e^{3x}\right) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0 \quad \forall\, x \in I$$

Therefore $y_1, y_2,$ and y_3 form a fundamental solution of the differential equation on $(-\infty, \infty)$. We conclude that

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

is the general solution of the differential equation on the interval $(-\infty, \infty)$.

Non-Homogeneous Equations

A function y_p that satisfies the non-homogeneous differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

and is free of parameters is called the particular solution of the differential equation

Example 1

Suppose that

$$y_p = 3$$

Then $\quad y_p'' = 0$

So that

$$y_p'' + 9y_p = 0 + 9(3)$$
$$= 27$$

Therefore

$$y_p = 3$$

is a particular solution of the differential equation

$$y_p'' + 9y_p = 27$$

Example 2

Suppose that

$$y_p = x^3 - x$$

Then $\quad y_p' = 3x^2 - 1, \quad y_p'' = 6x$

Therefore $\quad x^2 y_p'' + 2xy_p' - 8y_p = x^2(6x) + 2x\left(3x^2 - 1\right) - 8\left(x^3 - x\right)$

$$= 4x^3 + 6x$$

Therefore

$$y_p = x^3 - x$$

is a particular solution of the differential equation

$$x^2 y'' + 2xy' - 8y = 4x^3 + 6x$$

Complementary Function

The general solution

$$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

of the homogeneous linear differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

is known as the complementary function for the non-homogeneous linear differential equation.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

General Solution of Non-Homogeneous Equations

Suppose that

- The particular solution of the non-homogeneous equation

  $$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

  is y_p.

- The complementary function of the non-homogeneous differential equation

  $$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

  is

  $$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n .$$

- Then general solution of the non-homogeneous equation on the interval I is given by

  $$y = y_c + y_p$$

  or

  $$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x) = y_c(x) + y_p(x)$$

Hence

General Solution = Complementary solution + any particular solution.

Example

Suppose that

$$y_p = -\frac{11}{12} - \frac{1}{2}x$$

Then $$y'_p = -\frac{1}{2},\ y''_p = 0 = y'''_p$$

$\therefore$ $$\frac{d^3y_p}{dx^3} - 6\frac{d^2y_p}{dx^2} + 11\frac{dy_p}{dx} - 6y_p = 0 - 0 - \frac{11}{2} + \frac{11}{2} + 3x = 3x$$

Hence

$$y_p = -\frac{11}{12} - \frac{1}{2}x$$

is a particular solution of the non-homogeneous equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 3x$$

Now consider

$$y_c = c_1e^x + c_2e^{2x} + c_3e^{3x}$$

Then

$$\frac{dy_c}{dx} = c_1e^x + 2c_2e^{2x} + 3c_3e^{3x}$$
$$\frac{d^2y_c}{dx^2} = c_1e^x + 4c_2e^{2x} + 9c_3e^{3x}$$
$$\frac{d^3y_c}{dx^3} = c_1e^x + 8c_2e^{2x} + 27c_3e^{3x}$$

Since,

$$\frac{d^3y_c}{dx^3} - 6\frac{d^2y_c}{dx^2} + 11\frac{dy_c}{dx} - 6y_c$$
$$= c_1e^x + 8c_2e^{2x} + 27c_3e^{3x} - 6\left(c_1e^x + 4c_2e^{2x} + 9c_3e^{3x}\right)$$
$$+11\left(c_1e^x + 2c_2e^{2x} + 3c_3e^{3x}\right) - 6\left(c_1e^x + c_2e^{2x} + c_3e^{3x}\right)$$
$$= 12c_1e^x - 12c_1e^x + 30c_2e^{2x} - 30c_2e^{2x} + 60c_3e^{3x} - 60c_3e^{3x}$$
$$= 0$$

Thus y_c is general solution of associated homogeneous differential equation

$$\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y=0$$

Hence general solution of the non-homogeneous equation is

$$y=y_c+y_p=c_1e^x+c_2e^{2x}+c_3e^{3x}-\frac{11}{12}-\frac{1}{2}x$$

Superposition Principle for Non-homogeneous Equations

Suppose that

$$y_{p_1},y_{p_2},\ldots,y_{p_k}$$

denote the particular solutions of the k differential equation

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_1(x)y'+a_0(x)y=g_i(x),$$

$i=1,2,\ldots k$, on an interval I. Then

$$y_p=y_{p_1}(x)+y_{p_2}(x)+\cdots+y_{p_k}(x)$$

is a particular solution of

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_1(x)y'+a_0(x)y=g_1(x)+g_2(x)+\cdots+g_k(x)$$

Example
Consider the differential equation

$$y''-3y'+4y=-16x^2+24x-8+2e^{2x}+2xe^x-e^x$$

Suppose that

$$y_{p_1}=-4x^2,\qquad y_{p_2}=e^{2x},\qquad y_{p_3}=xe^x$$

Then $\qquad y''_{p_1}-3y'_{p_1}+4y_{p_1}=-8+24x-16x^2$

Therefore $\qquad y_{p_1}=-4x^2$

is a particular solution of the non-homogenous differential equation

$$y''-3y'+4y=-16x^2+24x-8$$

Similarly, it can be verified that

$$y_{p_2}=e^{2x}\ \text{ and }\ y_{p_3}=xe^x$$

are particular solutions of the equations:

$$y''-3y'+4y=2e^{2x}$$

and $\qquad y''-3y'+4y=2xe^x-e^x$

respectively.

Hence $y_p = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^x$

is a particular solution of the differential equation

$$y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^x - e^x$$

Exercise

Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Form the general solution.

1. $y'' - y' - 12y = 0;\ \ e^{-3x}, e^{4x},\ \ (-\infty, \infty)$

2. $y'' - 2y' + 5y = 0;\ \ e^{x}\cos 2x, e^{x}\sin 2x,\ \ (-\infty, \infty)$

3. $x^2 y'' + xy' + y = 0;\ \ \cos(\ln x), \sin(\ln x),\ \ (0, \infty)$

4. $4y'' - 4y' + y = 0;\ \ e^{x/2}, xe^{x/2},\ \ (-\infty, \infty)$

5. $x^2 y'' - 6xy' + 12y = 0;\ \ x^3,\ x^4\ \ (0, \infty)$

6. $y'' - 4y = 0;\ \ \cosh 2x,\ \sinh 2x,\ \ (-\infty, \infty)$

Verify that the given two-parameter family of functions is the general solution of the non-homogeneous differential equation on the indicated interval.

7. $y'' + y = \sec x, \quad y = c_1 \cos x + c_2 \sin x + x\sin x + (\cos x)\ln(\cos x),\ \ (-\pi/2, \pi/2).$

8. $y'' - 4y' + 4y = 2e^{2x} + 4x - 12, \quad y = c_1 e^{2x} + c_2 xe^{2x} + x^2 e^{2x} + x - 2$

9. $y'' - 7y' + 10y = 24e^{x}, \quad y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x},\ \ (-\infty, \infty)$

10. $x^2 y'' + 5xy' + y = x^2 - x, \quad y = c_1 x^{-1/2} + c_2 x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x,\ \ (0, \infty)$

Lecture 4

Construction of a Second Solution

General Case

Consider the differential equation

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

We divide by $a_2(x)$ to put the above equation in the form

$$y''+P(x)y'+Q(x)y=0$$

Where $P(x)$ and $Q(x)$ are continuous on some interval I .

Suppose that $y_1(x)\neq 0,\ \forall\, x\in \mathrm{I}$ is a solution of the differential equation

Then $$y_1''+P\,y_1'+Q\,y_1=0$$

We define $y=u(x)y_1\ (x)$ then

$$y'=uy_1'+y_1u',\qquad y''=uy_1''+2y_1'u_1'+y_1u''$$

$$y''+Py'+Qy=u\underbrace{[y_1''+Py_1'+Qy_1]}_{zero}+y_1u''+(2y_1'+Py_1)u'=0$$

This implies that we must have

$$y_1u''+(2y_1'+Py_1)u'=0$$

If we suppose $w=u'$, then

$$y_1w'+(2y_1'+Py_1)w=0$$

The equation is separable. Separating variables we have from the last equation

$$.\frac{dw}{w}+(2\frac{y_1'}{y_1}+P)dx=0$$

Integrating

$$\ln|w|+2\ln|y_1|=-\int Pdx+c$$

$$\ln\left|wy_1{}^2\right|=-\int Pdx+c$$

$$wy_1{}^2=c_1e^{-\int Pdx}$$

$$w=\frac{c_1e^{-\int Pdx}dx}{y_1{}^2}$$

or $$u^{/} = \frac{c_1 e^{-\int P dx}}{y_1^{\,2}}$$

Integrating again, we obtain

$$u = c_1 \int \frac{e^{-\int P dx}}{y_1^{\,2}} dx + c_2$$

Hence $$y = u(x) y_1(x) = c_1 y_1(x) \int \frac{e^{-\int P dx}}{y_1^{\,2}} dx + c_2 y_1(x).$$

Choosing $c_1 = 1$ and $c_2 = 0$, we obtain a second solution of the differential equation

$$y_2 = y_1(x) \int \frac{e^{-\int P dx}}{y_1^{\,2}} dx$$

The Woolskin

$$W(y_1(x), y_2(x)) = \begin{vmatrix} y_1 & y_1 \int \frac{e^{-\int P dx}}{y_1^{\,2}} dx \\ y_1' & \frac{e^{-\int P dx}}{y_1} + y_1' \int \frac{e^{-\int P dx}}{y_1^{\,2}} dx \end{vmatrix}$$

$$= e^{-\int P dx} \neq 0, \forall x$$

Therefore $y_1(x)$ and $y_2(x)$ are linear independent set of solutions. So that they form a fundamental set of solutions of the differential equation

$$y^{//} + P(x) y^{/} + Q(x) y = 0$$

Hence the general solution of the differential equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Example 1

Given that

$$y_1 = x^2$$

is a solution of

$$x^2 y^{//} - 3xy^{/} + 4y = 0$$

Find general solution of the differential equation on the interval $(0, \infty)$.

Solution:

The equation can be written as

$$y^{//} - \frac{3}{x} y^{/} + \frac{4}{x^2} y = 0,$$

The 2nd solution y_2 is given by

$$y_2 = y_1(x) \int \frac{e^{-\int P dx}}{y_1^{\,2}} dx$$

or $$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4} dx = x^2 \int \frac{e^{\ln x^3}}{x^4} dx$$

$$y_2 = x^2 \int \frac{1}{x} dx = x^2 \ln x$$

Hence the general solution of the differential equation on $(0, \infty)$ is given by

$$y = c_1 \, y_1 + c_2 y_2$$

or $$y = c_1 \, x^2 + c_2 x^2 \ln x$$

Example 2

Verify that

$$y_1 = \frac{\sin x}{\sqrt{x}}$$

is a solution of

$$x^2 y^{//} + xy^{/} + (x^2 - 1/4)y = 0$$

on $(0, \pi)$. Find a second solution of the equation.

Solution:

The differential equation can be written as

$$y'' + \frac{1}{x}y' + (1 - \frac{1}{4x^2})y = 0$$

The 2nd solution is given by

$$y_2 = y_1 \int \frac{e^{-\int P dx}}{y_1^{\,2}} dx$$

Therefore

$$y_2 = \frac{\sin x}{\sqrt{x}} \int \frac{e^{-\int \frac{dx}{x}}}{(\frac{\sin x}{\sqrt{x}})^2} dx$$

$$= \frac{-\sin x}{\sqrt{x}} \int \frac{x}{x \sin^2 x} dx$$

$$= \frac{-\sin x}{\sqrt{x}} \int \csc^2 x dx$$

$$= \frac{-\sin x}{\sqrt{x}}(-\cot x) = \frac{\cos x}{\sqrt{x}}$$

Thus the second solution is

$$y_2 = \frac{\cos x}{\sqrt{x}}$$

Hence, general solution of the differential equation is

$$y = c_1\left(\frac{\sin x}{\sqrt{x}}\right) + c_2\left(\frac{\cos x}{\sqrt{x}}\right)$$

Order Reduction

Example 3

Given that

$$y_1 = x^3$$

is a solution of the differential equation

$$x^2 y'' - 6y = 0,$$

Find second solution of the equation

Solution

We write the given equation as:

$$y^{//} - \frac{6}{x^2} y = 0$$

So that $$P(x) = -\frac{6}{x^2}$$

Therefore

$$y_2 = y_1 \int \frac{e^{-\int P dx}}{{y_1}^2} dx$$

$$y_2 = x^3 \int \frac{e^{-\int \frac{6}{x^2}}}{x^6} dx$$

$$y_2 = x^3 \int \frac{e^{\frac{6}{x}}}{x^6} dx$$

Therefore, using the formula

$$y_2 = y_1 \int \frac{e^{-\int P dx}}{{y_1}^2} dx$$

We encounter an integral that is difficult or impossible to evaluate.

Hence, we c onclude s ometimes us e of t he f ormula t o f ind a s econd solution i s not suitable. We need to try something else.

Alternatively, we can try the reduction of order to find y_2 . For this purpose, we again define

$$y(x) = u(x) y_1(x) \quad \text{or} \quad y = u(x).x^3$$

then

$$y' = 3x^2 u + x^3 u'$$
$$y'' = x^3 u'' + 6x^2 u^{/} + 6xu$$

Substituting the values of y, y'' in the given differential equation

$$x^2 y'' - 6y = 0$$

we have

$$x^2 (x^3 u'' + 6x^2 u' + 6xu) - 6ux^3 = 0$$

or $$x^5u'' + 6x^4u' = 0$$

or $$u'' + \frac{6}{x}u' = 0,$$

If we take $w = u'$ then

$$w' + \frac{6}{x}w = 0$$

This is separable as well as linear first order differential equation in w. For using the latter, we find the integrating factor

$$I.F = e^{6\int \frac{1}{x}dx} = e^{6\ln x} = x^6$$

Multiplying with the $IF = x^6$, we obtain

$$x^6w' + 6x^5w = 0$$

or $$\frac{d}{dx}(x^6w) = 0$$

Integrating w.r.t. ' x ', we have

$$x^6w = c_1$$

or $$u' = \frac{c_1}{x^6}$$

Integrating once again, gives

$$u = -\frac{c_1}{5x^5} + c_2$$

Therefore $$y = ux^3 = \frac{-c_1}{5x^2} + c_2x^3$$

Choosing $c_2 = 0$ and $c_1 = -5$, we obtain

$$y_2 = \frac{1}{x^2}$$

Thus the second solution is given by

$$y_2 = \frac{1}{x^2}$$

Hence, general solution of the given differential equation is

$$y = c_1y_1 + c_2y_2$$

i.e. $$y = c_1x^3 + c_2\left(1/x^2\right)$$

Where c_1 and c_2 are constants.

Exercise

Find the 2nd solution of each of Differential equations by reducing order or by using the formula.

1. $y'' - y' = 0; \qquad y_1 = 1$

2. $y'' + 2y' + y = 0; \qquad y_1 = xe^{-x}$

3. $y'' + 9y = 0; \qquad y_1 = \sin x$

4. $y'' - 25y = 0; \qquad y_1 = e^{5x}$

5. $6y'' + y' - y = 0; \qquad y_1 = e^{x/2}$

6. $x^2 y'' + 2xy' - 6y = 0; \qquad y_1 = x^2$

7. $4x^2 y'' + y = 0; \qquad y_1 = x^{1/2} \ln x$

8. $(1 - x^2) y'' - 2xy' = 0; \qquad y_1 = 1$

9. $x^2 y'' - 3xy' + 5y = 0; \qquad y_1 = x^2 \cos(\ln x)$

10. $(1 + x) y'' + xy' - y = 0; \qquad y_1 = x$

Lecture 5

Homogeneous Linear Equations with Constant Coefficients

We know that the linear first order differential equation

$$\frac{dy}{dx} + my = 0$$

m being a constant, has the exponential solution on $(-\infty, \infty)$

$$y = c_1 e^{-mx}$$

The question?

- The question is whether or not the exponential solutions of the higher-order differential equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y^{//} + a_1 y^{/} + a_0 y = 0,$$

exist on $(-\infty, \infty)$.

- In fact all the solutions of this equation are exponential functions or constructed out of exponential functions.

Recall

That the linear differential of order n is an equation of the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

Method of Solution

Taking $n = 2$, the nth-order differential equation becomes

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

This equation can be written as

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

We now try a solution of the exponential form

$$y = e^{mx}$$

Then

$$y' = me^{mx} \text{ and } y'' = m^2 e^{mx}$$

Substituting in the differential equation, we have

$$e^{mx}(am^2 + bm + c) = 0$$

Since $\qquad e^{mx} \neq 0, \quad \forall x \in (-\infty, \infty)$

Therefore $\qquad am^2 + bm + c = 0$

This algebraic equation is known as the Auxiliary equation (AE).The solution of the auxiliary equation determines the solutions of the differential equation.

Case 1: *Distinct Real Roots*

If the auxiliary equation has distinct real roots m_1 and m_2 then we have the following two solutions of the differential equation.

$$y_1 = e^{m_1 x} \text{ and } y_2 = e^{m_2 x}$$

These solutions are linearly independent because

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = (m_2 - m_1) e^{(m_1 + m_2)x}$$

Since $m_1 \neq m_2$ and $e^{(m_1+m_2)x} \neq 0$

Therefore $W(y_1, y_2) \neq 0 \;\; \forall x \in (-\infty, \infty)$

Hence

- y_1 and y_2 form a fundamental set of solutions of the differential equation.
- The general solution of the differential equation on $(-\infty, \infty)$ is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Case 2. *Repeated Roots*

If the auxiliary equation has real and equal roots i.e

$$m = m_1, m_2 \quad \text{with} \quad m_1 = m_2$$

Then we obtain only one exponential solution

$$y = c_1 e^{mx}$$

To construct a second solution we rewrite the equation in the form

$$y'' + \frac{b}{a} y' + \frac{c}{a} y = 0$$

Comparing with $y'' + Py' + Qy = 0$

We make the identification

$$P = \frac{b}{a}$$

Thus a second solution is given by

$$y_2 = y_1 \int \frac{e^{-\int Pdx}}{y_1{}^2} dx = e^{mx} \int \frac{e^{-\frac{b}{a}x}}{e^{2mx}} dx$$

Since the auxiliary equation is a quadratic algebraic equation and has equal roots

Therefore, $Disc. = b^2 - 4ac = 0$

We know from the quadratic formula

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we have $2m = -\frac{b}{a}$

Therefore

$$y_2 = e^{mx} \int \frac{e^{2mx}}{e^{2mx}} dx = xe^{mx}$$

Hence the general solution is

$$y = c_1 e^{mx} + c_2 x e^{mx} = (c_1 + c_2 x) e^{mx}$$

Case 3: *Complex Roots*

If the auxiliary equation has complex roots $\alpha \pm i\beta$ then, with

$$m_1 = \alpha + i\beta \text{ and } m_2 = \alpha - i\beta$$

Where α >0 and β >0 are real, the general solution of the differential equation is

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

First we choose the following two pairs of values of c_1 and c_2

$$c_1 = c_2 = 1$$

$$c_1 = 1, c_2 = -1$$

Then we have

$$y_1 = e^{(\alpha + i\beta)x} + e^{(\alpha - i\beta)x}$$
$$y_2 = e^{(\alpha + i\beta)x} - e^{(\alpha - i\beta)x}$$

We know by the Euler's Formula that

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathrm{R}$$

Using this formula, we can simplify the solutions y_1 and y_2 as

$$y_1 = e^{\alpha x}(e^{i\beta x} + e^{-i\beta x}) = 2e^{\alpha x}\cos\beta x$$
$$y_2 = e^{\alpha x}(e^{i\beta x} - e^{-i\beta x}) = 2ie^{\alpha x}\sin\beta x$$

We can drop constant to write

$$y_1 = e^{\alpha x}\cos\beta x\,,\quad y_2 = e^{\alpha x}\sin\beta x$$

The Wronskian

$$W\left(e^{\alpha x}\cos\beta x\,,\ e^{\alpha x}\sin\beta x\right) = \beta e^{2\alpha x} \neq 0 \quad \forall\ x$$

Therefore, $e^{\alpha x}\cos(\beta\ x),\ e^{\alpha x}\sin(\beta\ x)$

form a fundamental set of solutions of the differential equation on $(-\infty,\infty)$.

Hence general solution of the differential equation is

$$y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$$

or $$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

Example:

Solve

$$2y'' - 5y' - 3y = 0$$

Solution:

The given differential equation is

$$2y'' - 5y' - 3y = 0$$

Put $y = e^{mx}$

Then $y' = me^{mx},\ \ y'' = m^2 e^{mx}$

Substituting in the give differential equation, we have

$$\left(2m^2 - 5m - 3\right)e^{mx} = 0$$

Since $e^{mx} \neq 0\ \ \forall\ x$, the auxiliary equation is

$$2m^2 - 5m - 3 = 0 \qquad \text{as} \quad e^{mx} \neq 0$$
$$(2m+1)(m-3) = 0 \Rightarrow m = -\frac{1}{2},\ 3$$

Therefore, the auxiliary equation has distinct real roots

$$m_1 = -\frac{1}{2} \ \text{ and } \ m_2 = 3$$

Hence the general solution of the differential equation is

$$y = c_1 e^{(-1/2)x} + c_2 e^{3x}$$

Example 2

Solve $y'' - 10y' + 25y = 0$

Solution:

We put $y = e^{mx}$

Then $y' = me^{mx}, y'' = m^2 e^{mx}$

Substituting in the given differential equation, we have

$$(m^2 - 10m + 25)e^{mx} = 0$$

Since $e^{mx} \neq 0 \; \forall \; x$, the auxiliary equation is

$$m^2 - 10m + 25 = 0$$

$$(m-5)^2 = 0 \Rightarrow m = 5, 5$$

Thus the auxiliary equation has repeated real roots i.e

$$m_1 = 5 = m_2$$

Hence general solution of the differential equation is

$$y = c_1 e^{5x} + c_2 x e^{5x}$$

or $$y = (c_1 + c_2 x)e^{5x}$$

Example 3

Solve the initial value problem

$$y'' - 4y' + 13y = 0$$
$$y(0) = -1, \; y'(0) = 2$$

Solution:

Given that the differential equation

$$y'' - 4y' + 13y = 0$$

Put $y = e^{mx}$

Then $$y' = me^{mx},\ y'' = m^2 e^{mx}$$

Substituting in the given differential equation, we have:

$$(m^2 - 4m + 13)e^{mx} = 0$$

Since $e^{mx} \neq 0 \forall x$, the auxiliary equation is

$$m^2 - 4m + 13 = 0$$

By quadratic formula, the solution of the auxiliary equation is

$$m = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i$$

Thus the auxiliary equation has complex roots

$$m_1 = 2 + 3i, \qquad m_2 = 2 - 3i$$

Hence general solution of the differential equation is

$$y = e^{2x}\left(c_1 \cos 3x + c_2 \sin 3x\right)$$

Example 4

Solve the differential equations

(a) $$y'' + k^2 y = 0$$

(b) $$y'' - k^2 y = 0$$

Solution

First consider the differential equation

$$y'' + k^2 y = 0,$$

Put $$y = e^{mx}$$

Then $$y' = me^{mx} \text{ and } y'' = m^2 e^{mx}$$

Substituting in the given differential equation, we have:

$$\left(m^2 + k^2\right) e^{mx} = 0$$

Since $e^{mx} \neq 0 \forall x$, the auxiliary equation is

$$m^2 + k^2 = 0$$

or $$m = \pm ki,$$

Therefore, the auxiliary equation has complex roots

$$m_1 = 0 + ki, \quad m_2 = 0 - ki$$

Hence general solution of the differential equation is

$$y = c_1 \cos kx + c_2 \sin kx$$

Next consider the differential equation

$$\frac{d^2 y}{dx^2} - k^2 y = 0$$

Substituting values y and y'', we have.

$$\left(m^2 - k^2\right)e^{mx} = 0$$

Since $e^{mx} \neq 0$, the auxiliary equation is

$$m^2 - k^2 = 0$$
$$\Rightarrow \quad m = \pm k$$

Thus the auxiliary equation has distinct real roots

$$m_1 = +k, \; m_2 = -k$$

Hence the general solution is

$$y = c_1 e^{kx} + c_2 e^{-kx}.$$

Higher Order Equations

If we consider nth order homogeneous linear differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = 0$$

Then, the auxiliary equation is an nth degree polynomial equation

$$a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0 = 0$$

Case 1: *Real distinct roots*

If the roots $m_1, m_2, \ldots, m_n$ of the auxiliary equation are all real and distinct, then the general solution of the equation is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$$

Case 2: *Real & repeated roots*

We suppose that m_1 is a root of multiplicity n of the auxiliary equation, then it can be shown that

$$e^{m_1 x}, xe^{m_1 x}, \ldots, x^{n-1} e^{m_1 x}$$

are n linearly independent solutions of the differential equation. Hence general solution of the differential equation is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + \ldots + c_n x^{n-1} e^{m_1 x}$$

Case 3: *Complex roots*

Suppose that coefficients of the auxiliary equation are real.

- We fix n at 6, all roots of the auxiliary are complex, namely
  $\alpha_1 \pm i\beta_1, \quad \alpha_2 \pm i\beta_2, \; \alpha_3 \pm i\beta_3$
  - Then the general solution of the differential equation

$$y = e^{\alpha_1 x}(c_1 \cos \beta_1 x + c_2 \sin \beta_1 x) + e^{\alpha_2 x}(c_3 \cos \beta_2 x + c_4 \sin \beta_2 x)$$
$$+ e^{\alpha_3 x}(c_5 \cos \beta_3 x + c_6 \sin \beta_3 x)$$

- If $n = 6$, two roots of the auxiliary equation are real and equal and the remaining 4 are complex, namely $\alpha_1 \pm i\beta_1, \quad \alpha_2 \pm i\beta_2$
  Then the general solution is
  $$y = e^{\alpha_1 x}(c_1 \cos \beta_1 x + c_2 \sin \beta_1 x) + e^{\alpha_2 x}(c_3 \cos \beta_2 x + c_4 \sin \beta_2 x) + c_5 e^{m_1 x} + c_6 x e^{m_1 x}$$
- If $m_1 = \alpha + i\beta$ is a complex root of multiplicity k of the auxiliary equation. Then its conjugate $m_2 = \alpha - i\beta$ is also a root of multiplicity k. Thus from Case 2 **,** the

differential equation has $2k$ solutions

$$e^{(\alpha+i\beta)x},\ xe^{(\alpha+i\beta)x},\ x^2e^{(\alpha+i\beta)x},\ldots,x^{k-1}e^{(\alpha+i\beta)x}$$
$$e^{(\alpha-i\beta)x},\ xe^{(\alpha-i\beta)x},\ x^2e^{(\alpha-i\beta)x},\ldots,x^{k-1}e^{(\alpha-i\beta)x}$$

- By using the Euler's formula, we conclude that the general solution of the differential equation is a linear combination of the linearly independent solutions

$$e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x,\ x^2e^{\alpha x}\cos\beta x,\ldots,x^{k-1}e^{\alpha x}\cos\beta x$$
$$e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x,\ x^2e^{\alpha x}\sin\beta x,\ldots,x^{k-1}e^{\alpha x}\sin\beta x$$

- Thus if $k=3$ then

$$y=e^{\alpha x}\left[\left(c_1+c_2x+c_3x^2\right)\cos\beta x+\left(d_1+d_2x+d_3x^2\right)\sin\beta x\right]$$

Solving the Auxiliary Equation

Recall that the auxiliary equation of nth degree differential equation is nth degree polynomial equation

- Solving the auxiliary equation could be difficult

$$P_n(m)=0,\quad \text{n}>2$$

- One way to solve this polynomial equation is to guess a root m_1. Then $m-m_1$ is a factor of the polynomial $P_n(m)$.
- Dividing with $m-m_1$ synthetically or otherwise, we find the factorization

$$P_n(m)=(m-m_1)\,Q(m)$$

- We then try to find roots of the quotient i.e. roots of the polynomial equation

$$Q(m)=0$$

- Note that if $m_1=\dfrac{p}{q}$ is a rational real root of the equation

$$P_n(m)=0,\quad \text{n}>2$$

  then p is a factor of a_0 and q of a_n.
- By using this fact we can construct a list of all possible rational roots of the auxiliary equation and test each of them by synthetic division.

Example 1

Solve the differential equation

$$y'''+3y''-4y=0$$

Solution:

Given the differential equation

$$y'''+3y''-4y=0$$

Put $y = e^{mx}$

Then $y' = me^{mx}, y'' = m^2 e^{mx}$ and $y''' = m^3 e^{mx}$

Substituting this in the given differential equation, we have

$$(m^3 + 3m^2 - 4)e^{mx} = 0$$

Since $e^{mx} \neq 0$

Therefore $m^3 + 3m^2 - 4 = 0$

So that the auxiliary equation is

$$m^3 + 3m^2 - 4 = 0$$

Solution of the AE

If we take $m = 1$ then we see that

$$m^3 + 3m^2 - 4 = 1 + 3 - 4 = 0$$

Therefore $m = 1$ satisfies the auxiliary equations so that m-1 is a factor of the polynomial

$$m^3 + 3m^2 - 4$$

By synthetic division, we can write

$$m^3 + 3m^2 - 4 = (m-1)(m^2 + 4m + 4)$$

or $m^3 + 3m^2 - 4 = (m-1)(m+2)^2$

Therefore $m^3 + 3m^2 - 4 = 0$

$$\Rightarrow (m-1)(m+2)^2 = 0$$

or $m = 1, -2, -2$

Hence solution of the differential equation is

$$y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}$$

Example 2

Solve

$$3y''' + 5y'' + 10y' - 4y = 0$$

Solution:

Given the differential equation

$$3y''' + 5y'' + 10y' - 4y = 0$$

Put $y = e^{mx}$

Then $y' = me^{mx}, y'' = m^2 e^{mx}$ and $y''' = m^3 e^{mx}$

Therefore the auxiliary equation is

$$3m^3 + 5m^2 + 10m - 4 = 0$$

Solution of the auxiliary equation:

a) $a_\circ = -4$ and all its factors are:

$p: \quad \pm 1, \pm 2, \pm 4$

b) $a_n = 3$ and all its factors are:

$q: \quad \pm 1, \ \pm 3$

c) List of possible rational roots of the auxiliary equation is

$$\frac{p}{q}: \quad -1, 1, -2, 2, -4, 4, \frac{-1}{3}, \frac{1}{3}, \frac{-2}{3}, \frac{2}{3}, \frac{-4}{3}, \frac{4}{3}$$

d) Testing each of these successively by synthetic division we find

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & 5 & 10 & -4 \\ & & 1 & 2 & 4 \\ \hline & 3 & 6 & 12 & \big| \, 0 \end{array}$$

Consequently a root of the auxiliary equation is

$$m = 1/3$$

The coefficients of the quotient are

3 6 12

Thus we can write the auxiliary equation as:

$$(m - 1/3)\left(3m^2 + 6m + 12\right) = 0$$

$$m - \frac{1}{3} = 0 \quad \text{or} \quad 3m^2 + 6m + 12 = 0$$

Therefore $\quad m = 1/3 \quad$ or $\quad m = -1 \pm i\sqrt{3}$

Hence solution of the given differential equation is

$$y = c_1 e^{(1/3)x} + e^{-x}\left(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right)$$

Example 3

Solve the differential equation

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$$

Solution:

Given the differential equation

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$$

Put $\quad y = e^{mx}$

Then $y' = me^{mx},\ y'' = m^2 e^{mx}$

Substituting in the differential equation, we obtain

$$\left(m^4 + 2m^2 + 1\right) e^{mx} = 0$$

Since $e^{mx} \neq 0$, the auxiliary equation is

$$m^4 + 2m^2 + 1 = 0$$

$$(m^2 + 1)^2 = 0$$

$$\Rightarrow m = \pm i,\ \pm i$$

$$m_1 = m_3 = i \quad \text{and} \quad m_2 = m_4 = -i$$

Thus i is a root of the auxiliary equation of multiplicity 2 and so is $-i$.

Now $\alpha = 0$ and $\beta = 1$

Hence the general solution of the differential equation is

$$y = e^{0x}\left[(c_1 + c_2 x)\cos x + (d_1 + d_2 x)\sin x\right]$$

or $$y = c_1 \cos x + d_1 \sin x + c_2 x \cos x + d_2 x \sin x$$

Exercise

Find the general solution of the given differential equations.

1. $y'' - 8y = 0$
2. $y'' - 3y' + 2y = 0$
3. $y'' + 4y' - y = 0$
4. $2y'' - 3y' + 4y = 0$
5. $4y''' + 4y'' + y' = 0$
6. $y''' + 5y'' = 0$
7. $y''' + 3y'' - 4y' - 12y = 0$

Solve the given differential equations subject to the indicated initial conditions.

8. $y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = y'(0) = 0,\ y''(0) = 1$
9. $\dfrac{d^4 y}{dx^4} = 0,\ y(0) = 2,\ y'(0) = 3,\ y''(0) = 4,\ y'''(0) = 5$

10. $\dfrac{d^4y}{dx^4} - y = 0$, $y(0) = y^{/}(0) = y^{//}(0) = 0, y^{///}(0) = 1$

Lecture 6

Method of Undetermined Coefficients-Superposition Approach

Recall

1. That a non-homogeneous linear differential equation of order n is an equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

The co efficients $a_0, a_1, \ldots, a_n$ can be f unctions o f x . H owever, w e w ill di scuss equations with constant coefficients.

2. That to obtain the general solution of a non-homogeneous linear differential equation we must find**:**

- The c omplementary f unction y_c , w hich i s general s olution of t he a ssociated homogeneous differential equation.
- Any particular solution y_p of the non-homogeneous differential equation.

3. That the general solution of the non-homogeneous linear differential equation is given by

General solution = Complementary function + Particular Integral

Finding

Complementary f unction ha s be en di scussed i n t he pr evious l ecture. In t he ne xt three lectures w e w ill d iscuss me thods f or f inding a p articular in tegral f or t he non - homogeneous equation, namely

- The method of undetermined coefficients-*superposition approach*
- The method undetermined coefficients-*annihilator operator approach*.
- The method of variation of parameters.

The Method of Undetermined Coefficient

The method of undetermined coefficients developed here is limited to non-homogeneous linear differential equations

- That have constant coefficients, and
- Where the function $g(x)$ has a specific form.

The form of $g(x)$

The input function $g(x)$ can have one of the following forms:

- A constant function k.
- A polynomial function
- An exponential function e^x
- The trigonometric functions $\sin(\beta x)$, $\cos(\beta x)$
- Finite sums and products of these functions.

Otherwise, we cannot apply the method of undetermined coefficients.

The method

Consist of performing the following steps.

Step 1 Determine the form of the input function $g(x)$.

Step 2 Assume the general form of y_p according to the form of $g(x)$

Step 3 Substitute in the given non-homogeneous differential equation.

Step 4 Simplify and equate coefficients of like terms from both sides.

Step 5 Solve the resulting equations to find the unknown coefficients.

Step 6 Substitute the calculated values of coefficients in assumed y_p

Restriction on g **?**

The input function g is restricted to have one of the above stated forms because of the reason**:**

- The derivatives of sums and products of polynomials, exponentials etc are again sums and products of similar kind of functions.
- The expression $ay_p'' + by_p' + cy_p$ has to be identically equal to the input function $g(x)$.

Therefore, to make an educated guess, y_p is assured to have the same form as g.

Caution!

- In addition to the form of the input function $g(x)$, the educated guess for y_p must take into consideration the functions that make up the complementary function y_c.
- No function in the assumed y_p must be a solution of the associated homogeneous differential equation. This means that the assumed y_p should not contain terms that duplicate terms in y_c.

Taking for granted that no function in the assumed y_p is duplicated by a function in y_c, some forms of g and the corresponding forms of y_p are given in the following table.

Trial particular solutions

Number	The input function $g(x)$	The assumed particular solution y_p
1	Any constant e.g. 1	A
2	$5x+7$	$Ax+B$
3	$3x^2-2$	Ax^2+Bx+c
4	x^3-x+1	Ax^3+Bx^2+Cx+D
5	$\sin 4x$	$A\cos 4x+B\sin 4x$
6	$\cos 4x$	$A\cos 4x+B\sin 4x$
7	e^{5x}	Ae^{5x}
8	$(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
9	x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
10	$e^{3x}\sin 4x$	$A\,e^{3x}\cos 4x+B\,e^{3x}\sin 4x$
11	$5x^2\sin 4x$	$(A_1x^2+B_1x+C_1)\cos 4x+(A_2x^2+B_2x+C_2)\sin 4x$
12	$xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x+(Cx+D)e^{3x}\sin 4x$

If $g(x)$ equals a sum?

Suppose that

- The input function $g(x)$ consists of a sum of m terms of the kind listed in the above table i.e.
$$g(x)=g_1(x)+g_2(x)+\cdots+g_m(x).$$
- The trial forms corresponding to $g_1(x), g_2(x),\ldots, g_m(x)$ be $y_{p_1}, y_{p_2},\ldots, y_{p_m}$.

Then the particular solution of the given non-homogeneous differential equation is
$$y_p=y_{p_1}+y_{p_2}+\cdots+y_{p_m}$$
In other words, *the form of y_p is a linear combination of all the linearly independent functions generated by repeated differentiation of the input function $g(x)$.*

Example 1

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$

Solution:

Complementary function

To find y_c, we first solve the associated homogeneous equation

$$y'' + 4y' - 2y = 0$$

We put $y = e^{mx}, \quad y' = me^{mx}, \quad y'' = m^2e^{mx}$

Then the associated homogeneous equation gives

$$(m^2 + 4m - 2)e^{mx} = 0$$

Therefore, the auxiliary equation is

$$m^2 + 4m - 2 = 0 \quad \text{as} \quad e^{mx} \neq 0, \quad \forall\, x$$

Using the quadratic formula, roots of the auxiliary equation are

$$m = -2 \pm \sqrt{6}$$

Thus we have real and distinct roots of the auxiliary equation

$$m_1 = -2 - \sqrt{6} \quad \text{and} \quad m_2 = -2 + \sqrt{6}$$

Hence the complementary function is

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}$$

Next we find a particular solution of the non-homogeneous differential equation.

Particular Integral

Since the input function

$$g(x) = 2x^2 - 3x + 6$$

is a quadratic polynomial. Therefore, we assume that

$$y_p = Ax^2 + Bx + C$$

Then $y_p' = 2Ax + B$ and $y_p'' = 2A$

Therefore $y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C$

Substituting in the given equation, we have

$$2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6$$

or $-2Ax^2 + (8A - 2B)x + (2A + 4B - 2C) = 2x^2 - 3x + 6$

Equating the coefficients of the like powers of x, we have

$$-2A = 2, \quad 8A - 2B = -3, \quad 2A + 4B - 2C = 6$$

Solving this system of equations leads to the values

$$A = -1, \quad B = -5/2, \quad C = -9.$$

Thus a particular solution of the given equation is

$$y_p = -x^2 - \frac{5}{2}x - 9 .$$

Hence, the general solution of the given non-homogeneous differential equation is given by

$$y = y_c + y_p$$

or $$y = -x^2 - \frac{5}{2}x - 9 + c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}$$

Example 2

Solve the differential equation

$$y'' - y' + y = 2\sin 3x$$

Solution:

Complementary function

To find y_c, we solve the associated homogeneous differential equation

$$y'' - y' + y = 0$$

Put $$y = e^{mx}, \quad y' = me^{mx}, \quad y'' = m^2 e^{mx}$$

Substitute in the given differential equation to obtain the auxiliary equation

$$m^2 - m + 1 = 0 \qquad \text{or} \qquad m = \frac{1 \pm i\sqrt{3}}{2}$$

Hence, the auxiliary equation has complex roots. Hence the complementary function is

$$y_c = e^{(1/2)x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

Particular Integral

Since successive differentiation of

$$g(x) = \sin 3x$$

produce $\sin 3x$ and $\cos 3x$

Therefore, we include both of these terms in the assumed particular solution, see table

$$y_p = A\cos 3x + B\sin 3x.$$

Then $y'_p = -3A\sin 3x + 3B\cos 3x.$

$y''_p = -9A\cos 3x - 9B\sin 3x.$

Therefore $y_p'' - y_p' + y_p = (-8A - 3B)\cos 3x + (3A - 8B)\sin 3x.$

Substituting in the given differential equation

$$(-8A - 3B)\cos 3x + (3A - 8B)\sin 3x = 0\cos 3x + 2\sin 3x.$$

From the resulting equations

$$-8A - 3B = 0,\ 3A - 8B = 2$$

Solving these equations, we obtain

$$A = 6/73, B = -16/73$$

A particular solution of the equation is

$$y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x$$

Hence the general solution of the given non-homogeneous differential equation is

$$y = e^{(1/2)x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x$$

Example 3

Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$

Solution:

Complementary function

To find y_c, we solve the associated homogeneous equation

$$y'' - 2y' - 3y = 0$$

Put $y = e^{mx},\ y' = me^{mx},\ y'' = m^2e^{mx}$

Substitute in the given differential equation to obtain the auxiliary equation

$$m^2 - 2m - 3 = 0$$

$$\Rightarrow (m+1)(m-3) = 0$$

$$m = -1, 3$$

Therefore, the auxiliary equation has real distinct root

$$m_1 = -1, m_2 = 3$$

Thus the complementary function is

$$y_c = c_1e^{-x} + c_2e^{3x}.$$

Particular integral

Since $g(x) = (4x - 5) + 6xe^{2x} = g_1(x) + g_2(x)$

Corresponding to $g_1(x)$ $y_{p_1} = Ax + B$

Corresponding to $g_2(x)$ $y_{p_2} = (Cx + D)e^{2x}$

The superposition principle suggests that we assume a particular solution

$$y_p = y_{p_1} + y_{p_2}$$

i.e. $$y_p = Ax + B + (Cx + D)\, e^{2x}$$

Then $$y'_p = A + 2(Cx + D)\, e^{2x} + Ce^{2x}$$

$$y''_p = 4(Cx + D)\, e^{2x} + 4Ce^{2x}$$

Substituting in the given

$$y_p{}'' - 2y_p{}' - 3y_p = 4Cxe^{2x} + 4De^{2x} + 4Ce^{2x} - 2A - 4Cxe^{2x}$$
$$- 4De^{2x} - 2Ce^{2x} - 3Ax - 3B - 3Cxe^{2x} - 3De^{2x}$$

Simplifying and grouping like terms

$$y_p{}'' - 2y_p{}' - 3y_p = -3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3D)e^{2x} = 4x - 5 + 6xe^{2x}.$$

Substituting in the non-homogeneous differential equation, we have

$$-3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3D)e^{2x} = 4x - 5 + 6xe^{2x} + 0e^{2x}$$

Now equating constant terms and coefficients of x, xe^{2x} and e^{2x}, we obtain

$$-2A - 3B = -5, \qquad -3A = 4$$
$$-3C = 6, \qquad 2C - 3D = 0$$

Solving these algebraic equations, we find

$$A = -4/3, \quad B = 23/9$$
$$C = -2, \quad D = -4/3$$

Thus, a particular solution of the non-homogeneous equation is

$$y_p = -(4/3)x + (23/9) - 2\, xe^{2x} - (4/3)e^{2x}$$

The general solution of the equation is

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{3x} - (4/3)x + (23/9) - 2\, x\, e^{2x} - (4/3)e^{2x}$$

Duplication between y_p and y_c?

- If a function in the assumed y_p is also present in y_c then this function is a solution of the associated homogeneous differential equation. In this case the obvious assumption for the form of y_p is not correct**.**
- In this case we suppose that the input function is made up of terms of n kinds i.e.

  $$g(x) = g_1(x) + g_2(x) + \cdots + g_n(x)$$

  and corresponding to this input function the assumed particular solution y_p is

  $$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_n}$$
- If a y_{p_i} contain terms that duplicate terms in y_c, then that y_{p_i} must be multiplied with x^n, n being the least positive integer that eliminates the duplication.

Example 4

Find a particular solution of the following non-homogeneous differential equation

$$y'' - 5y' + 4y = 8e^x$$

Solution:

To find y_c, we solve the associated homogeneous differential equation

$$y'' - 5y' + 4y = 0$$

We put $y = e^{mx}$ in the given equation, so that the auxiliary equation is

$$m^2 - 5m + 4 = 0 \Rightarrow m = 1, 4$$

Thus $\quad y_c = c_1 e^x + c_2 e^{4x}$

Since $\quad g(x) = 8e^x$

Therefore, $\quad y_p = Ae^x$

Substituting in the given non-homogeneous differential equation, we obtain

$$Ae^x - 5Ae^x + 4Ae^x = 8e^x$$

So $\quad 0 = 8e^x$

Clearly we have made a wrong assumption for y_p, as we did not remove the duplication.

Since Ae^x is present in y_c. Therefore, it is a solution of the associated homogeneous differential equation

$$y'' - 5y' + 4y = 0$$

To avoid this we find a particular solution of the form

$$y_p = Axe^x$$

We notice that there is no duplication between y_c and this new assumption for y_p

Now $\quad y_p' = Axe^x + Ae^x, \quad y_p'' = Axe^x + 2Ae^x$

Substituting in the given differential equation, we obtain

$$Axe^x + 2Ae^x - 5Axe^x - 5Ae^x + 4Axe^x = 8e^x.$$

or $\quad -3Ae^x = 8e^x \Rightarrow A = -8/3.$

So that a particular solution of the given equation is given by

$$y_p = -(8/3)e^x$$

Hence, the general solution of the given equation is

$$y = c_1 e^x + c_2 e^{4x} - (8/3)\, x\, e^x$$

Example 5

Determine the form of the particular solution

(a) $y'' - 8y' + 25y = 5x^3e^{-x} - 7e^{-x}$

(b) $y'' + 4y = x\cos x.$

Solution:

(a) To find y_c we solve the associated homogeneous differential equation

$$y'' - 8y' + 25y = 0$$

Put $y = e^{mx}$

Then the auxiliary equation is

$$m^2 - 8m + 25 = 0 \Rightarrow m = 4 \pm 3i$$

Roots of the auxiliary equation are complex

$$y_c = e^{4x}(c_1 \cos 3x + c_2 \sin 3x)$$

The input function is

$$g(x) = 5x^3e^{-x} - 7e^{-x} = (5x^3 - 7)e^{-x}$$

Therefore, we assume a particular solution of the form

$$y_p = (Ax^3 + Bx^2 + Cx + D)e^{-x}$$

Notice th at th ere is ***no duplication*** between t he t erms i n y_p and th e te rms in y_c . Therefore, while proceeding further we can easily calculate the value A, B, C and D.

(b) Consider the associated homogeneous differential equation

$$y'' + 4y = 0$$

Since $g(x) = x\cos x$

Therefore, we assume a particular solution of the form

$$y_p = (Ax + B)\cos x + (Cx + D)\sin x$$

Again observe that there is ***no duplication*** of terms between y_c and y_p

Example 6
Determine the form of a particular solution of

$$y'' - y' + y = 3x^2 - 5\sin 2x + 7xe^{6x}$$

Solution:
To find y_c, we solve the associated homogeneous differential equation

$$y'' - y' + y = 0$$

Put $y = e^{mx}$

Then the auxiliary equation is

$$m^2 - m + 1 = 0 \Rightarrow m = \frac{1 \pm i\sqrt{3}}{2}$$

Therefore $$y_c = e^{(1/2)x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

Since $$g(x) = 3x^2 - 5\sin 2x + 7xe^{6x} = g_1(x) + g_2(x) + g_3(x)$$

Corresponding to $g_1(x) = 3x^2$: $y_{p_1} = Ax^2 + Bx + C$

Corresponding to $g_2(x) = -5\sin 2x$: $y_{p_2} = D\cos 2x + E\sin 2x$

Corresponding to $g_3(x) = 7xe^{6x}$: $y_{p_3} = (Fx + G)\, e^{6x}$

Hence, the assumption for the particular solution is

$$y_p = y_{p_1} + y_{p_2} + y_{p_3}$$

or $$y_p = Ax^2 + Bx + C + D\cos 2x + E\sin 2x + (Fx + G)e^{6x}$$

No term in this assumption duplicate any term in the complementary function

$$y_c = c_1 e^{2x} + c_2 e^{7x}$$

Example 7
Find a particular solution of

$$y'' - 2y' + y = e^x$$

Solution:
Consider the associated homogeneous equation

$$y'' - 2y' + y = 0$$

Put $y = e^{mx}$

Then the auxiliary equation is

$$m^2 - 2m + 1 = (m-1)^2 = 0$$

$$\Rightarrow m = 1,\ 1$$

Roots of the auxiliary equation are real and equal. Therefore,

$$y_c = c_1 e^x + c_2 x e^x$$

Since $g(x) = e^x$

Therefore, we assume that

$$y_p = Ae^x$$

This assumption fails because of duplication between y_c and y_p. We multiply with x

Therefore, we now assume

$$y_p = Axe^x$$

However, the duplication is still there. Therefore, we again multiply with x and assume

$$y_p = Ax^2e^x$$

Since there is no duplication, this is acceptable form of the trial y_p

$$y_p = \frac{1}{2}x^2e^x$$

Example 8

Solve the initial value problem

$$y^{//} + y = 4x + 10\sin x,$$

$$\text{y}(\pi) = 0, \text{y}^{/}(\pi) = 2$$

Solution

Consider the associated homogeneous differential equation

$$y^{//} + y = 0$$

Put $y = e^{mx}$

Then the auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

The roots of the auxiliary equation are complex. Therefore, the complementary function is

$$y_c = c_1 \cos x + c_2 \sin x$$

Since $g(x) = 4x + 10\sin x = g_1(x) + g_2(x)$

Therefore, we assume that

$$y_{p_1} = Ax + B, \quad y_{p_2} = \text{C}\cos x + D\sin x$$

So that $\text{y}_\text{p} = Ax + B + \text{C}\cos x + D\sin x$

Clearly, there is duplication of the functions $\cos x$ and $\sin x$. To remove this duplication we multiply y_{p_2} with x. Therefore, we assume that

$$y_p = Ax + B + \text{C x}\cos x + Dx\sin x.$$

$$y_p'' = -2C\sin x - Cx\cos x + 2D\cos x - Dx\sin x$$

So that $\text{y}_\text{p}^{//} + \text{y}_\text{p} = Ax + B - 2\text{C}\sin x + 2D\text{x}\cos x$

Substituting into the given non-homogeneous differential equation, we have

$$Ax + B - 2\text{C}\sin x + 2D\text{x}\cos x = 4x + 10\sin x$$

Equating constant terms and coefficients of x, $\sin x$, $x\cos x$, we obtain

$$B = 0,\ A = 4,\ -2C = 10,\ 2D = 0$$

So that $A=4, B=0, C=-5,\ D=0$

Thus $y_p = 4x - 5x\cos x$

Hence the general solution of the differential equation is

$$y = y_c + y_p = c_1\cos x + c_2\sin x + 4\text{ x - }5x\cos x$$

We now apply the initial conditions to find c_1 and c_2.

$$y(\pi)=0 \Rightarrow c_1\cos\pi + c_2\sin\pi + 4\pi - 5\pi\cos\pi = 0$$

Since $\sin\pi = 0, \cos\pi = -1$

Therefore $c_1 = 9\pi$

Now $y' = -9\pi\sin x + c_2\cos x + 4 + 5x\sin x - 5\cos x$

Therefore $y'(\pi) = 2 \Rightarrow -9\pi\sin\pi + c_2\cos\pi + 4 + 5\pi\sin\pi - 5\cos\pi = 2$

$\therefore$ $c_2 = 7.$

Hence the solution of the initial value problem is

$$y = 9\pi\cos x + 7\sin x + 4x - 5x\cos x.$$

Example 9

Solve the differential equation

$$y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$$

Solution:

The associated homogeneous differential equation is

$$y'' - 6y' + 9y = 0$$

Put $y = e^{mx}$

Then the auxiliary equation is

$$m^2 - 6m + 9 = 0 \Rightarrow m = 3, 3$$

Thus the complementary function is

$$y_c = c_1 e^{3x} + c_2 x e^{3x}$$

Since $g(x) = (x^2+2) - 12e^{3x} = g_1(x) + g_2(x)$

We assume that

Corresponding to $g_1(x) = x^2 + 2$**:** $y_{p_1} = Ax^2 + Bx + C$

Corresponding to $g_2(x) = -12e^{3x}$**:** $y_{p_2} = De^{3x}$

Thus the assumed form of the particular solution is

$$y_p = Ax^2 + Bx + C + De^{3x}$$

The function e^{3x} in y_{p_2} is duplicated between y_c and y_p. Multiplication with x does not remove t his dupl ication. H owever, i f w e m ultiply y_{p_2} with x^2 , t his dupl ication i s removed.

Thus the operative from of a particular solution is

$$y_p = Ax^2 + Bx + C + Dx^2e^{3x}$$

Then $$y'_p = 2Ax + B + 2Dxe^{3x} + 3Dx^2e^{3x}$$

and $$y''_p = 2A + 2De^{3x} + 6Dxe^{3x} + 9Dx^2e^{3x}$$

Substituting into the given differential equation and collecting like term, we obtain

$$y_p{}'' - 6y_p{}' + y_p = 9Ax^2 + (-12A + 9B)x + 2A - 6B + 9C + 2De^{3x} = 6x^2 + 2 - 12e^{3x}$$

Equating constant terms and coefficients of x, x^2 and e^{3x} yields

$$2A - 6B + 9C = 2, \quad -12A + 9B = 0$$
$$9A = 6, \quad 2D = -12$$

Solving these equations, we have the values of the unknown coefficients

$$A = 2/3, B = 8/9, C = 2/3 \text{ and } D = -6$$

Thus $$y_p = \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2e^{3x}$$

Hence the general solution

$$y = y_c + y_p = c_1e^{3x} + c_2xe^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2e^{3x}.$$

Higher –Order Equation

The method of undetermined coefficients can also be used for higher order equations of the form

$$a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1\frac{dy}{dx} + a_0y = g(x)$$

with constant coefficients. The only requirement is that $g(x)$ consists of the proper kinds of functions as discussed earlier.

Example 10

Solve $$y''' + y'' = e^x \cos x$$

Solution:

To find the complementary function we solve the associated homogeneous differential equation

$$y''' + y'' = 0$$

Put $$y = e^{mx}, y' = me^{mx}, y'' = m^2e^{mx}$$

Then the auxiliary equation is

$$m^3 + m^2 = 0$$

or $$m^2(m+1) = 0 \Rightarrow m = 0, 0, -1$$

The auxiliary equation has equal and distinct real roots. Therefore, the complementary function is

$$y_c = c_1 + c_2x + c_3e^{-x}$$

Since $$g(x) = e^x \cos x$$

Therefore, we assume that

$$y_p = Ae^x \cos x + Be^x \sin x$$

Clearly, there is no duplication of terms between y_c and y_p.

Substituting the derivatives of y_p in the given differential equation and grouping the like terms, we have

$$y_p''' + y_p'' = (-2A+4B)e^x \cos x + (-4A-2B)e^x \sin x = e^x \cos x.$$

Equating the coefficients, of $e^x \cos x$ and $e^x \sin x$, yields

$$-2A+4B=1, -4A-2B=0$$

Solving these equations, we obtain

$$A=-1/10, B=1/5$$

So that a particular solution is

$$y_p = c_1 + c_2 x + c_3 e^{-x} - (1/10)e^x \cos x + (1/5)e^x \sin x$$

Hence the general solution of the given differential equation is

$$y_p = c_1 + c_2 x + c_3 e^{-x} - (1/10)e^x \cos x + (1/5)e^x \sin x$$

Example 12

Determine the form of a particular solution of the equation

$$y'''' + y''' = 1 - e^{-x}$$

Solution:

Consider the associated homogeneous differential equation

$$y'''' + y''' = 0$$

The auxiliary equation is

$$m^4 + m^3 = 0 \Rightarrow m = 0, 0, 0, -1$$

Therefore, the complementary function is

$$y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x}$$

Since $\quad g(x) = 1 - e^{-x} = g_1(x) + g_2(x)$

Corresponding to $g_1(x) = 1$: $\quad y_{p_1} = A$

Corresponding to $g_2(x) = -e^{-x}$: $\quad y_{p_2} = Be^{-x}$

Therefore, the normal assumption for the particular solution is

$$y_p = A + Be^{-x}$$

Clearly there is duplication of

(i) The constant function between y_c and y_{p_1}.

(ii) The exponential function e^{-x} between y_c and y_{p_2}.

To remove this duplication, we multiply y_{p_1} with x^3 and y_{p_2} with x. This duplication can't be removed by multiplying with x and x^2. Hence, the correct assumption for the particular solution y_p is

$y_p = Ax^3 + Bxe^{-x}$

Exercise

Solve the following differential equations using the undetermined coefficients.

1. $\frac{1}{4}y'' + y' + y = x^2 + 2x$

2. $y'' - 8y' + 20y = 100x^2 - 26xe^x$

3. $y'' + 3y = -48x^2e^{3x}$

4. $4y'' - 4y' - 3y = \cos 2x$

5. $y'' + 4y = (x^2 - 3)\sin 2x$

6. $y'' - 5y' = 2x^3 - 4x^2 - x + 6$

7. $y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x)$

Solve the following initial value problems.

8. $y'' + 4y' + 4y = (3 + x)e^{-2x}, \quad y(0) = 2, y'(0) = 5$

9. $\frac{d^2x}{dt^2} + \omega^2 x = F_0 \cos \gamma t, \qquad x(0) = 0, x'(0) = 0$

10. $y''' + 8y = 2x - 5 + 8e^{-2x}, \quad y(0) = -5, \; y'(0) = 3, y''(0) = -4$

Lecture 7

Undetermined Coefficient: Annihilator Operator Approach

Recall

1. That a non-homogeneous linear differential equation of order n is an equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

The following differential equation is called the associated homogeneous equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$$

The co efficients $a_0, a_1, \ldots, a_n$ can be f unctions o f x . H owever, w e w ill di scuss equations with constant coefficients.

2. That to obtain the general solution of a non-homogeneous linear differential equation we must find**:**

- The c omplementary f unction y_c , w hich i s general s olution of t he a ssociated homogeneous differential equation.
- Any particular solution y_p of the non-homogeneous differential equation.

3. That the general solution of the non-homogeneous linear differential equation is given by

General Solution = Complementary Function + Particular Integral

- Finding t he c omplementary f unction ha s be en c ompletely d iscussed i n an earlier <u>lecture</u>
- <u>In the previous lecture</u>, we studied a method for finding particular integral of the non-homogeneous e quations. T his w as t he *method of undetermined coefficients developed from the viewpoint of superposition* principle.
- <u>In t he p resent l ecture</u>, we w ill le arn to f ind particular in tegral o f th e n on-homogeneous equations by the same method utilizing the concept of differential annihilator operators.

Differential Operators

- In calculus, t he d ifferential co efficient d/dx is of ten de noted b y t he capital letter D. So that

$$\frac{dy}{dx} = Dy$$

The symbol D is known as differential operator.

- This operator transforms a differentiable function into another function, e.g.

$$D(e^{4x}) = 4e^{4x},\ D(5x^3 - 6x^2) = 15x^2 - 12x,\ D(\cos 2x) = -2\sin 2x$$

- The differential operator D possesses the property of linearity. This means that if f, g are two differentiable functions, then

$$D\{af(x) + bg(x)\} = aDf(x) + bDg(x)$$

Where a and b are constants. Because of this property, we say that D is a linear differential operator.

- Higher order derivatives can be expressed in terms of the operator D in a natural manner:

$$\frac{d^2y}{d^2x} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = D(Dy) = D^2y$$

Similarly

$$\frac{d^3y}{dx^3} = D^3y, \ldots, \frac{d^ny}{d^nx} = D^ny$$

- The f ollowing pol ynomial e xpression of de gree n involving t he ope rator D

$$a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0$$

is also a linear differential operator.

For example, the following expressions are all linear differential operators

$$D + 3,\ D^2 + 3D - 4,\ 5D^3 - 6D^2 + 4D$$

Differential Equation in Terms of D

Any linear differential equation can be expressed in terms of the notation D. Consider a 2nd order equation with constant coefficients

$$ay'' + by' + cy = g(x)$$

Since

$$\frac{dy}{dx} = Dy, \frac{d^2y}{dx^2} = D^2y$$

Therefore the equation can be written as

$$aD^2y + bDy + cy = g(x)$$

or $$(aD^2 + bD + c)y = g(x)$$

Now, we define another differential operator L as

$$L = aD^2 + bD + c$$

Then the equation can be compactly written as

$$L(y) = g(x)$$

The operator L is a second-order linear differential operator with constant coefficients.

Example 1

Consider the differential equation

$$y'' + y' + 2y = 5x - 3$$

Since $$\frac{dy}{dx} = Dy, \frac{d^2y}{dx^2} = D^2y$$

Therefore, the equation can be written as

$$(D^2 + D + 2)y = 5x - 3$$

Now, we define the operator L as

$$L = D^2 + D + 2$$

Then the given differential can be compactly written as

$$L(y) = 5x - 3$$

Factorization of a differential operator

- An nth-order linear differential operator

  $$L = a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0$$

  with co nstant co efficients can b e f actorized, w henever t he characteristics polynomial equation

  $$L = a_nm^n + a_{n-1}m^{n-1} + \cdots + a_1m + a_0$$

  can be factorized.

- The factors of a linear differential operator with constant coefficients commute.

Example 2

(a) Consider the following 2nd order linear differential operator

$$D^2 + 5D + 6$$

If we treat D as an algebraic quantity, then the operator can be factorized as

$$D^2 + 5D + 6 = (D + 2)(D + 3)$$

(b) To illu strate th e c ommutative p roperty o f th e factors, w e co nsider a t wice-differentiable function $y = f(x)$. Then we can write

$$(D^2 + 5D + 6)y = (D + 2)(D + 3)y = (D + 3)(D + 2)y$$

To verify this we let

$$w = (D+3)y = y' + 3y$$

Then

$$(D+2)w \;= Dw + 2w$$

or $$(D+2)w \;= (y'' + 3y') + (2y' + 6y)$$

or $$(D+2)w = y'' + 5y' + 6y$$

or $$(D+2)(D+3)y = y'' + 5y' + 6y$$

Similarly if we let

$$w = (D+2)y = (y' + 2y)$$

Then $$(D+3)w = Dw + 3w = (y'' + 2y') + (3y' + 6y)$$

or $$(D+3)w = y'' + 5y' + 6y$$

or $$(D+3)(D+2)y = y'' + 5y' + 6y$$

Therefore, we can write from the two expressions that

$$(D+3)(D+2)y = (D+2)(D+3)y$$

Hence $$(D+3)(D+2)y = (D+2)(D+3)y$$

Example 3

(a) The operator $D^2 - 1$ can be factorized as

$$D^2 - 1 = (D+1)(D-1)\,.$$

or $$D^2 - 1 = (\mathrm{D} - 1)(D+1)$$

(b) The operator $D^2 + D + 2$ does not factor with real numbers.

Example 4

The differential equation

$$y'' + 4y' + 4y = 0$$

can be written as

$$(D^2 + 4D + 4)y = 0$$

or $$(D+2)(D+2)y = 0$$

or $$(D+2)^2\, y = 0.$$

Annihilator Operator

Suppose that

- ***L*** is a linear differential operator with constant coefficients.
- *y = f(x)* defines a sufficiently differentiable function.
- The function *f* is such that $L(y)=0$

Then the differential operator *L* is said to be an **annihilator operator** of the function *f*.

Example 5
Since

$$Dx = 0,\ D^2 x = 0,\ D^3 x^2 = 0,\ D^4 x^3 = 0,\ \ldots$$

Therefore, the differential operators

$$D,\ D^2,\ D^3,\ D^4,\ \ldots$$

are annihilator operators of the following functions

$$k(\text{a constant}),\ \ x,\ \ x^2,\ \ x^3,\ \ldots$$

In general, the differential operator D^n annihilates each of the functions

$$1, x, x^2, \ldots, x^{n-1}$$

Hence, we conclude that the polynomial function

$$c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}$$

can be annihilated by finding an operator that annihilates the highest power of x.

Example 6
Find a differential operator that annihilates the polynomial function

$$y = 1 - 5x^2 + 8x^3.$$

Solution

Since $\qquad D^4 x^3 = 0,$

Therefore $\qquad D^4 y = D^4\left(1 - 5x^2 + 8x^3\right) = 0.$

Hence, D^4 is the differential operator that annihilates the function *y*.

Note that the functions that are annihilated by an nth-order linear differential operator L are simply those functions that can be obtained from the general solution of the homogeneous differential equation

$$L(y) = 0.$$

Example 7
Consider the homogeneous linear differential equation of order n

$$(D - \alpha)^n y = 0$$

The auxiliary equation of the differential equation is

$$(m-\alpha)^n = 0$$

$\Rightarrow$ $$m = \alpha, \alpha, \ldots, \alpha \ (n \text{ times})$$

Therefore, the auxiliary equation has a real root α of multiplicity n. So that the differential equation has the following linearly independent solutions:

$$e^{\alpha x}, xe^{\alpha x}, x^2 e^{\alpha x}, \ldots, x^{n-1} e^{\alpha x}.$$

Therefore, the general solution of the differential equation is

$$y = c_1 e^{\alpha x} + c_2 x e^{\alpha x} + c_3 x^2 e^{\alpha x} + \cdots + c_n x^{n-1} e^{\alpha x}$$

So that the differential operator

$$(D-\alpha)^n$$

annihilates each of the functions

$$e^{\alpha x}, xe^{\alpha x}, x^2 e^{\alpha x}, \ldots, x^{n-1} e^{\alpha x}$$

Hence, as a consequence of the fact that the differentiation can be performed term by term, the differential operator

$$(D-\alpha)^n$$

annihilates the function

$$y = c_1 e^{\alpha x} + c_2 x e^{\alpha x} + c_3 x^2 e^{\alpha x} + \cdots + c_n x^{n-1} e^{\alpha x}$$

Example 8

Find an annihilator operator for the functions

(a) $f(x) = e^{5x}$

(b) $g(x) = 4e^{2x} - 6xe^{2x}$

Solution

(a) Since

$$(D-5)e^{5x} = 5e^{5x} - 5e^{5x} = 0.$$

Therefore, the annihilator operator of function f is given by

$$L = D - 5$$

We notice that in this case $\alpha = 5,\ n = 1$.

(b) Similarly

$$(D-2)^2\left(4e^{2x} - 6xe^{2x}\right) = (D^2 - 4D + 4)(4e^{2x}) - (D^2 - 4D + 4)(6xe^{2x})$$

or $$(D-2)^2\left(4e^{2x} - 6xe^{2x}\right) = 32e^{2x} - 32e^{2x} + 48xe^{2x} - 48xe^{2x} + 24e^{2x} - 24e^{2x}$$

or $$(D-2)^2\left(4e^{2x} - 6xe^{2x}\right) = 0$$

Therefore, the annihilator operator of the function g is given by

$$L = (D-2)^2$$

We notice that in this case $\alpha = 2 = n$.

Example 9
Consider the differential equation

$$\left(D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right)^n y = 0$$

The auxiliary equation is

$$\left(m^2 - 2\alpha m + \left(\alpha^2 + \beta^2\right)\right)^n = 0$$

$$\Rightarrow \qquad m^2 - 2\alpha m + \left(\alpha^2 + \beta^2\right) = 0$$

Therefore, when α, β are real numbers, we have from the quadratic formula

$$m = \frac{2\alpha \pm \sqrt{4\alpha^2 - 4\left(\alpha^2 + \beta^2\right)}}{2} = \alpha \pm i\beta$$

Therefore, the auxiliary equation has the following two complex roots of multiplicity n.

$$m_1 = \alpha + i\beta, \quad m_2 = \alpha - i\beta$$

Thus, the general solution of the differential equation is a linear combination of the following linearly independent solutions

$$e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x,\ x^2 e^{\alpha x}\cos\beta x,\ ;\cdots\ x^{n-1}e^{\alpha x}\cos\beta x$$
$$e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x,\ x^2 e^{\alpha x}\sin\beta x,\ ;\cdots\ x^{n-1}e^{\alpha x}\sin\beta x$$

Hence, the differential operator

$$\left(D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right)^n$$

is the annihilator operator of the functions

$$e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x,\ x^2 e^{\alpha x}\cos\beta x,\ ;\cdots\ x^{n-1}e^{\alpha x}\cos\beta x$$
$$e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x,\ x^2 e^{\alpha x}\sin\beta x,\ ;\cdots\ x^{n-1}e^{\alpha x}\sin\beta x$$

Example 10
If we take

$$\alpha = -1, \ \beta = 2, \ n = 1$$

Then the differential operator

$$\left(D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right)^n$$

becomes $\qquad D^2 + 2D + 5$.

Also, it can be verified that

$$\left(D^2 + 2D + 5\right)e^{-x}\cos 2x = 0$$
$$\left(D^2 + 2D + 5\right)e^{-x}\sin 2x = 0$$

Therefore, the linear differential operator

$$D^2 + 2D + 5$$

annihilates the functions

$$y_1(x) = e^{-x}\cos 2x$$

$$y_2(x) = e^{-x}\sin 2x$$

Now, consider the differential equation

$$\left(D^2 + 2D + 5\right)y = 0$$

The auxiliary equation is

$$m^2 + 2m + 5 = 0$$

$$\Rightarrow m = -1 \pm 2\,i$$

Therefore, the functions

$$y_1(x) = e^{-x}\cos 2x$$

$$y_2(x) = e^{-x}\sin 2x$$

are the two linearly independent solutions of the differential equation

$$\left(D^2 + 2D + 5\right)y = 0,$$

Therefore, the operator also annihilates a linear combination of y_1 and y_2, e.g.

$$5y_1 - 9y_2 = 5e^{-x}\cos 2x - 9e^{-x}\sin 2x.$$

Example 11
If we take

$$\alpha = 0,\ \beta = 1,\ n = 2$$

Then the differential operator

$$\left(D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right)^n$$

becomes

$$(D^2 + 1)^2 = D^4 + 2D^2 + 1$$

Also, it can be verified that

$$\left(D^4 + 2D^2 + 1\right)\cos x = 0$$

$$\left(D^4 + 2D^2 + 1\right)\sin x = 0$$

and

$$\left(D^4 + 2D^2 + 1\right)x\cos x = 0$$

$$\left(D^4 + 2D^2 + 1\right)x\sin x = 0$$

Therefore, the linear differential operator

$$D^4 + 2D^2 + 1$$

annihilates the functions

$$\cos x, \quad \sin x$$
$$x\cos x, \quad x\sin x$$

Example 12

Taking $\alpha = 0,\ n = 1$, the operator

$$\left(D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right)^n$$

becomes

$$D^2 + \beta^2$$

Since
$$\left(D^2 + \beta^2\right)\cos\beta x = -\beta^2 \cos\beta x + \beta^2 \cos\beta x = 0$$
$$\left(D^2 + \beta^2\right)\sin\beta x = -\beta^2 \sin\beta x + \beta^2 \sin\beta x = 0$$

Therefore, the differential operator annihilates the functions

$$f(x) = \cos\beta x, \quad g(x) = \sin\beta x$$

Note that

- If a linear differential operator with constant coefficients is such that
$$L(y_1) = 0, \qquad L(y_2) = 0$$
i.e. the operator L annihilates the functions y_1 and y_2. Then the operator L annihilates their linear combination.
$$L[c_1 y_1(x) + c_2 y_2(x)] = 0.$$
This result follows from the linearity property of the differential operator L.

- Suppose that L_1 and L_2 are linear operators with constant coefficients such that
$$L_1(y_1) = 0, \quad L_2(y_2) = 0$$
and
$$L_1(y_2) \neq 0, \quad L_2(y_1) \neq 0$$
then the product of these differential operators L_1L_2 annihilates the linear sum
$$y_1(x) + y_2(x)$$
So that
$$L_1L_2[y_1(x) + y_2(x)] = 0$$

  To demonstrate this fact we use the linearity property for writing
$$L_1L_2(y_1 + y_2) = L_1L_2(y_1) + L_1L_2(y_2)$$

  Since
$$L_1L_2 = L_2L_1$$
therefore
$$L_1L_2(y_1 + y_2) = L_2L_1(y_1) + L_1L_2(y_2)$$
or
$$L_1L_2(y_1 + y_2) = L_2[L_1(y_1)] + L_1[L_2(y_2)]$$

But we know that $L_1(y_1) = 0, \quad L_2(y_2) = 0$

Therefore $L_1 L_2(y_1 + y_2) = L_2[0] + L_1[0] = 0$

Example 13

Find a differential operator that annihilates the function

$$f(x) = 7 - x + 6\sin 3x$$

Solution

Suppose that

$$y_1(x) = 7 - x, \quad y_2(x) = 6\sin 3x$$

Then

$$D^2 y_1(x) \quad = D^2(7 - x) \quad = 0$$

$$(D^2 + 9)y_2(x) = \left(D^2 + 9\right)\sin 3x = 0$$

Therefore, $D^2(D^2 + 9)$ annihilates the function $f(x)$.

Example 14

Find a differential operator that annihilates the function

$$f(x) = e^{-3x} + xe^x$$

Solution

Suppose that

$$y_1(x) = e^{-3x}, \quad y_2(x) = xe^x$$

Then

$$(D+3)\,y_1 \;=\; (D+3)\,e^{-3x} \;=0,$$

$$(D-1)^2\; y_2 = \;(D-1)^2\, xe^x \;= 0.$$

Therefore, the product of two operators

$$(D+3)(D-1)^2$$

annihilates the given function $f(x) = e^{-3x} + xe^x$

Note that

- The differential operator that annihilates a function is not unique. For example,

  $$(D-5)\,e^{5x} = 0,$$

  $$(D-5)(D+1)e^{5x} = 0,$$

  $$(D-5)D^2e^{5x} = 0$$

  Therefore, there are 3 annihilator operators of the functions, namely

  $$(D-5),\ (D-5)(D+1),\ (D-5)D^2$$

- When we seek a differential annihilator for a function, we want the operator of lowest possible order that does the job.

Exercises

Write the given differential equation in the form $L(y) = g(x)$, where L is a differential operator with constant coefficients.

1. $\dfrac{dy}{dx} + 5y = 9\sin x$
2. $4\dfrac{dy}{dx} + 8y = x + 3$
3. $\dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} = 4x$
4. $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} - 6y = 1 - \sin x$

Factor the given differentiable operator, if possible.

5. $9D^2 - 4$
6. $D^2 - 5$
7. $D^3 + 2D^2 - 13D + 10$
8. $D^4 - 8D^2 + 16$

Verify that the given differential operator annihilates the indicated functions

9. $2D - 1;\quad y = 4e^{x/2}$
10. $D^4 + 64;\quad y = 2\cos 8x - 5\sin 8x$

Find a differential operator that annihilates the given function.

11. $x + 3xe^{6x}$
12. $1 + \sin x$

Lecture 8

Undetermined Coefficients: Annihilator Operator Approach

The method of undetermined coefficients that utilizes the concept of annihilator operator approach is also limited to non-homogeneous linear differential equations

- That have constant coefficients, and
- Where the function $g(x)$ has a specific form.

The form of $g(x)$: The input function $g(x)$ has to have one of the following forms:

- A constant function k .
- A polynomial function
- An exponential function e^{x}
- The trigonometric functions $\sin(\beta x),\ \cos(\beta x)$
- Finite sums and products of these functions.

Otherwise, we cannot apply the method of undetermined coefficients.

The Method

Consider the following non-homogeneous linear differential equation with constant coefficients of order n

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = g(x)$$

If L denotes the following differential operator

$$L = a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$$

Then the non-homogeneous linear differential equation of order n can be written as

$$L(y) = g(x)$$

The function $g(x)$ should consist of finite sums and products of the proper kind of functions as already explained.

The method of undetermined coefficients, annihilator operator approach, for finding a particular integral of the non-homogeneous equation consists of the following steps**:**

Step 1 Write the given non-homogeneous linear differential equation in the form

$$L(y) = g(x)$$

Step 2 Find the complementary solution y_c by finding the general solution of the associated homogeneous differential equation**:**

$$L(y) = 0$$

Step 3 Operate on both sides of the non-homogeneous equation with a differential operator L_1 that annihilates the function g(x).

Step 4 Find the general solution of the higher-order homogeneous differential equation

$$L_1L(y)=0$$

Step 5 Delete all those terms from the solution in step 4 that are duplicated in the complementary solution y_c, found in step 2.

Step 6 Form a linear combination y_p of the terms that remain. This is the form of a particular solution of the non-homogeneous differential equation

$$L(y)=g(x)$$

Step 7 Substitute y_p found in step 6 into the given non-homogeneous linear differential equation

$$L(y)=g(x)$$

Match coefficients of various functions on each side of the equality and solve the resulting system of equations for the unknown coefficients in y_p.

Step 8 With the particular integral found in step 7, form the general solution of the given differential equation as:

$$y=y_c+y_p$$

Example 1

Solve $$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=4x^2.$$

Solution:

Step 1 Since $$\frac{dy}{dx}=Dy,\ \frac{d^2y}{dx^2}=D^2y$$

Therefore, the given differential equation can be written as

$$\left(D^2+3D+2\right)y=4x^2$$

Step 2 To find the complementary function y_c, we consider the associated homogeneous differential equation

$$\left(D^2+3D+2\right)y=0$$

The auxiliary equation is

$$m^2+3m+2=(m+1)(m+2)=0$$

$$\Rightarrow\quad m=-1,-2$$

Therefore, the auxiliary equation has two distinct real roots.

$$m_1=-1,\ m_2=-2,$$

Thus, the complementary function is given by

$$y_c=c_1e^{-x}+c_2e^{-2x}$$

Step 3 In this case the input function is

$$g(x)=4x^2$$

Further $$D^3g(x)=4D^3x^2=0$$

Therefore, the differential operator D^3 annihilates the function g. Operating on both sides of the equation in step 1, we have

$$D^3(D^2+3D+2)y = 4D^3x^2$$
$$D^3(D^2+3D+2)y = 0$$

This is the homogeneous equation of order 5. Next we solve this higher order equation.

Step 4 The auxiliary equation of the differential equation in step 3 is

$$m^3(m^2+3m+2)=0$$
$$m^3(m+1)(m+2)=0$$
$$m = 0, 0, 0, -1, -2$$

Thus its general solution of the differential equation must be

$$y = c_1 + c_2x + c_3x^2 + c_4e^{-x} + c_5e^{-2x}$$

Step 5 The following terms constitute y_c

$$c_4e^{-x} + c_5e^{-2x}$$

Therefore, we remove these terms and the remaining terms are

$$c_1 + c_2x + c_3x^2$$

Step 6 This means that the basic structure of the particular solution y_p is

$$y_p = A + Bx + Cx^2,$$

Where the constants c_1, c_2 and c_3 have been replaced, with A, *B*, and *C*, respectively.

Step 7 Since
$$y_p = A + Bx + Cx^2$$
$$y'_p = B + 2Cx,$$
$$y''_p = 2C$$

Therefore
$$y''_p + 3y'_p + 2y_p = 2C + 3B + 6Cx + 2A + 2Bx + 2Cx^2$$

or
$$y''_p + 3y'_p + 2y_p = (2C)x^2 + (2B+6C)x + (2A+3B+2C)$$

Substituting into the given differential equation, we have

$$(2C)x^2 + (2B+6C)x + (2A+3B+2C) = 4x^2 + 0x + 0$$

Equating the coefficients of x^2, x and the constant terms, we have

$$\begin{aligned} 2C &= 4 \\ 2B + 6C &= 0 \\ 2A+3B+2C &= 0 \end{aligned}$$

Solving these equations, we obtain

$$A = 7, \quad B = -6, \quad C = 2$$

Hence $$y_p = 7 - 6x + 2x^2$$

Step 8 The general solution of the given non-homogeneous differential equation is

$$y = y_c + y_p$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + 7 - 6x + 2x^2.$$

Example 2

Solve $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 8e^{3x} + 4\sin x$$

Solution:

Step 1 Since $$\frac{dy}{dx} = Dy,\ \frac{d^2y}{dx^2} = D^2 y$$

Therefore, the given differential equation can be written as

$$\left(D^2 - 3D\right)y = 8e^{3x} + 4\sin x$$

Step 2 We first consider the associated homogeneous differential equation to find y_c
The auxiliary equation is

$$m(m-3) = 0 \Rightarrow m = 0, 3$$

Thus the auxiliary equation has real and distinct roots. So that we have

$$y_c = c_1 + c_2 e^{3x}$$

Step 3 In this case the input function is given by

$$g(x) = 8e^{3x} + 4\sin x$$

Since $$(D-3)(8e^{3x}) = 0,\ (D^2+1)(4\sin x) = 0$$

Therefore, the operators $D-3$ and D^2+1 annihilate $8e^{3x}$ and $4\sin x$, respectively. So the operator $(D-3)(D^2+1)$ annihilates the input function $g(x)$. This means that

$$(D-3)(D^2+1)g(x) = (D-3)(D^2+1)(8e^{3x} + \sin x) = 0$$

We apply $(D-3)(D^2+1)$ to both sides of the differential equation in step 1 to obtain

$$(D-3)(D^2+1)(D^2-3D)y = 0.$$

This is homogeneous differential equation of order 5.

Step 4 The auxiliary equation of the higher order equation found in step 3 is

$$(m-3)(m^2+1)(m^2-3m) = 0$$

$$m(m-3)^2(m^2+1) = 0$$

$$\Rightarrow m = 0,\ 3,\ 3,\ \pm i$$

Thus, the general solution of the differential equation

$$y = c_1 + c_2 e^{3x} + c_3 x e^{3x} + c_4 \cos x + c_5 \sin x$$

Step 5 First two terms in this solution are already present in y_c

$$c_1 + c_2 e^{3x}$$

Therefore, we eliminate these terms. The remaining terms are

$$c_3 x e^{3x} + c_4 \cos x + c_5 \sin x$$

Step 6 Therefore, the basic structure of the particular solution y_p must be

$$y_p = Axe^{3x} + B\cos x + C\sin x$$

The c onstants c_3, c_4 and c_5 have be en r eplaced w ith t he c onstants A, B and C, respectively.

Step 7 Since $y_p = Axe^{3x} + B\cos x + C\sin x$

Therefore $y_p'' - 3y_p' = 3Ae^{3x} + (-B - 3C)\cos x + (3B - C)\sin x$

Substituting into the given differential equation, we have

$$3Ae^{3x} + (-B - 3C)\cos x + (3B - C)\sin x = 8e^{3x} + 4\sin x.$$

Equating coefficients of e^{3x}, $\cos x$ and $\sin x$, we obtain

$$3A = 8,\ -B - 3C = 0,\ 3B - C = 4$$

Solving these equations we obtain

$$A = 8/3,\quad B = 6/5,\quad C = -2/5$$

$$y_p = \frac{8}{3}xe^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x.$$

Step 8 The general solution of the differential equation is then

$$y = c_1 + c_2 e^{3x} + \frac{8}{3}xe^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x.$$

Example 3

Solve $$\frac{d^2y}{dx^2} + 8y = 5x + 2e^{-x}.$$

Solution:

Step 1 The given differential equation can be written as

$$(D^2 + 8)y = 5x + 2e^{-x}$$

Step 2 The associated homogeneous differential equation is

$$(D^2 + 8)y = 0$$

Roots of the auxiliary equation are complex

$$m = \pm 2\sqrt{2}\, i$$

Therefore, the complementary function is

$$y_c = c_1 \cos 2\sqrt{2}\, x + c_2 \sin 2\sqrt{2}\, x$$

Step 3 Since $D^2 x = 0, \quad (D+1)e^{-x} = 0$

Therefore the operators D^2 and $D+1$ annihilate the functions $5x$ and $2e^{-x}$. We apply $D^2(D+1)$ to the non-homogeneous differential equation

$$D^2(D+1)(D^2+8)y = 0 .$$

This is a homogeneous differential equation of order 5.

Step 4 The auxiliary equation of this differential equation is

$$m^2(m+1)(m^2+8) = 0$$

$$\Rightarrow m = 0, 0, -1, \pm 2\sqrt{2}\, i$$

Therefore, the general solution of this equation must be

$$y = c_1 \cos 2\sqrt{2}x + c_2 \sin 2\sqrt{2}x + c_3 + c_4 x + c_5 e^{-x}$$

Step 5 Since the following terms are already present in y_c

$$c_1 \cos 2\sqrt{2}x + c_2 \sin 2\sqrt{2}x$$

Thus we remove these terms. The remaining ones are

$$c_3 + c_4 x + c_5 e^{-x}$$

Step 6 The basic form of the particular solution of the equation is

$$y_p = A + Bx + Ce^{-x}$$

The constants c_3, c_4 and c_5 have been replaced with A, B and C.

Step 7 Since $y_p = A + Bx + Ce^{-x}$

Therefore $y_p'' + 8y_p = 8A + 8Bx + 9Ce^{-x}$

Substituting in the given differential equation, we have

$$8A + 8Bx + 9Ce^{-x} = 5x + 2e^{-x}$$

Equating coefficients of x, e^{-x} and the constant terms, we have

$$A = 0, B = 5/8, \ C = 2/9$$

Thus $y_p = \frac{5}{8}x + \frac{2}{9}e^{-x}$

Step 8 Hence, the general solution of the given differential equation is

$$y = y_c + y_p$$

or $y = c_1 \cos 2\sqrt{2}x + c_2 \sin 2\sqrt{2}x + \frac{5}{8}x + \frac{2}{9}e^{-x}.$

Example 4

Solve $$\frac{d^2y}{dx^2}+y=x\cos x-\cos x$$

Solution:

Step 1 The given differential equation can be written as

$$(D^2+1)y=x\cos x-\cos x$$

Step 2 Consider the associated differential equation

$$(D^2+1)y=0$$

The auxiliary equation is

$$m^2+1=0 \;\Rightarrow m=\pm i$$

Therefore $$y_c=c_1\cos x+c_2\sin x$$

Step 3 Since $$(D^2+1)^2(x\cos x)=0$$

$$(D^2+1)^2\cos x=0\ ; \qquad \because x\neq 0$$

Therefore, the operator $(D^2+1)^2$ annihilates the input function

$$x\cos x-\cos x$$

Thus operating on both sides of the non-homogeneous equation with $(D^2+1)^2$, we have

$$(D^2+1)^2(D^2+1)y=0$$

or $$(D^2+1)^3y=0$$

This is a homogeneous equation of order 6.

Step 4 The auxiliary equation of this higher order differential equation is

$$(m^2+1)^3=0\Rightarrow m=i,i,i,-i,-i,-i$$

Therefore, the auxiliary equation has complex roots i, and $-i$ both of multiplicity 3. We conclude that

$$y=c_1\cos x+c_2\sin x+c_3x\cos x+c_4x\sin x+c_5x^2\cos x+c_6x^2\sin x$$

Step 5 Since first two terms in the above solution are already present in y_c

$$c_1\cos x+c_2\sin x$$

Therefore, we remove these terms.

Step 6 The basic form of the particular solution is

$$y_p=Ax\cos x+Bx\sin x+Cx^2\cos x+Ex^2\sin x$$

Step 7 Since $$y_p=Ax\cos x+Bx\sin x+Cx^2\cos x+Ex^2\sin x$$

Therefore

$$y_p''+y_p=4Ex\cos x-4Cx\sin x+(2B+2C)\cos x+(-2A+2E)\sin x$$

Substituting in the given differential equation, we obtain

$$4Ex\cos x - 4Cx\sin x + (2B + 2C)\cos x + (-2A + 2E)\sin x = x\cos x - \cos x$$

Equating coefficients of $x\cos x, x\sin x, \cos x$ and $\sin x$, we obtain

$$4E \quad = \quad 1, \quad -4C \quad = 0$$

$$2B + 2C = -1, \quad -2A + 2E = 0$$

Solving these equations we obtain

$$A = 1/4,\ B = -1/2,\ C = 0,\ E = 1/4$$

Thus $$y_p = \frac{1}{4}x\cos x - \frac{1}{2}x\sin x + \frac{1}{4}x^2\sin x$$

Step 8 Hence the general solution of the differential equation is

$$y = c_1\cos x + c_2\sin x + \frac{1}{4}x\cos x - \frac{1}{2}x\sin x + \frac{1}{4}x^2\sin x.$$

Example 5

Determine the form of a particular solution for

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 10e^{-2x}\cos x$$

Solution

Step 1 The given differential equation can be written as

$$(D^2 - 2D + 1)y = 10e^{-2x}\cos x$$

Step 2 To find the complementary function, we consider

$$y'' - 2y' + y = 0$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$$

The complementary function for the given equation is

$$y_c = c_1e^x + c_2xe^x$$

Step 3 Since $(D^2 + 4D + 5)e^{-2x}\cos x = 0$

Applying the operator $(D^2 + 4D + 5)$ to both sides of the equation, we have

$$(D^2 + 4D + 5)(D^2 - 2D + 1)y = 0$$

This is homogeneous differential equation of order 4.

Step 4 The auxiliary equation is

$$(m^2 + 4m + 5)(m^2 - 2m + 1) = 0$$

$$\Rightarrow m = -2 \pm i, 1, 1$$

Therefore, general solution of the 4th order homogeneous equation is

$$y = c_1e^x + c_2xe^x + c_3e^{-2x}\cos x + c_4e^{-2x}\sin x$$

Step 5 Since the terms $c_1e^x + c_2xe^x$ are already present in y_c, therefore, we remove these and the remaining terms are $c_3e^{-2x}\cos x + c_4e^{-2x}\sin x$

Step 6 Therefore, the form of the particular solution of the non-homogeneous equation is

$$\therefore \qquad y_p = Ae^{-2x}\cos x + Be^{-2x}\sin x$$

Note that the s teps 7 and 8 a re not ne eded, as w e don' t h ave t o solve t he g iven differential equation.

Example 6

Determine the form of a particular solution for

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 5x^2 - 6x + 4x^2e^{2x} + 3e^{5x}.$$

Solution:

Step 1 The given differential can be rewritten as

$$\left(D^3 - 4D^2 + 4D\right)y = 5x^2 - 6x + 4x^2e^{2x} + 3e^{5x}$$

Step 2 To find the complementary function, we consider the equation

$$\left(D^3 - 4D^2 + 4D\right)y = 0$$

The auxiliary equation is

$$m^3 - 4m^2 + 4m = 0$$
$$m(m^2 - 4m + 4) = 0$$
$$m(m-2)^2 = 0 \Rightarrow m = 0, 2, 2$$

Thus the complementary function is

$$y_c = c_1 + c_2e^{2x} + c_3xe^{2x}$$

Step 3 Since $\qquad g(x) = 5x^2 - 6x + 4x^2e^{2x} + 3e^{5x}$

Further $\qquad D^3(5x^2 - 6x) = 0$

$$(D-2)^3x^2e^{2x} = 0$$
$$(D-5)e^{5x} = 0$$

Therefore the following operator must annihilate the input function $g(x)$. Therefore, applying the operator $D^3(D-2)^3(D-5)$ to both sides of the non-homogeneous equation, we have

$$D^3(D-2)^3(D-5)(D^3 - D^2 + 4D)y = 0$$

or $\qquad D^4(D-2)^5(D-5)y = 0$

This is homogeneous differential equation of order 10.

Step 4 The auxiliary equation for the 10th order differential equation is

$$m^4(m-2)^5(m-5)=0$$

$$\Rightarrow m = 0, 0, 0, 0, 2, 2, 2, 2, 2, 5$$

Hence the general solution of the 10th order equation is

$$y = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5e^{2x} + c_6xe^{2x} + c_7x^2e^{2x} + c_8x^3e^{2x} + c_9x^4e^{2x} + c_{10}e^{5x}$$

Step 5 Since the following terms constitute the complementary function y_c, we remove these $c_1 + c_5e^{2x} + c_6xe^{2x}$

Thus the remaining terms are

$$c_2x + c_3x^2 + c_4x^3 + c_7x^2e^{2x} + c_8x^3e^{2x} + c_9x^4e^{2x} + c_{10}e^{5x}$$

Hence, the form of the particular solution of the given equation is

$$y_p = Ax + Bx^2 + Cx^3 + Ex^2e^{2x} + Fx^3e^{2x} + Gx^4e^{2x} + He^{5x}$$

Exercise

Solve the given differential equation by the undetermined coefficients.

1. $2y'' - 7y' + 5y = -29$
2. $y'' + 3y' = 4x - 5$
3. $y'' + 2y' + 2y = 5e^{6x}$
4. $y'' + 4y = 4\cos x + 3\sin x - 8$
5. $y'' + 2y' + y = x^2 e^{-x}$
6. $y'' + y = 4\cos x - \sin x$
7. $y''' - y'' + y' - y = xe^x - e^{-x} + 7$
8. $y'' + y = 8\cos 2x - 4\sin x$, $y(\pi/2) = -1$, $y'(\pi/2) = 0$
9. $y''' - 2y'' + y' = xe^x + 5$, y(0)=2, $y'(0) = 2$, $y''(0) = -1$
10. $y^{(4)} - y''' = x + e^x$, y(0)=0, $y'(0) = 0$, $y''(0) = 0$, $y'''(0) = 0$

Lecture 9

Variation of Parameters

Recall

- That a non-homogeneous linear differential equation with constant coefficients is an equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

- The general solution of such an equation is given by

General Solution = Complementary Function + Particular Integral

- Finding the complementary function has already been completely discussed.

- In the last two lectures, we learnt how to find the particular integral of the non-homogeneous equations by using the undetermined coefficients.

- That the general solution of a linear first order differential equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)$$

is given by $$y = e^{-\int Pdx}. \int e^{\int Pdx} f(x)\,dx + c_1 e^{-\int Pdx}$$

Note that

- In this last equation, the 2nd term

$$y_c = c_1 e^{-\int Pdx}$$

is solution of the associated homogeneous equation:

$$\frac{dy}{dx} + P(x)y = 0$$

- Similarly, the 1st term

$$y_p = e^{-\int Pdx}.\int e^{\int Pdx}.f(x)dx$$

is a particular solution of the first order non-homogeneous linear differential equation.

- Therefore, the solution of the first order linear differential equation can be written in the form

$$y = y_c + y_p$$

In this lecture, we will use the variation of parameters to find the particular integral of the non-homogeneous equation.

The Variation of Parameters

First order equation

The particular solution y_p of the first order linear differential equation is given by

$$y_p = e^{-\int P dx} . \int e^{\int P dx} . f(x) dx$$

This formula can also be derived by another method, known as the variation of parameters. The basic procedure is same as discussed in the lecture on construction of a second solution

Since $\quad y_1 = e^{-\int P dx}$

is the solution of the homogeneous differential equation

$$\frac{dy}{dx} + P(x) y = 0,$$

and the equation is linear. Therefore, the general solution of the equation is

$$y = c_1 y_1(x)$$

The variation of parameters consists of finding a function $u_1(x)$ such that

$$y_p = u_1(x)\, y_1(x)$$

is a particular solution of the non-homogeneous differential equation

$$\frac{dy}{dx} + P(x)\, y = f(x)$$

Notice that the parameter c_1 has been replaced by the variable u_1. We substitute y_p in the given equation to obtain

$$u_1\left[\frac{dy_1}{dx} + P(x) y_1\right] + y_1 \frac{du_1}{dx} = f(x)$$

Since y_1 is a solution of the non-homogeneous differential equation. Therefore we must have

$$\frac{dy_1}{dx} + P(x)\, y_1 = 0$$

So that we obtain

$$\therefore \quad y_1 \frac{du_1}{dx} = f(x)$$

This is a variable separable equation. By separating the variables, we have

$$du_1 = \frac{f(x)}{y_1(x)} dx$$

Integrating the last expression *w.r.to* x, we obtain

$$u_1(x) = \int \frac{f(x)}{y_1} dx = \int e^{\int Pdx} \cdot f(x) dx$$

Therefore, the particular solution y_p of the given first-order differential equation is .

$$y = u_1(x) y_1$$

or

$$y_p = e^{-\int Pdx} . \int e^{\int Pdx} . f(x) dx$$

$$u_1 = \int \frac{f(x)}{y_1(x)} dx$$

Second Order Equation

Consider the 2nd order linear non-homogeneous differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

By dividing with $a_2(x)$, we can write this equation in the standard form

$$y'' + P(x)y' + Q(x)y = f(x)$$

The functions $P(x)$, $Q(x)$ and $f(x)$ are continuous on some interval I. For the complementary function we consider the associated homogeneous differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

Complementary function

Suppose that y_1 and y_2 are two linearly independent solutions of the homogeneous equation. Then y_1 and y_2 form a fundamental set of solutions of the homogeneous equation on the interval I. Thus the complementary function is

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

Since y_1 and y_2 are solutions of the homogeneous equation. Therefore, we have

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0$$

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0$$

Particular Integral

For finding a particular solution y_p, we replace the parameters c_1 and c_2 in the complementary function with the unknown variables $u_1(x)$ and $u_2(x)$. So that the assumed particular integral is

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

Since we seek to determine two unknown functions u_1 and u_2, we need two equations involving these unknowns. One of these two equations results from substituting the assumed y_p in the given differential equation. We impose the other equation to simplify the first derivative and thereby the 2nd derivative of y_p.

$$y'_p = u_1 y'_1 + y_1 u'_1 + u_2 y'_2 + u'_2 y_2 = u_1 y'_1 + u_2 y'_2 + u'_1 y_1 + u'_2 y_2$$

To avoid 2nd derivatives of u_1 and u_2, we impose the condition

$$u'_1 y_1 + u'_2 y_2 = 0$$

Then $$y'_p = u_1 y'_1 + u_2 y'_2$$

So that

$$y''_p = u_1 y''_1 + u'_1 y'_1 + u_2 y''_2 + u'_2 y'_2$$

Therefore

$$\begin{aligned} y''_p + P\,y'_p + Q\,y_p = \; & u_1 y''_1 + u'_1 y'_1 + u_2 y''_2 + u'_2 y'_2 \\ & + P u_1 y'_1 + P u_2 y'_2 + Q u_1 y_1 + Q u_2 y_2 \end{aligned}$$

Substituting in the given non-homogeneous differential equation yields

$$u_1 y''_1 + u'_1 y'_1 + u_2 y''_2 + u'_2 y'_2 + P u_1 y'_1 + P u_2 y'_2 + Q u_1 y_1 + Q u_2 y_2 = f(x)$$

or $$u_1[y''_1 + P\,y'_1 + Q\,y_1] + u_2[y''_2 + P y'_2 + Q y_2] + u'_1 y'_1 + u'_2 y'_2 = f(x)$$

Now making use of the relations

$$y''_1 + P(x) y'_1 + Q(x) y_1 = 0$$
$$y''_2 + P(x) y'_2 + Q(x) y_2 = 0$$

we obtain

$$u'_1 y'_1 + u'_2 y'_2 = f(x)$$

Hence u_1 and u_2 must be functions that satisfy the equations

$$u'_1 y_1 + u'_2 y_2 = 0$$
$$u'_1 y'_1 + u'_2 y'_2 = f(x)$$

By using the Cramer's rule, the solution of this set of equations is given by

$$u'_1 = \frac{W_1}{W}, \qquad u'_2 = \frac{W_2}{W}$$

Where W, W_1 and W_2 denote the following determinants

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}, \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y'_2 \end{vmatrix}, \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y'_1 & f(x) \end{vmatrix}$$

The determinant W can be identified as the Wronskian of the solutions y_1 and y_2. Since the solutions y_1 and y_2 are linearly independent on I. Therefore

$$W(y_1(x), y_2(x)) \neq 0, \ \forall \ x \in I.$$

Now integrating the expressions for u_1' and u_2', we obtain the values of u_1 and u_2, hence the particular solution of the non-homogeneous linear differential equation.

Summary of the Method

To solve the 2nd order non-homogeneous linear differential equation

$$a_2 y'' + a_1 y' + a_0 y = g(x),$$

using the variation of parameters, we need to perform the following steps:

Step 1 We find the complementary function by solving the associated homogeneous differential equation

$$a_2 y'' + a_1 y' + a_0 y = 0$$

Step 2 If the complementary function of the equation is given by

$$y_c = c_1 y_1 + c_2 y_2$$

then y_1 and y_2 are two linearly independent solutions of the homogeneous differential equation. Then compute the Wronskian of these solutions.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Step 3 By dividing with a_2, we transform the given non-homogeneous equation into the standard form

$$y'' + P(x)y' + Q(x)y = f(x)$$

and we identify the function $f(x)$.

Step 4 We now construct the determinants W_1 and W_2 given by

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Step 5 Next we determine the derivatives of the unknown variables u_1 and u_2 through the relations

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}$$

Step 6 Integrate the derivatives u_1' and u_2' to find the unknown variables u_1 and u_2. So that

$$u_1 = \int \frac{W_1}{W}\, d\,x\,, \quad u_2 = \int \frac{W_2}{W}\, d\,x$$

Step 7 Write a particular solution of the given non-homogeneous equation as

$$y_p = u_1 y_1 + u_2 y_2$$

Step 8 The general solution of the differential equation is then given by

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2\,.$$

Constants of Integration

We don't need to introduce the constants of integration, when computing the indefinite integrals in step 6 to find the unknown functions of u_1 and u_2. For, if we do introduce these constants, then

$$y_p = (u_1 + a_1) y_1 + (u_2 + b_1) y_2$$

So that the general solution of the given non-homogeneous differential equation is

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + (u_1 + a_1) y_1 + (u_2 + b_1) y_2$$

or $$y = (c_1 + a_1) y_1 + (c_2 + b_1) y_2 + u_1 y_1 + u_2 y_2$$

If we replace $c_1 + a_1$ with C_1 and $c_2 + b_1$ with C_2, we obtain

$$y = C_1 y_1 + C_2 y_2 + u_1 y_1 + u_2 y_2$$

This does not provide anything new and is similar to the general solution found in step 8, namely

$$y = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2$$

Example 1

Solve $$y'' - 4y' + 4y = (x+1) e^{2x}.$$

Solution:

Step 1 To find the complementary function

$$y'' - 4y' + 4y = 0$$

Put $$y = e^{mx}, y' = me^{mx}, y'' = m^2 e^{mx}$$

Then the auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0 \Rightarrow m = 2, 2$$

Repeated real roots of the auxiliary equation

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

Step 2 By the inspection of the complementary function y_c, we make the identification

$$y_1 = e^{2x} \text{ and } y_2 = xe^{2x}$$

Therefore $$W(y_1, y_2) = W\left(e^{2x}, xe^{2x}\right) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x} \neq 0, \forall x$$

Step 3 The given differential equation is

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Since this equation is already in the standard form

$$y'' + P(x)y' + Q(x)y = f(x)$$

Therefore, we identify the function $f(x)$ as

$$f(x) = (x+1)e^{2x}$$

Step 4 We now construct the determinants

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x+1)xe^{4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

Step 5 We determine the derivatives of the functions u_1 and u_2 in this step

$$u_1' = \frac{W_1}{W} = -\frac{(x+1)xe^{4x}}{e^{4x}} = -x^2 - x$$

$$u_2' = \frac{W_2}{W} = \frac{(x+1)e^{4x}}{e^{4x}} = x + 1$$

Step 6 Integrating the last two expressions, we obtain

$$u_1 = \int (-x^2 - x)dx = -\frac{x^3}{3} - \frac{x^2}{2}$$

$$u_2 = \int (x+1)\, dx \quad = \frac{x^2}{2} + x.$$

Remember! We don't have to add the constants of integration.
Step 7 Therefore, a particular solution of then given differential equation is

$$y_p = \left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}$$

or
$$y_p = \left(\frac{x^3}{6} + \frac{x^2}{2}\right)e^{2x}$$

Step 8 Hence**,** the general solution of the given differential equation is

$$y = y_c + y_p = c_1 e^{2x} + c_2 x e^{2x} + \left(\frac{x^3}{6} + \frac{x^2}{2}\right)e^{2x}$$

Example 2

Solve $$4y'' + 36y = \csc 3x.$$

Solution:

Step 1 To f ind t he c omplementary function w e s olve t he a ssociated hom ogeneous differential equation

$$4y'' + 36y = 0 \Rightarrow y'' + 9y = 0$$

The auxiliary equation is

$$m^2 + 9 = 0 \Rightarrow m = \pm 3\,i$$

Roots of the auxiliary equation are complex. Therefore, the complementary function is

$$y_c = c_1 \cos 3x + c_2 \sin 3x$$

Step 2 From the complementary function, we identify

$$y_1 = \cos 3x,\ \ y_2 = \sin 3x$$

as two linearly independent solutions of the associated homogeneous equation. Therefore

$$W(\cos 3x, \sin 3x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$$

Step 3 By dividing with 4 , we put the given equation in the following standard form

$$y'' + 9y = \frac{1}{4}\csc 3x.$$

So that we identify the function $f(x)$ as

$$f(x) = \frac{1}{4}\csc 3x$$

Step 4 We now construct the determinants W_1 and W_2

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -\frac{1}{4}\csc 3x \cdot \sin 3x = -\frac{1}{4}$$

$$W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}$$

Step 5 Therefore, the derivatives u_1' and u_2' are given by

$$u_1' = \frac{W_1}{W} = -\frac{1}{12}, \quad u_2' = \frac{W_2}{W} = \frac{1}{12}\frac{\cos 3x}{\sin 3x}$$

Step 6 Integrating the last two equations $w.r.to\ x$, we obtain

$$u_1 = -\frac{1}{12}x \quad \text{and} \quad u_2 = \frac{1}{36}\ln|\sin 3x|$$

Note that no constants of integration have been added.

Step 7 The particular solution of the non-homogeneous equation is

$$y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|$$

Step 8 Hence, the general solution of the given differential equation is

$$y = y_c + y_p = c_1\cos 3x + c_2\sin 3x - \frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|$$

Example 3

Solve $$y'' - y = \frac{1}{x}.$$

Solution:

Step 1 For the complementary function consider the associated homogeneous equation

$$y'' - y = 0$$

To solve this equation we put

$$y = e^{mx}, y' = m e^{mx}, y'' = m^2 e^{mx}$$

Then the auxiliary equation is:

$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

The roots of the auxiliary equation are real and distinct. Therefore, the complementary function is

$$y_c = c_1 e^x + c_2 e^{-x}$$

Step 2 From the complementary function we find

$$y_1 = e^x, \quad y_2 = e^{-x}$$

The functions y_1 and y_2 are two linearly independent solutions of the homogeneous equation. The Wronskian of these solutions is

$$W\left(e^x,\ e^{-x}\right) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Step 3 The given equation is already in the standard form

$$y'' + p(x)y' + Q(x)y = f(x)$$

Here $$f(x) = \frac{1}{x}$$

Step 4 We now form the determinants

$$\mathrm{W}_1 = \begin{vmatrix} 0 & e^{-x} \\ 1/x & -e^{-x} \end{vmatrix} = -e^{-x}(1/x)$$

$$\mathrm{W}_2 = \begin{vmatrix} e^x & 0 \\ e^x & 1/x \end{vmatrix} = e^x(1/x)$$

Step 5 Therefore, the derivatives of the unknown functions u_1 and u_2 are given by

$$u_1' = \frac{W_1}{W} = -\frac{e^{-x}(1/x)}{-2} = \frac{e^{-x}}{2x}$$

$$u_2' = \frac{W_2}{W} = \frac{e^{x}(1/x)}{-2} = -\frac{e^{x}}{2x}$$

Step 6 We integrate these two equations to find the unknown functions u_1 and u_2.

$$u_1 = \frac{1}{2}\int \frac{e^{-x}}{x}dx, \qquad u_2 = -\frac{1}{2}\int \frac{e^{x}}{x}dx$$

The integrals defining u_1 and u_2 cannot be expressed in terms of the elementary functions and it is customary to write such integral as:

$$u_1 = \frac{1}{2}\int_{x_\circ}^{x} \frac{e^{-t}}{t}dt, \quad u_2 = -\frac{1}{2}\int_{x_\circ}^{x} \frac{e^{t}}{t}dt$$

Step 7 A particular solution of the non-homogeneous equations is

$$y_p = \frac{1}{2}e^{x}\int_{x_\circ}^{x} \frac{e^{-t}}{t}dt - \frac{1}{2}e^{-x}\int_{x_\circ}^{x} \frac{e^{t}}{t}dt$$

Step 8 Hence, the general solution of the given differential equation is

$$y = y_c + y_p = c_1e^{x} + c_2e^{-x} + \frac{1}{2}e^{x}\int_{x_\circ}^{x} \frac{e^{-t}}{t}dt - \frac{1}{2}e^{-x}\int_{x_\circ}^{x} \frac{e^{t}}{t}dt$$

Lecture 10
Variation of Parameters Method for Higher-Order Equations

The method of the variation of parameters just examined for second-order differential equations can be generalized for an nth-order equation of the type.

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

The application of the method to n^{th} *order differential equations consists of performing the following steps.*

Step 1 *To find the complementary function we solve the associated homogeneous equation*

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$$

Step 2 *Suppose that the complementary function for the equation is*

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

Then $y_1, y_2, \ldots, y_n$ *are n linearly independent solutions of the homogeneous equation. Therefore, we compute Wronskian of these solutions.*

$$W(y_1, y_2, y_3, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Step 4 *We write the differential equation in the form*

$$y^{(n)} + P_{n-1}(x) y^{(n-1)} + \cdots + P_1(x) y' + P_\circ(x) y = f(x)$$

and compute the determinants W_k *;* $k = 1, 2, \ldots, n$*; by replacing the kth column of W by the column*

$$\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{matrix}$$

Step 5 *Next we find the derivatives* $u_1', u_2', \ldots, u_n'$ *of the unknown functions* $u_1, u_2, \ldots, u_n$ *through the relations*

$$u_k' = \frac{W_k}{W}, \qquad k = 1, 2, \ldots, n$$

Note that these derivatives can be found by solving the n *equations*

$$\begin{array}{lllllll} y_1 u_1' & + & y_2 u_2' & + \cdots + & y_n u_n' & = & 0 \\ y_1' u_1' & + & y_2' u_2' & + \cdots + & y_n' u_n' & = & 0 \\ \vdots & & \vdots & & \vdots & & \\ y_1^{(n-1)} u_1' & + & y_2^{(n-1)} u_2' & + \cdots + & y_n^{(n-1)} u_n' & = & f(x) \end{array}$$

Step 6 *Integrate the derivative functions computed in the step 5 to find the functions* u_k

$$u_k = \int \frac{W_k}{W} dx, \qquad k = 1, 2, \ldots, n$$

Step 7 *We write a particular solution of the given non-homogeneous equation as*

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x) + \cdots + u_n(x) y_n(x)$$

Step 8 *Having found the complementary function* y_c *and the particular integral* y_p, *we write the general solution by substitution in the expression*

$$y = y_c + y_p$$

Note that

- *The first* $n-1$ *equations in step 5 are assumptions made to simplify the first* $n-1$ *derivatives of* y_p. *The last equation in the system results from substituting the particular integral* y_p *and its derivatives into the given nth order linear differential equation and then simplifying.*

- *Depending upon how the integrals of the derivatives* u_k' *of the unknown functions are found, the answer for* y_p *may be different for different attempts to find* y_p *for the same equation.*

- *When asked to solve an initial value problem, we need to be sure to apply the initial conditions to the general solution and not to the complementary function alone, thinking that it is only* y_c *that involves the arbitrary constants.*

Example 1

Solve the differential equation by variation of parameters.

$$\frac{d^3y}{dx^3}+\frac{dy}{dx}=\csc x$$

Solution

Step 1: *The associated homogeneous equation is*

$$\frac{d^3y}{dx^3}+\frac{dy}{dx}=0$$

Auxiliary equation

$$m^3+m=0 \Rightarrow m\left(m^2+1\right)=0$$

$$m=0, \quad m=\pm i$$

Therefore the complementary function is

$$y_c=c_1+c_2\cos x+c_3\sin x$$

Step 2: *Since*

$$y_c=c_1+c_2\cos x+c_3\sin x$$

Therefore $y_1=1, \quad y_2=\cos x, \quad y_3=\sin x$

So that the Wronskian of the solutions y_1, y_2 and y_3

$$W\left(y_1,y_2,y_3\right)=\begin{vmatrix}1 & \cos x & \sin x\\ 0 & -\sin x & \cos x\\ 0 & -\cos x & -\sin x\end{vmatrix}$$

By the elementary row operation R_1+R_3*, we have*

$$=\begin{vmatrix}1 & 0 & 0\\ 0 & -\sin x & \cos x\\ 0 & -\cos x & -\sin x\end{vmatrix}$$

$$=\left(\sin^2 x+\cos^2 x\right)=1\neq 0$$

Step 3: *The given differential equation is already in the required standard form*

$$y''' + 0\,y'' + y' + 0\,y = \csc x$$

Step 4: *Next we find the determinants* W_1, W_2 and W_3 *by respectively, replacing 1st, 2nd and 3rd column of W by the column* $\begin{matrix} 0 \\ 0 \\ \csc x \end{matrix}$

$$W_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \csc x & -\cos x & -\sin x \end{vmatrix}$$

$$= \csc x\left(\sin^2 x + \cos^2 x\right) = \csc x$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \csc x & -\sin x \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \cos x \\ \csc x & -\sin x \end{vmatrix} = -\cos x \csc x = -\cot x$$

and

$$W_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \csc x \end{vmatrix} = \begin{vmatrix} -\sin x & 0 \\ -\cos x & \csc x \end{vmatrix} = -\sin x \csc x = -1$$

Step 5: *We compute the derivatives of the functions* u_1, u_2 and u_3 *as:*

$$u_1' = \frac{W_1}{W} = \csc x$$

$$u_2' = \frac{W_2}{W} = -\cot x$$

$$u_3' = \frac{W_3}{W} = -1$$

Step 6: *Integrate these derivatives to find* u_1, u_2 and u_3

$$u_1 = \int \frac{W_1}{W}dx = \int \csc x\,dx = \ln\left|\csc x - \cot x\right|$$

$$u_2 = \int \frac{W_2}{W} dx = \int -\cot x dx = \int \frac{-\cos x}{\sin x} dx = -\ln|\sin x|$$

$$u_3 = \int \frac{W_3}{W} dx = \int -1 dx = -x$$

Step 7: *A particular solution of the non-homogeneous equation is*

$$y_p = \ln|\csc x - \cot x| - \cos x \ln|\sin x| - x\sin x$$

Step 8: *The general solution of the given differential equation is:*

$$y = c_1 + c_2 \cos x + c_3 \sin x + \ln|\csc x - \cot x| - \cos x \ \ln|\sin x| - x\sin x$$

Example 2

Solve the differential equation by variation of parameters.

$$y''' + y' = \tan x$$

Solution

Step 1: *We find the complementary function by solving the associated homogeneous equation*

$$y''' + y' = 0$$

Corresponding auxiliary equation is

$$m^3 + m = 0 \Rightarrow m\left(m^2 + 1\right) = 0$$

$$m = 0, \quad m = \pm i$$

Therefore the complementary function is

$$y_c = c_1 + c_2 \cos x + c_3 \sin x$$

Step 2: *Since*

$$y_c = c_1 + c_2 \cos x + c_3 \sin x$$

Therefore $y_1 = 1, \quad y_2 = \cos x, \quad y_3 = \sin x$

Now we compute the Wronskian of y_1, y_2 and y_3

$$W\left(y_1, y_2, y_3\right) = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

By the elementary row operation $R_1 + R_3$, *we have*

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$= \left(\sin^2 x + \cos^2 x\right) = 1 \neq 0$$

Step 3: *The given differential equation is already in the required standard form*

$$y''' + 0 \cdot y'' + y' + 0 \cdot y = \tan x$$

Step 4: *The determinants* W_1, W_2 and W_3 *are found by replacing the 1*[st]*, 2*[nd] *and 3*[rd] *column of W by the column*

$$\begin{matrix} 0 \\ 0 \\ \tan x \end{matrix}$$

Therefore

$$W_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix}$$

$$= \tan x \left(\cos^2 x + \sin^2 x\right) = \tan x$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{vmatrix} = 1\left(0 - \cos x \tan x\right) = -\sin x$$

and

$$W_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{vmatrix} = 1\left(-\sin x \tan x\right) - 0 = -\sin x \tan x$$

Step 5: *We compute the derivatives of the functions* u_1, u_2 and u_3.

$$u_1' = \frac{W_1}{W} = \tan x$$

$$u_2' = \frac{W_2}{W} = -\sin x$$

$$u_3' = \frac{W_3}{W} = -\sin x \tan x$$

Step 6: *We integrate these derivatives to find* u_1, u_2 and u_3

$$u_1 = \int \frac{W_1}{W}\,dx = \int \tan x\,dx = -\int -\frac{\sin x}{\cos x}\,dx = -\ln|\cos x|$$

$$u_2 = \int \frac{W_2}{W}\,dx = \int -\sin x\,dx = \cos x$$

$$u_3 = \int \frac{W_3}{W}\,dx = \int -\sin x \tan x\,dx$$

$$= \int -\sin x \frac{\sin x}{\cos x}\,dx = \int -\sin^2 x \sec dx$$

$$= \int \left(\cos^2 x - 1\right)\sec x\,dx = \int \left(\cos^2 x \sec x - \sec x\right)dx$$

$$= \int \left(\cos x - \sec x\right)dx = \int \cos x\,dx - \int \sec x\,dx$$

$$= \sin x - \ln|\sec x + \tan x|$$

Step 7: *Thus, a particular solution of the non-homogeneous equation*

$$y_p = -\ln|\cos x| + \cos x\ \cos x + \left(\sin x - \ln|\sec x + \tan x|\right)\ (\sin x)$$

$$= -\ln|\cos x| + \cos^2 x + \sin^2 x - \sin x \ln|\sec x + \tan x|$$

$$= -\ln|\cos x| + 1 - \sin x \ln|\sec x + \tan x|$$

Step 8: *Hence, the general solution of the given differential equation is:*

$$y = c_1 + c_2 \cos x + c_3 \sin x - \ln|\cos x| + 1 - \sin x \ln|\sec x + \tan x|$$

or $$y = (c_1 + 1) + c_2 \cos x + c_3 \sin x - \ln|\cos x| - \sin x \ln|\sec x + \tan x|$$

or $$y = d_1 + c_2 \cos x + c_3 \sin x - \ln|\cos x| - \sin x \ln|\sec x + \tan x|$$

where d_1 *represents* $c_1 + 1$.

Example 3

Solve the differential equation by variation of parameters.

$$y''' - 2y'' - y' + 2y = e^{3x}$$

Solution

Step 1: *The associated homogeneous equation is*

$$y''' - 2y'' - y' + 2y = 0$$

The auxiliary equation of the homogeneous differential equation is

$$m^3 - 2m^2 - m + 2 = 0$$

$$\Rightarrow (m-2)\ \left(m^2 - 1\right) = 0$$

$$\Rightarrow m = 1, 2, -1$$

The roots of the auxiliary equation are real and distinct. Therefore y_c *is given by*

$$y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{-x}$$

Step 2: *From* y_c *we find that three linearly independent solutions of the homogeneous differential equation.*

$$y_1 = e^x, \; y_2 = e^{2x}, \; y_3 = e^{-x}$$

Thus the Wronskian of the solutions y_1, y_2 and y_3 *is given by*

$$W = \begin{vmatrix} e^x & e^{2x} & e^{-x} \\ e^x & 2e^{2x} & -e^{-x} \\ e^x & 4e^{2x} & e^{-x} \end{vmatrix} = e^x \cdot e^{2x} \cdot e^{-x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 4 & 1 \end{vmatrix}$$

By applying the row operations $R_2 - R_1, \; R_3 - R_1$*, we obtain*

$$W = e^{2x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 3 & 0 \end{vmatrix} = 6e^{2x} \neq 0$$

Step 3: *The given differential equation is already in the required standard form*

$$y''' - 2y'' - y' + 2y = e^{3x}$$

Step 4: *Next we find the determinants* W_1, W_2 and W_3 *by, respectively, replacing the 1st, 2nd and 3rd column of W by the column*

$$\begin{matrix} 0 \\ 0 \\ e^{3x} \end{matrix}$$

Thus

$$W_1 = \begin{vmatrix} 0 & e^{2x} & e^{-x} \\ 0 & 2e^{2x} & -e^{-x} \\ e^{3x} & 4e^{2x} & e^{-x} \end{vmatrix} = (-1)^{3+1} \begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix} e^{3x}$$

$$= e^{3x}\left(-e^x - 2e^x\right) = -3e^{4x}$$

$$W_2 = \begin{vmatrix} e^x & 0 & e^{-x} \\ e^x & 0 & -e^{-x} \\ e^x & e^{3x} & e^{-x} \end{vmatrix} = (-1)^{3+2} \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} e^{3x}$$

$$= -\left(-e^0 - e^0\right) e^{3x} = 2e^{3x}$$

and
$$W_3 = \begin{vmatrix} e^x & e^{2x} & 0 \\ e^x & 2e^{2x} & 0 \\ e^x & 4e^{2x} & e^{3x} \end{vmatrix} = e^{3x}\begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}$$
$$= e^{3x}\left(2e^{3x} - e^{3x}\right) = e^{6x}$$

Step 5: *Therefore, the derivatives of the unknown functions* $u_1,\ u_2$ and u_3 *are given by.*

$$u_1' = \frac{W_1}{W} = \frac{-3e^{4x}}{6e^{2x}} = -\frac{1}{2}e^{2x}$$
$$u_2' = \frac{W_2}{W} = \frac{2e^{3x}}{6\,e^{2x}} = \frac{1}{3}e^{x}$$
$$u_3' = \frac{W_3}{W} = \frac{e^{6x}}{6e^{2x}} = \frac{1}{6}e^{4x}$$

Step 6: *Integrate these derivatives to find* u_1, u_2 and u_3
$$u_1 = \int \frac{W_1}{W}dx = \int -\frac{1}{2}e^{2x}dx = -\frac{1}{2}\int e^{2x}dx = -\frac{1}{4}e^{2x}$$
$$u_2 = \int \frac{W_2}{W}dx = \int \frac{1}{3}e^{x}dx = \frac{1}{3}e^{x}$$
$$u_3 = \int \frac{W_3}{W}dx = \int \frac{1}{6}e^{4x}dx = \frac{1}{24}e^{4x}$$

Step 7: *A particular solution of the non-homogeneous equation is*
$$y_p = -\frac{1}{4}e^{3x} + \frac{1}{3}e^{3x} + \frac{1}{24}e^{3x}$$

Step 8: *The general solution of the given differential equation is:*
$$y = c_1e^{x} + c_2e^{2x} + c_3e^{-x} - \frac{1}{4}e^{3x} + \frac{1}{3}e^{3x} + \frac{1}{24}e^{3x}$$

Exercise

Solve the differential equations by variations of parameters.

1. $y'' + y = \tan x$
2. $y'' + y = \sec x \tan x$
3. $y'' + y = \sec^2 x$
4. $y'' - y = 9x / e^{3x}$
5. $y'' - 2y' + y = e^x / (1 + x^2)$
6. $4y'' - 4y' + y = e^{x/2}\sqrt{1 - x^2}$
7. $y''' + 4y' = \sec 2x$
8. $2y''' - 6y'' = x^2$

Solve the initial value problems.

9. $2y'' + y' - y = x + 1$
10. $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$

Lecture 11

Applications of Second Order Differential Equation

- A single differential equation can serve as mathematical model for many different phenomena in science and engineering.
- Different forms of the 2nd order linear differential equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

  appear in the analysis of problems in physics, chemistry and biology.
- In the present and next lecture we shall focus on one application**;** the motion of a mass attached to a spring.
- We shall see, what the individual terms $a\frac{d^2y}{dx^2}$, $b\frac{dy}{dx}$, cy and $f(x)$ means in the context of vibrational system.
- Except for the terminology and physical interpretation of the terms

$$a\frac{d^2y}{dx^2},\quad b\frac{dy}{dx},\quad cy,\ f(x)$$

  the mathematics of a series circuit is identical to that of a vibrating spring-mass system. Therefore we will discuss an *LRC* circuit in lecture.

Simple Harmonic Motion

When the Newton's 2nd law is combined with the Hook's Law, we can derive a differential equation governing the motion of a mass attached to spring–the simple harmonic motion.

Hook's Law

Suppose that

- A mass is attached to a flexible spring suspended from a rigid support, then
- The spring stretches by an amount '*s*'.
- The spring exerts a restoring F opposite to the direction of elongation or stretch.

The Hook's law states that the force F is proportional to the elongation s. i.e

$$F=ks$$

Where k is constant of proportionality, and is called spring constant.

Note That

- Different masses stretch a spring by different amount i. e s is different for different m .
- The spring is characterized by the spring constant k .
- For example if $W=10$ lbs and $s=\frac{1}{2}$ ft

  Then $F=ks$

  or $10=\left(\frac{1}{2}\right)k$

or $k = 20$ lbs/ft

If $W = 8$ lbs then $8 = 20(s) \Rightarrow s = 2/5$ ft

Newton's Second Law

When a force F acts upon a body, the acceleration a is produced in the direction of the force whose magnitude is proportional to the magnitude of force. i.e

$$F = ma$$

Where m is constant of proportionality and it represents mass of the body.

Weight

- The gravitational force exerted by the earth on a body of mass m is called weight of the body, denoted by W
- In the absence of air resistance, the only force acting on a freely falling body is its weight. Thus from Newton's 2nd law of motion

$$W = mg$$

Where m is measured in slugs, kilograms or grams and $g = 32\text{ft/s}^2$, $9.8m/s^2$ or $980\ \text{cm/s}^2$.

Differential Equation

- When a body of mass m is attached to a spring
- The spring stretches by an amount s and attains an equilibrium position.
- At the equilibrium position, the weight is balanced by the restoring force ks. Thus, the condition of equilibrium is

$$mg = ks \Rightarrow mg - ks = 0$$

- If the mass is displaced by an amount x from its equilibrium position and then released. The restoring force becomes $k(s + x)$. So that the resultant of weight and the restoring force acting on the body is given by

$$Resultant = -k(s + x) + mg.$$

By Newton's 2nd Law of motion, we can written

$$m\frac{d^2x}{dt^2} = -k(s + x) + mg$$

or $$m\frac{d^2x}{dt^2} = -kx - ks + mg$$

Since $$mg - ks = 0$$

Therefore $$m\frac{d^2x}{dt^2} = -kx$$

- The negative indicates that the restoring force of the spring acts opposite to the direction of motion.
- The displacements measured below the equilibrium position are positive.

- By dividing with m, the last equation can be written as:

$$\frac{d^2x}{dt^2}+\frac{k}{m}x=0$$

or

$$\frac{d^2x}{dt^2}+\omega^2x=0$$

Where $\omega^2=\frac{k}{m}$. This equation is known as the equation of simple harmonic motion or as the free un-damped motion.

Initial Conditions

Associated with the differential equation

$$\frac{d^2x}{dt^2}+\omega^2x=0$$

are the obvious initial conditions

$$x(0)=\alpha,\quad x'(0)=\beta$$

These initial conditions represent the initial displacement and the initial velocity. For example

- If $\alpha>0,\ \beta<0$ then the body starts from a point below the equilibrium position with an imparted upward velocity.
- If $\alpha<0,\ \beta=0$ then the body starts from rest $|\alpha|$ units above the equilibrium position.

Solution and Equation of Motion

Consider the equation of simple harmonic motion

$$\frac{d^2x}{dt^2}+\omega^2x=0$$

Put

$$x=e^{mx},\ \frac{d^2x}{dt^2}=m^2e^{mx}$$

Then the auxiliary equation is

$$m^2+\omega^2=0\quad\Rightarrow\quad m=\pm i\omega$$

Thus the auxiliary equation has complex roots.

$$m_1=\omega i,\ \ m_2=-\omega i$$

Hence, the general solution of the equation of simple harmonic motion is

$$x(t)=c_1\cos\omega t+c_2\sin\omega t$$

Alternative form of Solution

It is often convenient to write the above solution in a alternative simpler form. Consider

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

and suppose that $A,\ \phi \in R$ such that

$$c_1 = A\sin\phi,\ c_2 = A\cos\phi$$

Then $$A = \sqrt{c_1^{\,2} + c_2^{\,2}}\ ,\ \tan\phi = \frac{c_1}{c_2}$$

So that

$$x(t) = A\sin\omega t\ \cos\phi + B\cos\ \omega t\ \sin\phi$$

or $$x(t) = A\sin(\omega t + \phi)$$

The number ϕ is called the phase angle;

Note that

This form of the solution of the equation of simple harmonic motion is very useful because

- Amplitude of free vibrations becomes very obvious
- The times when the body crosses equilibrium position are given by

  $$x = 0 \Rightarrow \sin(\omega t + \phi) = 0$$

  or $$\omega t + \phi = n\pi$$

  Where n is a non-negative integer.

The Nature of Simple Harmonic Motion

Amplitude

- We know that the solution of the equation of simple harmonic motion can be written as

  $$x(t) = A\sin(\omega t + \phi)$$

- Clearly, the maximum distance that the suspended body can travel on either side of the equilibrium position is A.
- This maximum distance called the amplitude of motion and is given by

  $$Amplitude = A = \sqrt{c_1^{\,2} + c_2^{\,2}}$$

A Vibration or a Cycle

In travelling from $x = A$ to $x = -A$ and then back to A, the vibrating body completes one vibration or one cycle.

Period of Vibration

The simple harmonic motion of the suspended body is periodic and it repeats its position after a specific time period T. We know that the distance of the mass at any time t is given by

$$x = A\sin(\omega t + \phi)$$

Since

$$A\sin\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right]$$

$$= A\sin\left[(\omega t + \phi + 2\pi)\right]$$

$$= A\sin\left[(\omega t + \phi)\right]$$

Therefore, the distances of the suspended body from the equilibrium position at the times t and $t + \frac{2\pi}{\omega}$ are same

Further, velocity of the body at any time t is given by

$$\frac{dx}{dt} = A\omega\cos(\omega t + \phi)$$

$$A\omega\cos\left(\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right)$$

$$= A\omega\cos[\omega t + \phi + 2\pi]$$

$$= A\omega\cos(\omega t + \phi)$$

Therefore the velocity of the body remains unaltered if t is increased by $2\pi/\omega$. Hence the time period of free vibrations described by the 2nd order differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

is given by

$$T = \frac{2\pi}{\omega}$$

Frequency

The number of vibration /cycle completed in a unit of time is known as frequency of the free vibrations, denoted by f. Since the cycles completed in time T is 1. Therefore, the number of cycles completed in a unit of time is $1/T$

Hence

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Example 1

Solve and interpret the initial value problem

$$\frac{d^2x}{dt^2} + 16x = 0$$

$$x(0) = 10,\ x'(0) = 0.$$

Interpretation

Comparing the initial conditions

$$x(0) = 10,\ x'(0) = 0.$$

With

$$x(0) = \alpha,\ x'(0) = \beta$$

We see that

$$\alpha = 10\ ,\ \beta = 0$$

Thus the problem is equivalent to

- Pulling the mass on a spring 10 units below the equilibrium position.
- Holding it there until time $t = 0$ and then releasing the mass from rest.

Solution

Consider the differential equation

$$\frac{d^2x}{dt^2} + 16x = 0$$

Put $$x = e^{mt},\ \frac{d^2x}{dt^2} = m^2 e^{mt}$$

Then, the auxiliary equation is

$$m^2 + 16 = 0$$

$$\Rightarrow\ m = 0 \pm 4i$$

Therefore, the general solution is:

$$x(t) = c_1 \cos 4t + c_2 \sin 4t$$

Now we apply the initial conditions.

$$x(0) = 10 \Rightarrow c_1.1 + c_2.0 = 10$$

Thus $$c_1 = 10$$

So that $$x(t) = 10\cos 4t + c_2 \sin 4t$$

$$\frac{dx}{dt} = -40\sin 4t + 4c_2 \cos 4t$$

Therefore $x'(0)=0 \quad \Rightarrow -40(0)+4c_2.1=0$

Thus $c_2=0$

Hence, the solution of the initial value problem is

$$x(t)=10\cos 4t$$

Note that

- Clearly, the solution shows that once the system is set into motion, it stays in motion with mass bouncing back and forth with amplitude being 10 *units*.
- Since $\omega=4$. Therefore, the period of oscillation is

$$T=\frac{2\pi}{4}=\frac{\pi}{2} \text{ seconds}$$

Example 2

A mass weighing 2lbs stretches a spring 6 inches. At *t = 0* the mass is released from a point 8 inches below the equilibrium position with an upward velocity of $\frac{4}{3}\, ft/s$. Determine the function *x (t)* that describes the subsequent free motion.

Solution

For consistency of units with the engineering system, we make the following conversions

$$6 \text{ inches}=\frac{1}{2}\text{ foot}$$

$$8 \text{ inches}=\frac{2}{3}\text{ foot}.$$

Further weight of the body is given to be

$$\text{W}=2 \text{ lbs}$$

But $\text{W}=mg$

Therefore $m=\frac{\text{W}}{g}=\frac{2}{32}$

or $m=\frac{1}{16}$ slugs.

Since $Stretch=s=\frac{1}{2}$ foot

Therefore by Hook's Law, we can write

$$2=k\left(\frac{1}{2}\right) \Rightarrow k=4 \text{ lbs/ft}$$

Hence the equation of simple harmonic motion

$$m\frac{d^2x}{dt^2} = -kx$$

becomes

$$\frac{1}{16}\frac{d^2x}{dt^2} = -4x$$

or $$\frac{d^2x}{dt^2} + 64x = 0.$$

Since the initial displacement is $8\ inches = \frac{2}{3}$ ft and the initial velocity is $\frac{-4}{3}$ ft/s, the initial conditions are:

$$x(0) = \frac{2}{3}, \quad x'(0) = -\frac{4}{3}$$

The negative sign indicates that the initial velocity is given in the upward i.e negative direction. Thus, we need to solve the initial value problem.

Solve $$\frac{d^2x}{dt^2} + 64x = 0$$

Subject to $$x(0) = \frac{2}{3}, \quad x'(0) = -\frac{4}{3}$$

Putting $$x = e^{mt}, \ \frac{d^2x}{dt^2} = m^2e^{mt}$$

We obtain the auxiliary equation

$$m^2 + 64 = 0$$

or $$m = \pm 8i$$

The general solution of the equation is

$$x(t) = c_1 \cos 8t + c_2 \sin 8t$$

Now, we apply the initial conditions.

$$x(0) = \frac{2}{3} \quad \Rightarrow c_1.1 + c_2.0 = \frac{2}{3}$$

Thus $$c_1 = \frac{2}{3}$$

So that $$x(t) = \frac{2}{3}\cos 8t + c_2 \sin 8t$$

Since

$$x'(t) = -\frac{16}{3}\sin 8t + 8c_2 \cos 8t\,.$$

Therefore

$$x'(0) = -\frac{4}{3} \;\Rightarrow -\frac{16}{3}.0 + 8c_2.1 = -\frac{4}{3}$$

Thus

$$c_2 = -\frac{1}{6}.$$

Hence, solution of the initial value problem is

$$x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t.$$

Example 3

Write the solution of the initial value problem discussed in the previous example in the form

$$x(t) = A\sin(\omega t + \phi\ ).$$

Solution

The initial value discussed in the previous example is:

Solve $$\frac{d^2x}{dt^2} + 64x = 0$$

Subject to $$x(0) = \frac{2}{3},\quad x'(0) = -\frac{4}{3}$$

Solution of the problem is

$$x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t$$

Thus amplitude of motion is given by

$$A = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{6}\right)^2} = \frac{\sqrt{17}}{6} \approx 0.69 \text{ ft}$$

and the phase angle is defined by

$$\sin\phi = \frac{2/3}{\sqrt{17}/6} = \frac{4}{\sqrt{17}} > 0$$

$$\cos\phi = \frac{-1/6}{\sqrt{17}/6} = -\frac{1}{\sqrt{17}} < 0$$

Therefore

$$\tan\phi = -4$$

or $$\tan^{-1}(-4) = -1.326 \text{ radians}$$

Since $\sin\phi > 0,\ \cos\phi < 0,$ the phase angle ϕ must be in 2nd quadrant.

Thus

$$\phi = \pi - 1.326 = 1.816 \text{ radians}$$

Hence the required form of the solution is

$$x(t) = \frac{\sqrt{17}}{6}\sin(8t + 1.816)$$

Example 4

For the motion described by the initial value problem

Solve $$\frac{d^2x}{dt^2} + 64x = 0$$

Subject to $$x(0) = \frac{2}{3},\quad x'(0) = -\frac{4}{3}$$

Find the first value of time for which the mass passes through the equilibrium position heading downward.

Solution

We know that the solution of initial value problem is

$$x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t\,.$$

This solution can be written in the form

$$x(t) = \frac{\sqrt{17}}{6}\sin(8t + 1.816)$$

The values of t for which the mass passes through the equilibrium position i.e for which $x = 0$ are given by

$$wt + \phi = n\pi$$

Where $n = 1,2,\ldots$, therefore, we have

$$8t_1 + 1.816 = \pi,\quad 8t_2 + 1.816 = 2\pi,\quad 8t_3 + 1.816 = 3\pi,\ldots$$

or $$t_1 = 0.166,\quad t_2 = 0.558,\quad t_3 = 0.951,\ \ldots$$

Hence, the mass passes through the equilibrium position

$$x = 0$$

heading downward first time at $t_2 = 0.558$ seconds.

Exercise

State in words a possible physical interpretation of the given initial-value problems.

1. $\frac{4}{32}x'' + 3x = 0, \qquad x(0) = -3, \quad x'(0) = -2$
2. $\frac{1}{16}x'' + 4x = 0, \qquad x(0) = 0.7, \quad x'(0) = 0$

Write the solution of the given initial-value problem in the form $x(t) = A\sin(\varpi t + \phi)$

3. $x'' + 25x = 0, \qquad x(0) = -2, \quad x'(0) = 10$
4. $\frac{1}{2}x'' + 8x = 0, \qquad x(0) = 1, \quad x'(0) = -2$
5. $x'' + 2x = 0, \qquad x(0) = -1, \quad x'(0) = -2\sqrt{2}$
6. $\frac{1}{4}x'' + 16x = 0, \qquad x(0) = 4, \quad x'(0) = 16$
7. $0.1x'' + 10x = 0, \qquad x(0) = 1, \quad x'(0) = 1$
8. $x'' + x = 0, \qquad x(0) = -4, \quad x'(0) = 3$
9. The period of free undamped oscillations of a mass on a spring is $\pi/4$ seconds. If the spring constant is 16 lb/ft, what is the numerical value of the weight?
10. A 4-lb weight is attached to a spring, whose spring constant is 16 lb/ft. What is period of simple harmonic motion?
11. A 24-lb weight, attached to the spring, stretches it 4 inches. Find the equation of the motion if the weight is released from rest from a point 3 inches above the equilibrium position.
12. A 20-lb weight stretches a spring 6 inches. The weight is released from rest 6 inches below the equilibrium position.

    a) Find the position of the weight at $t = \frac{\pi}{12}, \frac{\pi}{8}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{9\pi}{32}$ seconds.

    b) What is the velocity of the weight when $t = 3\pi/16$ seconds? In which direction is the weight heading at this instant?

    c) At what times does the weight pass through the equilibrium position?

Lecture 12

Differential Equations with Variable Coefficients

So far we have been solving Linear Differential Equations with constant coefficients.

We will now discuss the Differential Equations with non-constant (variable) coefficients.

These equations normally arise in applications such as temperature or potential u in the region bounded between two concentric spheres. Then under some circumstances we have to solve the differential equation:

$$r\frac{d^2u}{dr^2}+2\frac{du}{dr}=0$$

where the variable $r>0$ represents the radial distance measured outward from the center of the spheres.

Differential equations with variable coefficients such as

$$x^2y''+xy'+(x^2-v^2)y=0$$

$$(1-x^2)y''-2xy'+n(n+1)y=0$$

and $y''-2xy'+2ny=0$

occur in applications ranging from potential problems, temperature distributions and vibration phenomena to quantum mechanics.

The differential equations with variable coefficients cannot be solved so easily.

Cauchy- Euler Equation:

Any linear differential equation of the form

$$a_nx^n\frac{d^ny}{dx^n}+a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1x\frac{dy}{dx}+a_0y=g(x)$$

where $a_n,a_{n-1},\cdots,a_0$ are constants, is said to be a *Cauchy-Euler* equation or equi-dimensional equation. The degree of each monomial coefficient matches the order of differentiation i.e x^n is the coefficient of nth derivative of y, x^{n-1} of (n-1)th derivative of y, etc.

For convenience we consider a homogeneous second-order differential equation

$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=0,\quad x\neq 0$$

The solution of higher-order equations follows analogously.

Also, we can solve the non-homogeneous equation

$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=g(x),\quad x\neq 0$$

by variation of parameters after finding the complementary function $y_c(x)$.

We find the general solution on the interval $(0,\infty)$ and the solution on $(0,-\infty)$ can be obtained by substituting $t=-x$ in the differential equation.

Method of Solution:

We try a solution of the form $y=x^m$, where m is to be determined. The first and second derivatives are, respectively,

$$\frac{dy}{dx}=mx^{m-1}\quad\text{and}\quad\frac{d^2y}{dx^2}=m(m-1)x^{m-2}$$

Consequently the differential equation becomes

$$\begin{aligned}ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy&=ax^2\cdot m(m-1)x^{m-2}+bx\cdot mx^{m-1}+cx^m\\&=am(m-1)x^m+bmx^m+cx^m\\&=x^m(am(m-1)+bm+c)\end{aligned}$$

Thus $y=x^m$ is a solution of the differential equation whenever m is a solution of the ***auxiliary equation***

$$(am(m-1)+bm+c)=0\ \text{or}\ \ am^2+(b-a)m+c=0$$

The solution of the differential equation depends on the roots of the AE.

Case-I: Distinct Real Roots

Let m_1 and m_2 denote the real roots of the auxiliary equation such that $m_1\neq m_2$. Then $y=x^{m_1}$ and $y=x^{m_2}$ form a fundamental set of solutions.

Hence the general solution is

$$y=c_1x^{m_1}+c_2x^{m_2}.$$

Example 1

Solve $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=0$

Solution:

Suppose that $y=x^m$, then

$$\frac{dy}{dx}=mx^{m-1} \quad , \quad \frac{d^2y}{dx^2}=m(m-1)x^{m-2}$$

Now substituting in the differential equation, we get:

$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^2\cdot m(m-1)x^{m-2}-2x\cdot mx^{m-1}-4x^m$$

$$=x^m(m(m-1)-2m-4)$$

$$x^m(m^2-3m-4)=0 \quad \text{if } m^2-3m-4=0$$

This implies $m_1=-1, m_2=4$; roots are real and distinct.

So the solution is $y=c_1x^{-1}+c_2x^4$.

Case II: Repeated Real Roots

If the roots of the auxiliary equation are repeated, that is, then we obtain only one solution $y=x^{m_1}$.

To construct a second solution y_2, we first write the *Cauchy-Euler equation* in the form

$$\frac{d^2y}{dx^2}+\frac{b}{ax}\frac{dy}{dx}+\frac{c}{ax^2}y=0$$

Comparing with

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

We make the identification $P(x)=\frac{b}{ax}$. Thus

$$y_2=x^{m_1}\int\frac{e^{\int\frac{b}{ax}dx}}{(x^{m_1})^2}dx$$

$$=x^{m_1}\int\frac{e^{-(\frac{b}{a})\ln x}}{x^{2m_1}}dx$$

$$=x^{m_1}\int x^{-\frac{b}{a}}.x^{-2m_1}dx$$

Since roots of the AE $am^2+(b-a)m+c=0$ are equal, therefore discriminant is zero

$$\text{i.e } m_1 = -\frac{(b-a)}{2a} \quad \text{or} \quad -2m_1 = +\frac{(b-a)}{a}$$

$$y_2 = x^{m_1} \int x^{\frac{-b}{a}} . x^{\frac{b-a}{a}} \, dx$$

$$y_2 = x^{m_1} \int \frac{dx}{x} = x^{m_1} \ln x.$$

The general solution is then

$$y = c_1 x^{m_1} + c_2 x^{m_1} \ln x$$

Example 2

Solve $$4x^2 \frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = 0.$$

Solution:

Suppose that $y = x^m$, then

$$\frac{dy}{dx} = mx^{m-1} \quad , \quad \frac{d^2y}{dx^2} = m(m-1)x^{m-2}.$$

Substituting in the differential equation, we get:

$$4x^2 \frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = x^m(4m(m-1) + 8m + 1) = x^m(4m^2 + 4m + 1) = 0$$

if $4m^2 + 4m + 1 = 0$ or $(2m+1)^2 = 0$.

Since $m_1 = -\frac{1}{2}$, the general solution is

$$y = c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln x .$$

For higher order equations, if m_1 is a root of multiplicity k, then it can be shown that:

$x^{m_1}, x^{m_1} \ln x, x^{m_1} (\ln x)^2, \cdots, x^{m_1} (\ln x)^{k-1}$ are k linearly independent solutions.

Correspondingly, the general solution of the differential equation must then contain a linear combination of these k solutions.

Case III Conjugate Complex Roots

If the roots of the auxiliary equation are the conjugate pair

$$m_1 = \alpha + i\beta, \quad m_2 = \alpha - i\beta$$

where α and β >0 are real, then the solution is

$$y = c_1 x^{\alpha + i\beta} + c_2 x^{\alpha - i\beta}.$$

But, as in the case of equations with constant coefficients, when the roots of the auxiliary equation are complex, we wish to write the solution in terms of real functions only. We note the identity

$$x^{i\beta} = (e^{\ln x})^{i\beta} = e^{i\beta \ln x},$$

which, by *Euler's formula*, is the same as

$$x^{i\beta} = \cos(\beta \ln x) + i\sin(\beta \ln x)$$

Similarly we have

$$x^{-i\beta} = \cos(\beta \ln x) - i\sin(\beta \ln x)$$

Adding and subtracting last two results yields, respectively,

$$x^{i\beta} + x^{-i\beta} = 2\cos(\beta \ln x)$$

and $$x^{i\beta} - x^{-i\beta} = 2i\sin(\beta \ln x)$$

From the fact that $y = c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta}$ is the solution of $ax^2 y'' + bxy' + cy = 0$, for any values of constants c_1 and c_2, we see that

$$y_1 = x^{\alpha}(x^{i\beta} + x^{-i\beta}), \quad (c_1 = c_2 = 1)$$

$$y_2 = x^{\alpha}(x^{i\beta} - x^{-i\beta}), \quad (c_1 = 1, c_2 = -1)$$

or $$y_1 = 2x^{\alpha}(\cos(\beta \ln x))$$

$$y_2 = 2x^{\alpha}(\sin(\beta \ln x))$$ are also solutions.

Since $W(x^{\alpha}\cos(\beta \ln x), x^{\alpha}\sin(\beta \ln x)) = \beta x^{2\alpha-1} \neq 0; \beta > 0$, on the interval $(0,\infty)$, we conclude that

$$y_1 = x^{\alpha}\cos(\beta \ln x) \text{ and } y_2 = x^{\alpha}\sin(\beta \ln x)$$

constitute a fundamental set of real solutions of the differential equation.

Hence the general solution is

$$y_1 = x^{\alpha}[c_1\cos(\beta \ln x) + c_2\sin(\beta \ln x)]$$

Example 3

Solve the initial value problem

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 3y = 0, \qquad y(1) = 1, y'(1) = -5$$

Solution:

Let us suppose that: $y = x^m$, then $\frac{dy}{dx} = mx^{m-1}$ and $\frac{d^2y}{dx^2} = m(m-1)x^{m-2}$.

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 3y = x^m(m(m-1) + 3m + 3) = x^m(m^2 + 2m + 3) = 0$$

if $\quad m^2 + 2m + 3 = 0$.

From the quadratic formula we find that $m_1 = -1 + \sqrt{2}i$ and $m_1 = -1 - \sqrt{2}i$. If we make the identifications $\alpha = -1$ and $\beta = \sqrt{2}$, so the general solution of the differential equation is

$$y_1 = x^{-1}[c_1 \cos(\sqrt{2}\ln x) + c_2 \sin(\sqrt{2}\ln x)].$$

By applying the conditions $y(1) = 1, y'(1) = -5$, we find that

$$c_1 = 1 \qquad \text{and } c_2 = -2\sqrt{2}.$$

Thus the solution to the initial value problem is

$$y_1 = x^{-1}[\cos(\sqrt{2}\ln x) - 2\sqrt{2}\sin(\sqrt{2}\ln x)]$$

Example 4

Solve the *third-order Cauchy-Euler differential equation*

$$x^3\frac{d^3y}{dx^3} + 5x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 8y = 0,$$

Solution

The first three derivative of $y = x^m$ are

$$\frac{dy}{dx} = mx^{m-1}, \quad \frac{d^2y}{dx^2} = m(m-1)x^{m-2}, \quad \frac{d^3y}{dx^3} = m(m-1)(m-2)x^{m-3},$$

so the given differential equation becomes

$$x^3\frac{d^3y}{dx^3} + 5x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 8y = x^3m(m-1)(m-2)x^{m-3} + 5x^2m(m-1)x^{m-2} + 7xmx^{m-1} + 8x^m,$$

$$= x^m(m(m-1)(m-2) + 5m(m-1) + 7m + 8)$$

$$= x^m(m^3 + 2m^2 + 4m + 8)$$

In this case we see that $y = x^m$ is a solution of the differential equation, provided m is a root of the cubic equation

$$m^3 + 2m^2 + 4m + 8 = 0$$

$$\text{or} \quad (m+2)(m^2+4) = 0$$

The roots are: $m_1 = -2, m_2 = 2i, m_3 = -2i$.

Hence the general solution is

$$y_1 = c_1x^{-2} + c_2\cos(2\ln x) + c_3\sin(2\ln x)$$

Example 5

Solve the non-homogeneous equation

$$x^2y'' - 3xy' + 3y = 2x^4e^x$$

Solution

Put $y = x^m$

$$\frac{dy}{dx} = mx^{m-1}, \quad \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

Therefore we get the auxiliary equation,

$$m(m-1) - 3m + 3 = 0 \text{ or } (m-1)(m-3) = 0 \text{ or } m = 1,3$$

Thus $y_c = c_1x + c_2x^3$

Before using variation of parameters to find the particular solution $y_p = u_1y_1 + u_2y_2$, recall that the formulas $u_1' = \frac{W_1}{W}$ and $u_2' = \frac{W_2}{W}$, where $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$, and W is the Wronskian of y_1 and y_2, were derived under the assumption that the differential equation has been put into special form . $y'' + P(x)y' + Q(x)y = f(x)$

Therefore we divide the given equation by x^2, and form $y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x$

we make the identification $f(x) = 2x^2e^x$. Now with $y_1 = x$, $y_2 = x^2$, and

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3, \quad W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x, \quad W_2 = \begin{vmatrix} x & x \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x$$

we find

$$u'_1 = \frac{2x^5e^x}{2x^3} = -x^2e^x \text{ and } \qquad u'_2 = \frac{2x^3e^x}{2x^3} = e^x$$

$$u_1 = -x^2e^x + 2xe^x - 2e^x \text{ and } u_2 = e^x.$$

Hence

$$y_p = u_1y_1 + u_2y_2$$

$$= (-x^2e^x + 2xe^x - 2e^x)x + e^xx^3 = 2x^2e^x - 2xe^x$$

Finally we have $y = y_c + y_p = c_1x + c_2x^3 + 2x^2e^x - 2xe^x$

Exercises

1. $4x^2 y'' + y = 0$
2. $xy'' - y' = 0$
3. $x^2 y'' + 5xy' + 3y = 0$
4. $4x^2 y'' + 4xy' - y = 0$
5. $x^2 y'' - 7xy' + 41y = 0$
6. $x^3 \dfrac{d^3 y}{dx^3} - 2x^2 \dfrac{d^2 y}{dx^2} + 4x \dfrac{dy}{dx} - 4y = 0$
7. $x^4 \dfrac{d^4 y}{dx^4} + 6x^3 \dfrac{d^3 y}{dx^3} + 9x^2 \dfrac{d^2 y}{dx^2} + 3x \dfrac{dy}{dx} + y = 0$
8. $x^2 y'' - 5xy' + 8y = 0; y(1) = 0, y'(1) = 4$
9. $x^2 y'' - 2xy' + 2y = x^3 \ln x$
10. $x^3 \dfrac{d^3 y}{dx^3} - 3x^2 \dfrac{d^2 y}{dx^2} + 6x \dfrac{dy}{dx} - 6y = 3 + \ln x^3$

Lecture 13

Cauchy-Euler Equation: Alternative Method of Solution

We reduce any Cauchy-Euler differential equation to a differential equation with constant coefficients through the substitution

$$x = e^t \quad \text{or} \quad t = \ln x$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \cdot \frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x} \cdot \frac{dy}{dt}\right) = \frac{1}{x} \cdot \frac{d}{dx}\left(\frac{dy}{dt}\right) - \frac{1}{x^2} \cdot \frac{dy}{dt}$$

or
$$\frac{d^2y}{dx^2} = \frac{1}{x} \cdot \frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} - \frac{1}{x^2} \cdot \frac{dy}{dt}$$

or
$$\frac{d^2y}{dx^2} = \frac{1}{x^2} \cdot \frac{d^2y}{dt^2} - \frac{1}{x^2} \cdot \frac{dy}{dt}$$

Therefore
$$x\frac{dy}{dx} = \frac{dy}{dt}, \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Now introduce the notation

$$D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2}, \text{ etc.}$$

and
$$\Delta = \frac{d}{dt}, \Delta^2 = \frac{d^2}{dt^2}, \quad \text{etc.}$$

Therefore, we have

$$xD = \Delta$$

$$x^2D^2 = \Delta^2 - \Delta = \Delta(\Delta - 1)$$

Similarly

$$x^3D^3 = \Delta(\Delta - 1)(\Delta - 2)$$

$$x^4D^4 = \Delta(\Delta - 1)(\Delta - 2)(\Delta - 3) \quad \text{so on so forth.}$$

This substitution in a given *Cauchy-Euler* differential equation will reduce it in to a differential equation with constant coefficients.

At this stage we suppose $y = e^{mt}$ to obtain an auxiliary equation and write the solution in terms of y and t. We then go back to x through $x = e^t$.

Example 1

Solve $x^2 \dfrac{d^2 y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = 0$

Solution

The given differential equation can be written as

$$(x^2 D^2 - 2xD - 4)y = 0$$

With the substitution $x = e^t$ or $t = \ln x$, we obtain

$$xD = \Delta\,, \quad x^2 D^2 = \Delta(\Delta - 1)$$

Therefore the equation becomes:

$$[\Delta(\Delta-1) - 2\Delta - 4]y = 0$$

or $\quad (\Delta^2 - 3\Delta - 4)y = 0$

or $\quad \dfrac{d^2 y}{dt^2} - 3\dfrac{dy}{dt} - 4y = 0$

Now substitute: $y = e^{mt}$ then $\dfrac{dy}{dt} = me^{mt}$, $\dfrac{d^2 y}{dt^2} = m^2 e^{mt}$

Thus $(m^2 - 3m - 4)e^{mt} = 0$ or $m^2 - 3m - 4 = 0$, which is the auxiliary equation.

$$(m+1)(m-4) = 0 \qquad m = -1, 4$$

The roots of the auxiliary equation are distinct and real, so the solution is

$$y = c_1 e^{-t} + c_2 e^{4t}$$

But $x = e^t$, therefore the answer will be

$$y = c_1 x^{-1} + c_2 x^4$$

Example 2

Solve $4x^2 \dfrac{d^2 y}{dx^2} + 8x\dfrac{dy}{dx} + y = 0$

Solution

The differential equation can be written as:

$$(4x^2 D^2 + 8xD + 1)y = 0$$

Where $D = \dfrac{d}{dx}, D^2 = \dfrac{d^2}{dx^2}$

Now with the substitution $x = e^t$ or $t = \ln x$, $xD = \Delta$, $x^2 D^2 = \Delta(\Delta - 1)$ where $\Delta = \dfrac{d}{dt}$

The equation becomes:

$$(4\Delta(\Delta-1) + 8\Delta + 1)y = 0 \qquad \text{or } (4\Delta^2 + 4\Delta + 1)y = 0$$

$$4\frac{d^2 y}{dt^2} + 4\frac{dy}{dt} + y = 0$$

Now substituting $y = e^{mt}$ then $\frac{dy}{dt} = me^{mt}$, $\frac{d^2y}{dt^2} = m^2e^{mt}$, we get

$$(4m^2 + 4m + 1)e^{mt} = 0$$

or $4m^2 + 4m + 1 = 0$ or $(2m+1)^2 = 0$

or $m = -\frac{1}{2}, -\frac{1}{2}$; the roots are real but repeated.

Therefore the solution is

$$y = (c_1 + c_2 t)e^{-\frac{1}{2}t}$$

or $y = (c_1 + c_2 \ln x)x^{-\frac{1}{2}}$

i-e $y = c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln x$

Example 3

Solve the initial value problem

$$x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 3y = 0, \quad y(1) = 1, y'(1) = -5$$

Solution

The given differential can be written as:

$$(x^2D^2 + 3xD + 3)y = 0$$

Now with the substitution $x = e^t$ or $t = \ln x$ we have:

$xD = \Delta$, $x^2D^2 = \Delta(\Delta - 1)$

Thus the equation becomes:

$(\Delta(\Delta-1) + 3\Delta + 3)y = 0$ or $(\Delta^2 + 2\Delta + 3)y = 0$

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = 0$$

Put $y = e^{mt}$ then the A.E. equation is:

or $m^2 + 2m + 3 = 0$

or $m = \frac{-2 \pm \sqrt{4-12}}{2} = -1 \pm i\sqrt{2}$

So that solution is:

$$y = e^{-t}(c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t)$$

or $y = x^{-1}(c_1 \cos\sqrt{2}\ln x + c_2 \sin\sqrt{2}\ln x)$

Now $y(1)=1$ gives, $1=(c_1\cos 0+c_2\sin 0)\ \Rightarrow c_1=1$

$$y'=-x^{-2}(c_1\cos\sqrt{2}\ln x+c_2\sin\sqrt{2}\ln x)+x^{-2}(-\sqrt{2}c_1\sin\sqrt{2}\ln x+\sqrt{2}c_2\cos\sqrt{2}\ln x)$$

$\therefore y'(1)=-5$ gives: $-5=-[c_1+0]+[\sqrt{2}c_2]$ or $\sqrt{2}c_2=c_1-5=-4$, $c_2=\dfrac{-4}{\sqrt{2}}=-2\sqrt{2}$

Hence solution of the IVP is:

$$y=x^{-1}[\cos(\sqrt{2}\ln x)-2\sqrt{2}\sin(\sqrt{2}\ln x)].$$

Example 4

Solve $x^3\dfrac{d^3y}{dx^3}+5x^2\dfrac{d^2y}{dx^2}+7x\dfrac{dy}{dx}+8y=0$

Solution

The given differential equation can be written as:

$$(x^3D^3+5x^2D^2+7xD+8)y=0$$

Now with the substitution $x=e^t$ or $t=\ln x$ we have:

$$xD=\Delta\ ,\ x^2D^2=\Delta(\Delta-1),\ x^3D^3=\Delta(\Delta-1)(\Delta-2)$$

So the equation becomes:

$$(\Delta(\Delta-1)(\Delta-2)+5\Delta(\Delta-1)+7\Delta+8)y=0$$

or $(\Delta^3-3\Delta^2+2\Delta+5\Delta^2-5\Delta+7\Delta+8)y=0$

or $(\Delta^3+2\Delta^2+4\Delta+8)y=0$

or $\dfrac{d^3y}{dt^3}+2\dfrac{d^2y}{dt^2}+4\dfrac{dy}{dt}+8y=0$

Put $y=e^{mt}$, then the auxiliary equation is:

$$m^3+2m^2+4m+8=0$$

or $(m^2+4)(m+2)=0$

$m=-2,$ or $\pm 2i$

So the solution is:

$$y=c_1e^{-2t}+c_2\cos 2t+c_3\sin 2t$$

or $y=c_1x^{-2}+c_2\cos(2\ln x)+c_3\sin(2\ln x)$

Example 5

Solve the non-homogeneous differential equation

$$x^2y''-3xy'+3y=2x^4e^x$$

Solution

First consider the associated homogeneous differential equation.

$$x^2y''-3xy'+3y=0$$

With the notation $\frac{d}{dx}=D, \frac{d^2}{dx^2}=D^2$, the differential equation becomes:

$$(x^2D^2-3xD+3)y=0$$

With the substitution $x=e^t$ or $t=\ln x$, we have:

$$xD=\Delta, \quad x^2D^2=\Delta(\Delta-1)$$

So the homogeneous differential equation becomes:

$$[\Delta(\Delta-1)-3\Delta+3]y=0$$

$$(\Delta^2-4\Delta+3)y=0$$

or $$\frac{d^2y}{dt^2}-4\frac{dy}{dt}+3y=0$$

Put $y=e^{mt}$ then the AE is:

$m^2-4m+3=0$ or $(m-3)(m-1)=0$, or $m=1,3$

$\therefore y_c=c_1e^t+c_2e^{3t}$, as $x=e^t$

$$y_c=c_1x+c_2x^3$$

For y_p we write the differential equation as:

$$y''-\frac{3}{x}y'+\frac{3}{x^2}y=2x^2e^x$$

$y_p=u_1x+u_2x^3$, where u_1 and u_2 are functions given by

$$u_1'=\frac{W_1}{W}, \quad u_2'=\frac{W_2}{W},$$

with

$W=\begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}=2x^3$, $W_1=\begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix}=-2x^5e^x$ and

$$W_2=\begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix}=2x^3e^x$$

So that $u'_1 = \frac{2x^5e^x}{2x^3} = -x^2e^x$ and $u'_2 = \frac{2x^3e^x}{2x^3} = e^x$

$$\therefore u_1 = -\int x^2e^x dx = -[x^2e^x - 2\int xe^x dx]$$
$$= -x^2e^x + 2[xe^x - \int e^x dx]$$
$$= -x^2e^x + 2xe^x - 2e^x$$

and $u_2 = \int e^x dx = e^x$.

Therefore

$$y_p = x(-x^2e^x + 2xe^x - 2e^x) + x^3e^x = 2x^2e^x - 2xe^x$$

Hence the general solution is:

$$y = y_c + y_p$$
$$y = c_1x + c_2x^3 + 2x^2e^x - 2xe^x$$

Example 6

Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \ln x$

Solution

Consider the associated homogeneous differential equation.

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

or $(x^2D^2 - xD + 1)y = 0$

With the substitution $x = e^t$, we have:

$$xD = \Delta\,,\;\; x^2D^2 = \Delta(\Delta - 1)$$

So the homogeneous differential equation becomes:

$$[\Delta(\Delta - 1) - \Delta + 1]y = 0$$
$$(\Delta^2 - 2\Delta + 1)y = 0$$

or $$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0$$

Putting $y = e^{mt}$, we get the auxiliary equation as:

$$m^2 - 2m + 1 = 0 \quad \text{or} \quad (m-1)^2 = 0 \quad \text{or} \quad m = 1,1$$

$$\therefore y_c = c_1 e^t + c_2 t e^t$$

or $$y_c = c_1 x + c_2 x \ln x \;.$$

Now the non-homogeneous differential equation becomes:

$$\frac{d^2 y}{dt^2} - 2\frac{dy}{dt} + y = t$$

By the method of undetermined coefficients we try a particular solution of the form $y_p = A + Bt$. This assumption leads to

$-2B + A + Bt = t$ so that $A=2$ and $B=1$.

Using $y = y_c + y_p$, we get

$$y_c = c_1 e^t + c_2 t e^t + 2 + t\,;$$

So the general solution of the original differential equation on the interval $(0,\infty)$ is

$$y_c = c_1 x + c_2 x \ln x + 2 + \ln x$$

Exercises

Solve using $x = e^t$

1. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$
2. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 4y = 0$
3. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} - 2y = 0$
4. $25x^2\dfrac{d^2y}{dx^2} + 25x\dfrac{dy}{dx} + y = 0$
5. $3x^2\dfrac{d^2y}{dx^2} + 6x\dfrac{dy}{dx} + y = 0$
6. $x\dfrac{d^4y}{dx^4} + 6\dfrac{d^3y}{dx^3} = 0$
7. $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} = 0, y(1) = 0, y'(0) = 4$
8. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0, y(1) = 1, y'(1) = 2$
9. $x^2\dfrac{d^2y}{dx^2} + 10x\dfrac{dy}{dx} + 8y = x^2$
10. $x^2\dfrac{d^2y}{dx^2} + 9x\dfrac{dy}{dx} - 20y = \dfrac{5}{x^3}$

Lecture 14

Power Series: An Introduction

- A standard technique for solving linear differential equations with variable coefficients is to find a solution as an infinite series. Often this solution can be found in the form of a power series.

- Therefore, in this lecture we discuss some of the more important facts about power series.

- However, for an in-depth review of the infinite series concept one should consult a standard calculus text.

Power Series

A power series in $(x-a)$ is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n\left(x-a\right)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$$

The coefficients $c_0,\ c_1\ \ c_2,\ldots$ and a are constants and x represents a variable. In this discussion we will only be concerned with the cases where the coefficients, x and a are real numbers. The number a is known as the centre of the power series.

Example 1

The infinite series

$$\sum_{n=1}^{\infty} \frac{\left(-1\right)^{n+1}}{n^2} x^n = x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \cdots$$

is a power series in x. This series is centered at zero.

Convergence and Divergence

- If we choose a specified value of the variable x then the power series becomes an infinite series of constants. If, for the given x, the sum of terms of the power series equals a finite real number, then the series is said to be convergent at x.
- A power series that is not convergent is said to be a divergent series. This means that the sum of terms of a divergent power series is not equal to a finite real number.

Example 2

(a) Consider the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Since for $x = 1$ the series become

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = e$$

Therefore, the power series converges $x = 1$ to the number e

(b) Consider the power series

$$\sum_{n=0}^{\infty} n!(x+2)^n = 1 + (x+2) + 2!(x+2)^2 + 3!(x+2)^3 + \cdots$$

The series diverges $\forall\ x$, except at $x = -2$. For instance, if we take $x = 1$ then the series becomes

$$\sum_{n=0}^{\infty} n!(x+2)^n = 1 + 3 + 18 + \cdots$$

Clearly the sum of all terms on right hand side is not a finite number. Therefore, the series is divergent at $x = 1$. Similarly, we can see its divergence at all other values of $x \neq -2$

The Ratio Test

To determine for which values of x a power series is convergent, one can often use the Ratio Test. The Ratio test states that if

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is a power series and

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| |x - a| = L$$

Then:

- The power series converges absolutely for those values of x for which $L < 1$.
- The power series diverges for those values of x for which $L > 1$ or $L = \infty$.
- The test is inconclusive for those values of x for which $L = 1$.

Interval of Convergence

The set of all real values of x for which a power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

converges is known as the interval of convergence of the power series.

Radius of Convergence
Consider a power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

Then exactly one of the following three possibilities is true:

- The series converges only at its center $x = a$.
- The series converges for all values of x.
- There is a number $R > 0$ such that the series converges absolutely $\forall x$ satisfying $|x-a| < R$ and diverges for $|x-a| > R$. This means that the series converges for $x \in (a-R, a+R)$ and diverges out side this interval.

The number R is called the radius of convergence of the power series. If first possibility holds then $R = 0$ and in case of 2nd possibility we write $R = \infty$.
From the Ratio test we can clearly see that the radius of convergence is given by

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

provided the limit exists.

Convergence at an Endpoint
If the radius of convergence of a power series is $R > 0$, then the interval of convergence of the series is one of the following

$$(a-R, a+R),\ (a-R, a+R],\ [a-R, a+R),\ \left[a-R, a+R\right]$$

To determine which of these intervals is the interval of convergence, we must conduct separate investigations for the numbers $x = a - R \, and \, x = a + R$.

Example 3
Consider the power series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} x^n$$

Then $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n} \right|$$

or $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\sqrt{\frac{n}{n+1}}\cdot x\right| = \lim_{n\to\infty}\left|\sqrt{\frac{n}{n+1}}\right||x| = |x|$$

Therefore, it follows from the Ratio Test that the power series converges absolutely for those values of x which satisfy

$$|x| < 1$$

This means that the power series converges if x belongs to the interval

$$(-1,1)$$

The series diverges outside this interval i.e. when $x > 1$ or $x < -1$. The convergence of the power series at the numbers 1 and -1 must be investigated separately by substituting into the power series.

a) When we substitute $x = 1$, we obtain
$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(1)^n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}+\cdots$$
which is a divergent p-series, with $p=\frac{1}{2}$.

b) When we substitute $x = -1$, we obtain
$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(-1)^n = -1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\cdots+\frac{(-1)^n}{\sqrt{n}}+\cdots$$
which converges, by alternating series test.

Hence, the interval of convergence of the power series is $[-1,1)$. This means that the series is convergent for those vales of x which satisfy

$$-1 \le x < 1$$

Example 4

Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{(x-3)^n}{2^n\cdot n}$$

Solution

The power series is centered at 3 and the radius of convergence of the series is

$$R = \lim_{n\to\infty}\frac{2^{n+1}(n+1)}{2^n\cdot n} = 2$$

Hence, the series converges absolutely for those values of x which satisfy the inequality

$$|x-3| < 2 \Rightarrow 1 < x < 5$$

(a) At the left endpoint we substitute $x = 1$ in the given power series to obtain the series of constants:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This series is convergent by the alternating series test.
(b) At the right endpoint we substitute $x = 5$ in the given series and obtain the following harmonic series of constants

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Since a harmonic series is always divergent, the above power series is divergent.

Hence, the series the interval of convergence of the given power series is a half open and half closed interval $[1, 5)$.

Absolute Convergence

Within its interval of convergence a power series converges absolutely. In other words, the series of absolute values

$$\sum_{n=o}^{\infty} |c_n| \left|(x-a)^n\right|$$

converges for all values x in the interval of convergence.

A Power Series Represent Functions

A power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ determines a function f whose domain is the interval of convergence of the power series. Thus for all x in the interval of convergence, we write

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots$$

If a function is f is defined in this way, we say that $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series representation for $f(x)$. We also say that f is represented by the power series

Theorem

Suppose that a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence $R > 0$ and for every x in the interval of convergence a function f is defined by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots$$

Then

- The function f is continuous, differentiable, and integrable on the interval $(a-R,\ a+R)$.
- Moreover, $f'(x)$ and $\int f(x)dx$ can be found from term-by-term differentiation and ntegration.

Therefore

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$$

$$\int f(x)\ dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The series obtained by differentiation and integration have same radius of convergence. However, the convergence at the end points $x = a - R$ and $x = a + R$ of the interval may change. This means that the interval of convergence may be different from the interval of convergence of the original series.

Example 5

Find a function f that is represented by the power series

$$1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$$

Solution

The given power series is a geometric series whose common ratio is $r = -x$. Therefore, if $|x| < 1$ then the series converges and its sum is

$$S = \frac{a}{1-r} = \frac{1}{1+x}$$

Hence we can write

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$$

This last expression is the power series representation for the function $f(x) = \dfrac{1}{1+x}$.

Series that are Identically Zero

If for all real numbers x in the interval of convergence, a power series is identically zero i.e.

$$\sum_{n=o}^{\infty} c_n (x-a)^n = 0, \qquad R > 0$$

Then all the coefficients in the power series are zero. Thus we can write

$$c_n = 0, \qquad \forall\ n = 0, 1, 2, \ldots$$

Analytic at a Point

A function f is said to be analytic at point a if the function can be represented by power series in ($x-a$) with a positive radius of convergence. The notion of analyticity at a point will be important in finding power series solution of a differential equation.

Example 6

Since the functions e^x, $\cos x$, and $\ln(1+x)$ can be represented by the power series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Therefore, these functions are analytic at the point $x = 0$.

Arithmetic of Power Series

- Power series can be combined through the operations of addition, multiplication, and division.
- The procedure for addition, multiplication and division of power series is similar to the way in which polynomials are added, multiplied, and divided.
- Thus we add coefficients of like powers of x, use the distributive law and collect like terms, and perform long division.

Example 7

If both of the following power series converge for $|x| < R$

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \quad g(x) = \sum_{n=0}^{\infty} b_n x^n$$

Then
$$f(x) + g(x) = \sum_{n=0}^{\infty} (c_n + b_n) x^n$$

and
$$f(x) \cdot g(x) = c_0 b_0 + (c_0 b_1 + c_1 b_0) x + (c_0 b_2 + c_1 b_1 + c_2 b_0) x^2 + \cdots$$

Lecture 15

Power Series: An Introduction

Example 8

Find the first four terms of a power series in x for the product $e^x \cos x$.

Solution

From calculus the Maclaurin series for e^x and $\cos x$ are, respectively,

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots.$$

Multiplying the two series and collecting the like terms yields

$$e^x \cos x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots.\right)$$

$$= 1 + (1)x + \left(-\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(-\frac{1}{2} + \frac{1}{6}\right)x^3 + \left(\frac{1}{24} - \frac{1}{4} + \frac{1}{24}\right)x^4 + \cdots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

The interval of convergence of the power series for both the functions e^x and $\cos x$ is $(-\infty, \infty)$. Consequently the interval of convergence of the power series for their product $e^x \cos x$ is also $(-\infty, \infty)$.

Example 9

Find the first four terms of a power series in x for the function $\sec x$.

Solution

We know that

$$\sec x = \frac{1}{\cos x}, \quad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$$

Therefore using long division, we have

$$1+\frac{x^2}{2}+\frac{5x^4}{24}+\frac{61x^6}{720}+\cdots$$

$$1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots \,\Big)\, 1$$

$$1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots$$

$$\frac{x^2}{2}-\frac{x^4}{24}+\frac{x^6}{720}-\cdots$$

$$\frac{x^2}{2}-\frac{x^4}{4}+\frac{x^6}{48}-\cdots$$

$$\frac{5x^4}{24}-\frac{7x^6}{360}+\cdots$$

$$\frac{5x^4}{24}-\frac{5x^6}{48}+\cdots$$

$$\frac{61x^6}{720}-\cdots$$

Hence, the power series for the function $f(x)=\sec x$ is

$$\sec x = 1+\frac{x^2}{2}+\frac{5x^4}{24}+\frac{61x^6}{720}+\cdots$$

The interval of convergence of this series is $(-\pi/2, \pi/2)$.

Note that

- The procedures illustrated in examples 2 and 3 are obviously tedious to do by hand.
- Therefore, problems of this sort can be done using a computer algebra system (CAS) such as Mathematica.
- When we type the command: Series$\left[Sec[x],\ \{x,\ 0,\ 8\}\right]$ and enter, the Mathematica immediately gives the result obtained in the above example.
- For finding power series solutions it is important that we become adept at simplifying the sum of two or more power series, each series expressed in summation (sigma) notation, to an expression with a single $\sum$. This often requires a shift of the summation indices.

- In order to add any two power series, we must ensure that:

  (a) That summation indices in both series start with the same number.
  (b) That the powers of x in each of the power series be "in phase".

  Therefore, if one series starts with a multiple of, say, x to the first power, then the other series must also start with the same power of the same power of x.

Example 10

Write the following sum of two series as one power series

$$\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6nc_n x^{n+1}$$

Solution

To write the given sum power series as one series, we write it as follows:

$$\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6nc_n x^{n+1} = 2\cdot 1c_1 x^0 + \sum_{n=2}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6nc_n x^{n+1}$$

The first series on right hand side starts with x^1 for $n=2$ and the second series also starts with x^1 for $n=0$. Both the series on the right side start with x^1.

To get the same summation index we are inspired by the exponents of x which is $n-1$ in the first series and $n+1$ in the second series. Therefore, we let

$$k = n-1, \quad k = n+1$$

in the first series and second series, respectively. So that the right side becomes:

$$2c_1 + \sum_{k=1}^{\infty} 2(k+1)c_{k+1}x^k + \sum_{k=1}^{\infty} 6(k-1)c_{k-1}x^k .$$

Recall that the summation index is a "dummy" variable. The fact that $k = n-1$ in one case and $k = n+1$ in the other should cause no confusion if you keep in mind that it is the *value* of the summation index that is important. In both cases k takes on the same successive values $1,2,3,\ldots$ for $n = 2,3,4,\ldots$ (for $k = n-1$) and $n = 0,1,2,\ldots$ (for $k = n+1$)

We are now in a position to add the two series in the given sum term by term:

$$\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6nc_n x^{n+1} = 2c_1 + \sum_{k=1}^{\infty}\left[2(k+1)c_{k+1} + 6(k-1)c_{k-1}\right]x^k$$

If you are not convinced, then write out a few terms on both series of the last equation.

Lecture 29

Power Series Solution of a Differential Equation

We know that the explicit solution of the linear first-order differential equation

$$\frac{dy}{dx} - 2xy = 0$$

is $$y = e^{x^2}$$

Also $$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

If we replace x by x^2 in the series representation of e^x, we can write the solution of the differential equation as

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

This last series converges for all real values of x. In other words, knowing the solution in advance, we were able to find an infinite series solution of the differential equation.

We now propose to obtain a **power series solution** of the differential equation directly; the method of attack is similar to the technique of undetermined coefficients.

Example 11

Find a solution of the differential equation

$$\frac{dy}{dx} - 2xy = 0$$

in the form of power series in x.

Solution

If we assume that a solution of the given equation exists in the form

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + \sum_{n=1}^{\infty} c_n x^n$$

The que stion i s t hat: C an w e de termine c oefficients c_n for w hich t he p ower s eries converges t o a f unction s atisfying th e d ifferential e quation? N ow te rm-by-term differentiation of the proposed series solution gives

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

Using the last result and the assumed solution, we have

$$\frac{dy}{dx} - 2xy = \sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} 2c_n x^{n+1}$$

We would like to add the two series in this equation. To this end we write

$$\frac{dy}{dx} - 2xy = 1 \cdot c_1 x^0 + \sum_{n=2}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} 2c_n x^{n+1}$$

and then proceed as in the previous example by letting

$$k = n - 1, \quad k = n + 1$$

in the first and second series, respectively. Therefore, last equation becomes

$$\frac{dy}{dx} - 2xy = c_1 + \sum_{k=1}^{\infty} (k+1) c_{k+1} x^k - \sum_{k=1}^{\infty} 2c_{k-1} x^k$$

After we add the series term wise, it follows that

$$\frac{dy}{dx} - 2xy = c_1 + \sum_{k=1}^{\infty} [(k+1)c_{k+1} - 2c_{k-1}]x^k$$

Substituting in the given differential equation, we obtain

$$c_1 + \sum_{k=1}^{\infty} \left[(k+1) c_{k+1} - 2c_{k-1} \right] x^k = 0$$

In order to have this true, it is necessary that all the coefficients must be zero. This means that

$$c_1 = 0, \quad (k+1) c_{k+1} - 2c_{k-1} = 0, \qquad k = 1, 2, 3, \ldots$$

This eq uation p rovides a r ecurrence r elation t hat d etermines t he co efficient c_k . S ince $k+1 \neq 0$ for all the indicated values of k, we can write as

$$c_{k+1} = \frac{2c_{k-1}}{k+1}$$

Iteration of this last formula then gives

$$k = 1, \qquad c_2 = \frac{2}{2} c_0 = c_0$$

$$k = 2, \qquad c_3 = \frac{2}{3} c_1 = 0$$

$$k = 3, \qquad c_4 = \frac{2}{4} c_2 = \frac{1}{2} c_0 = \frac{1}{2!} c_0$$

$$k = 4, \qquad c_5 = \frac{2}{5}c_3 = 0$$

$$k = 5, \qquad c_6 = \frac{2}{6}c_4 = \frac{1}{3 \cdot 2!}c_0 = \frac{1}{3!}c_0$$

$$k = 6, \qquad c_7 = \frac{2}{7}c_5 = 0$$

$$k = 7, \qquad c_8 = \frac{2}{8}c_6 = \frac{1}{4 \cdot 3!}c_0 = \frac{1}{4!}c_0$$

and so on. Thus from the original assumption (7), we find

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

$$= c_0 + 0 + c_0 x^2 + 0 + \frac{1}{2!}c_0 x^4 + 0 + \frac{1}{3!}c_0 x^6 + 0 + \cdots$$

$$= c_0\left[1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \cdots\right] = c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Since the coefficient c_0 remains completely undetermined, we have in fact found the general solution of the differential equation.

Note that

The differential equation in this example and the differential equation in the following example can be easily solved by the other methods. The point of these two examples is to prepare ourselves for finding the power series solution of the differential equations with variable coefficients.

Example 12

Find solution of the differential equation

$$4y'' + y = 0$$

in the form of a powers series in x.

Solution

We assume that a solution of the given differential equation exists in the form of

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + \sum_{n=1}^{\infty} c_n x^n$$

Then term by term differentiation of the proposed series solution yields

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + \sum_{n=2}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

Substituting the expression for y'' and y, we obtain

$$4y'' + y = \sum_{n=2}^{\infty} 4n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n$$

Notice that both series start with x^0. If we, respectively, substitute

$$k = n-2, \quad k = n, \quad k = 0,1,2,\ldots$$

in the first series and second series on the right hand side of the last equation. Then we after using, in turn, $n = k+2$ and $n = k$, we get

$$4y'' + y = \sum_{k=0}^{\infty} 4(k+2)(k+1)c_{k+2}x^k + \sum_{k=0}^{\infty} c_k x^k$$

or

$$4y'' + y = \sum_{k=0}^{\infty} \left[4(k+2)(k+1)c_{k+2} + c_k \right] x^k$$

Substituting in the given differential equation, we obtain

$$\sum_{k=0}^{\infty} \left[4(k+2)(k+1)c_{k+2} + c_k \right] x^k = 0$$

From this last identity we conclude that

$$4(k+2)(k+1)c_{k+2} + c_k = 0$$

or

$$c_{k+2} = \frac{-c_k}{4(k+2)(k+1)}, \quad k = 0,1,2,\ldots$$

From iteration of this recurrence relation it follows that

$$c_2 = \frac{-c_0}{4.2.1} = -\frac{c_0}{2^2.2!}$$

$$c_3 = \frac{-c_1}{4.3\cdot 2} = -\frac{c_1}{2^2.3!}$$

$$c_4 = \frac{-c_2}{4.4.3} = +\frac{c_0}{2^4.4!}$$

$$c_5 = \frac{-c_3}{4.5.4} = +\frac{c_1}{2^4.5!}$$

$$c_6 = \frac{-c_4}{4.6.5} = -\frac{c_0}{2^6.6!}$$

$$c_7 = \frac{-c_5}{4.7.6} = -\frac{c_1}{2^6.7!}$$

and so forth. This iteration leaves both c_0 and c_1 arbitrary. From the original assumption we have

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + \cdots$$
$$= c_0 + c_1 x - \frac{c_0}{2^2.2!}x^2 - \frac{c_1}{2^2.3!}x^3 + \frac{c_0}{2^4.4!}x^4 + \frac{c_1}{2^4.5!}x^5 - \frac{c_0}{2^6.6!}x^6 - \frac{c_1}{2^6.7!}x^7 + \cdots$$

or

$$y = c_0\left[1 - \frac{1}{2^2.2!}x^2 + \frac{1}{2^4.4!}x^4 - \frac{1}{2^6.6!}x^6 + \cdots\right] + c_1\left[x - \frac{1}{2^2.3!}x^3 + \frac{1}{2^4.5!}x^5 - \frac{1}{2^6.7!}x^7 + \cdots\right]$$

is a general solution. When the series are written in summation notation,

$$y_1(x) = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{x}{2}\right)^{2k} \quad \text{and} \quad y_2(x) = 2c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{x}{2}\right)^{2k+1}$$

the ratio test can be applied to show that both series converges for all x. You might also recognize the Maclaurin series as $y_2(x) = c_0 \cos(x/2)$ and $y_2(x) = 2c_1 \sin(x/2)$.

Exercise

Find the interval of convergence of the given power series.

1. $\sum_{k=1}^{\infty} \frac{2^k}{k} x^k$
2. $\sum_{n=1}^{\infty} \frac{(x+7)^n}{\sqrt{n}}$
3. $\sum_{k=0}^{\infty} k!2^k x^k$
4. $\sum_{k=0}^{\infty} \frac{k-1}{k^{2k}} x^k$

Find the first four terms of a power series in x for the given function.

5. $e^x \sin x$
6. $e^x \ln(1-x)$
7. $\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right)^2$

Solve each differential equation in the manner of the previous chapters and then compare the results with the solutions obtained by assuming a power series solution

$$y = \sum_{n=0}^{\infty} c_n x^n$$

8. $y' - x^2 y = 0$
9. $y'' + y = 0$
10. $2y'' + y' = 0$

Lecture 16
Solution about Ordinary Points

Analytic Function: A function *f* is said to be analytic at a point *a* if it can be represented by a power series in *(x-a)* with a positive radius of convergence.

Suppose the linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (1)$$

is put into the form

$$y'' + P(x)y' + Q(x)y = 0 \qquad (2)$$

by dividing by the leading coefficient $a_2(x)$.

Ordinary and singular points: A point x_0 is said to be a ***ordinary point*** of a differential equation (1) if both *P(x)* and *Q(x)* are analytic at x_0. A point that is not an ordinary point is said to be ***singular point*** of the equation.

Polynomial Coefficients:
If $a_2(x), a_1(x)$ and $a_0(x)$ are polynomials with no common factors, then $x = x_0$ is

(i) an ordinary point if $a_2(x) \neq 0$ or
(ii) a singular point if $a_2(x) = 0$.

Example

(a) The singular points of the equation $(x^2 - 1)y'' + 2xy' + 6y = 0$ are the solutions of $x^2 - 1 = 0$ or $x = \pm 1$. All other finite values of *x* are the ordinary points.

(b) The singular points need not be real numbers.
The equation $(x^2 + 1)y'' + 2xy' + 6y = 0$ has the singular points at the solutions of $x^2 + 1 = 0$, namely, $x = \pm i$.

All other finite values, real or complex, are ordinary points.

Example

The Cauchy-Euler equation $ax^2 y'' + bxy' + cy = 0$, where a, b and c are constants, has singular point at $x = 0$.
All other finite values of *x*, real or complex, are ordinary points.

THEOREM (Existence of Power Series Solution)

If $x = x_0$ is an ordinary point of the differential equation $y'' + P(x)y' + Q(x)y = 0$, we can always find two linearly independent solutions in the form of power series centered at x_0:

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

A series solution converges at least for $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point (real or complex).

Example

Solve $y'' - 2xy = 0$.

Solution

We see that $x = 0$ is an ordinary point of the equation. Since there are no finite singular points, there exist two solutions of the form $y = \sum_{n=0}^{\infty} c_n x^n$ convergent for $|x| < \infty$. Proceeding, we write

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$y'' - 2xy = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} 2 c_n x^{n+1}$$

$$= 2 \cdot 1 c_2 x^0 + \underbrace{\sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} 2 c_n x^{n+1}}$$

both series start with x

Letting $k = n - 2$ in the first series and $k = n + 1$ in the second, we have

$$y'' - 2xy = 2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=1}^{\infty} 2 c_{k-1} x^k$$

$$= 2c_2 + \sum_{k=1}^{\infty} [(k+2)(k+1) c_{k+2} - 2c_{k-1}] x^k = 0$$

$$2c_2 = 0 \quad \text{and} \quad (k+2)(k+1) c_{k+2} - 2c_{k-1} = 0$$

The last expression is same as

$$c_{k+2} = \frac{2c_{k-1}}{(k+2)(k+1)}, \quad k = 1,2,3,\cdots$$

Iteration gives

$$c_3 = \frac{2c_0}{3\cdot 2}$$

$$c_4 = \frac{2c_1}{4\cdot 3}$$

$$c_5 = \frac{2c_2}{5\cdot 4} = 0 \qquad \text{because } c_2 = 0$$

$$c_6 = \frac{2c_3}{6\cdot 5} = \frac{2^2}{6\cdot 5\cdot 3\cdot 2}c_0$$

$$c_7 = \frac{2c_4}{7\cdot 6} = \frac{2^2}{7\cdot 6\cdot 4\cdot 3}c_1$$

$$c_8 = \frac{2c_5}{8\cdot 7} = 0$$

$$c_9 = \frac{2c_6}{9\cdot 8} = \frac{2^3}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}c_0$$

$$c_{10} = \frac{2c_7}{10\cdot 9} = \frac{2^3}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}c_1$$

$$c_{11} = \frac{2c_8}{11\cdot 10} = 0 \qquad \text{, and so on.}$$

It is obvious that both c_0 and c_1 are arbitrary. Now

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + c_8x^8 + c_9x^9 + c_{10}x^{10} + c_{11}x^{11} + \cdots$$

$$y = c_0 + c_1x + 0 + \frac{2}{3\cdot 2}c_0x^3 + \frac{2}{4\cdot 3}c_1x^4 + 0 + \frac{2^2}{6\cdot 5\cdot 3\cdot 2}c_0x^6 + \frac{2^2}{7\cdot 6\cdot 4\cdot 3}c_1x^7 + 0$$
$$+ \frac{2^3}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}c_0x^9 + \frac{2^3}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}c_1x^{10} + 0 + \cdots$$

$$y = c_0[1 + \frac{2}{3\cdot 2}x^3 + \frac{2^2}{6\cdot 5\cdot 3\cdot 2}x^6 + \frac{2^3}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}x^9 + \cdots]$$
$$+ c_1[x + \frac{2}{4\cdot 3}x^4 + \frac{2^2}{7\cdot 6\cdot 4\cdot 3}x^7 + \frac{2^3}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}x^{10} + \cdots].$$

Example

Solve $(x^2+1)y''+xy'-y=0.$

Solution

Since the singular points are $x=\pm i$, $x=0$ is the ordinary point, a power series will converge at least for $|x|<1$. The assumption $y=\sum_{n=0}^{\infty} c_n x^n$ leads to

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}+x\sum_{n=1}^{\infty} nc_n x^{n-1}-\sum_{n=0}^{\infty} c_n x^n$$
$$=\sum_{n=2}^{\infty} n(n-1)c_n x^n+\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}+\sum_{n=1}^{\infty} nc_n x^n-\sum_{n=0}^{\infty} c_n x^n$$

$$=2c_2x^0-c_0x^0+6c_3x+c_1x-c_1x+\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{\boxed{k=n}}+\underbrace{\sum_{n=4}^{\infty} n(n-1)c_n x^{n-2}}_{\boxed{k=n-2}}+\underbrace{\sum_{n=2}^{\infty} nc_n x^n}_{\boxed{k=n}}-\underbrace{\sum_{n=2}^{\infty} c_n x^n}_{\boxed{k=n}}$$

$$=2c_2-c_0+6c_3x+\sum_{k=2}^{\infty}[k(k-1)c_k+(k+2)(k+1)c_{k+2}+kc_k-c_k]x^k=0$$

or
$$2c_2-c_0+6c_3x+\sum_{k=2}^{\infty}[(k+1)(k-1)c_k+(k+2)(k+1)c_{k+2}]x^k=0.$$

Thus
$$2c_2-c_0=0$$
$$c_3=0$$
$$(k+1)(k-1)c_k+(k+2)(k+1)c_{k+2}=0$$

This implies

$$c_2=\frac{1}{2}c_0$$
$$c_3=0$$
$$c_{k+2}=\frac{-(k-1)}{(k+2)}c_k,\quad k=2,3,\cdots$$

Iteration of the last formula gives

$$c_4 = -\frac{1}{4}c_2 = -\frac{1}{2\cdot 4}c_0 = -\frac{1}{2^2 2!}c_0$$

$$c_5 = -\frac{2}{5}c_3 = 0$$

$$c_6 = -\frac{3}{6}c_4 = \frac{3}{2\cdot 4\cdot 6}c_0 = \frac{1\cdot 3}{2^3 3!}c_0$$

$$c_7 = -\frac{4}{7}c_5 = 0$$

$$c_8 = -\frac{5}{8}c_6 = -\frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 8}c_0 = -\frac{1\cdot 3\cdot 5}{2^4 4!}c_0$$

$$c_9 = -\frac{6}{9}c_7 = 0$$

$$c_{10} = -\frac{7}{10}c_8 = -\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}c_0 = \frac{1\cdot 3\cdot 5\cdot 7}{2^5 5!}c_0 \text{ and so on.}$$

Therefore

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + \cdots$$

$$y = c_1 x + c_0[1 + \frac{1}{2}x^2 - \frac{1}{2^2 2!}x^4 + \frac{1\cdot 3}{2^3 3!}x^6 - \frac{1\cdot 3\cdot 5}{2^4 4!}x^8 + \frac{1\cdot 3\cdot 5\cdot 7}{2^5 5!}x^{10} - \cdots]$$

The solutions are

$$y_1(x) = c_0[1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n\, n!}x^{2n}], \quad |x| < 1$$

$$y_2(x) = c_1 x.$$

Example

If we seek a solution $y = \sum_{n=0}^{\infty} c_n x^n$ for the equation

$$y'' - (1+x)y = 0,$$

we obtain $c_2 = c_0/2$ and the three-term recurrence relation

$$c_{k+2} = \frac{c_k + c_{k-1}}{(k+1)(k+2)}, \quad k = 1,2,3,\cdots$$

To simplify the iteration we can first choose $c_0 \neq 0, c_1 = 0$; this yields one solution. The other solution follows from next choosing $c_0 = 0, c_1 \neq 0$. With the first assumption we find

$$c_2 = \frac{1}{2}c_0$$
$$c_3 = \frac{c_1 + c_0}{2 \cdot 3} = \frac{c_0}{2 \cdot 3} = \frac{1}{6}c_0$$
$$c_4 = \frac{c_2 + c_1}{3 \cdot 4} = \frac{c_0}{2 \cdot 3 \cdot 4} = \frac{1}{24}c_0$$
$$c_5 = \frac{c_3 + c_2}{4 \cdot 5} = \frac{c_0}{4 \cdot 5}[\frac{1}{2 \cdot 3} + \frac{1}{2}] = \frac{1}{30}c_0 \text{ and so on.}$$

Thus one solution is

$$y_1(x) = c_0[1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \cdots].$$

Similarly if we choose $c_0 = 0$, then

$$c_2 = 0$$
$$c_3 = \frac{c_1 + c_0}{2 \cdot 3} = \frac{c_1}{2 \cdot 3} = \frac{1}{6}c_1$$
$$c_4 = \frac{c_2 + c_1}{3 \cdot 4} = \frac{c_1}{3 \cdot 4} = \frac{1}{12}c_1$$
$$c_5 = \frac{c_3 + c_2}{4 \cdot 5} = \frac{c_1}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{1}{120}c_1 \text{ and so on.}$$

Hence another solution is

$$y_2(x) = c_1[x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \cdots].$$

Each series converges for all finite values of x.

Non-polynomial Coefficients

The next example illustrates how to find a power series solution about an ordinary point of a differential equation when its coefficients are not polynomials. In this example we see an application of multiplication of two power series that we discussed earlier.

Example

Solve $y'' + (\cos x)y = 0$

Solution:

The equation has no singular point.

Since $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, it is seen that $x = 0$ is an ordinary point.

Thus the assumption $y = \sum_{n=0}^{\infty} c_n x^n$ leads to

$$y'' + (\cos x)y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots)\sum_{n=0}^{\infty} c_n x^n$$

$$= (2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots) + (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots)(c_0 + c_1 x + c_2 x^2 + \cdots)$$

$$= 2c_2 + c_0 + (6c_3 + c_1)x + (12c_4 + c_2 - \frac{1}{2}c_0)x^2 + (20c_5 + c_3 - \frac{1}{2}c_1)x^3 + \cdots$$

If the last line be identically zero, we must have

$$2c_2 + c_0 = 0 \Rightarrow c_2 = -\frac{c_0}{2}$$

$$6c_3 + c_1 = 0 \Rightarrow c_3 = -\frac{c_1}{6}$$

$$12c_4 + c_2 - \frac{1}{2}c_0 = 0 \Rightarrow c_4 = \frac{c_0}{12}$$

$$20c_5 + c_3 - \frac{1}{2}c_1 = 0 \Rightarrow c_5 = \frac{c_1}{30}$$ and so on. c_0 and c_1 are arbitrary.

Now

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

or $$y = c_0 + c_1 x - \frac{c_0}{2}x^2 - \frac{c_1}{6}x^3 + \frac{c_0}{12}x^4 + \frac{c_1}{30}x^5 - \cdots$$

$$y = c_0(1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \cdots) + c_1(x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \cdots)$$

$y_1(x) = c_0[1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \cdots]$ and $y_2(x) = c_1[x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \cdots]$

Since the differential equation has no singular point, both series converge for all finite values of x.

Exercise

In each of the following problems find two linearly independent power series solutions about the ordinary point $x = 0$.

1. $y'' + x^2 y = 0$
2. $y'' - xy' + 2y = 0$
3. $y'' + 2xy' + 2y = 0$
4. $(x+2)y'' + xy' - y = 0$
5. $(x^2 + 2)y'' - 6y = 0$

Lecture 17
Solution about Singular Points

If $x = x_0$ is singular point, it is not always possible to find a solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n \text{ for the equation } a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

However, we may be able to find a solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$, where r is constant to be determined.

To define regular/irregular singular points, we put the given equation into the standard form

$$y'' + P(x)y' + Q(x)y = 0$$

Definition: Regular and Irregular Singular Points
A Singular point $x = x_0$ of the given equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ is said to be a ***regular singular point*** if both $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at x_0. A singular point that is not regular is said to be an ***irregular singular point*** of the equation.

Polynomial Coefficients
If the coefficients in the given differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ are polynomials with no common factors, above definition is equivalent to the following:

*Let $a_2(x_0) = 0$ Form $P(x)$ and $Q(x)$ by reducing $\dfrac{a_1(x)}{a_2(x)}$ and $\dfrac{a_0(x)}{a_2(x)}$ to lowest terms, respectively. If the factor $(x - x_0)$ appears at most to the first powers in the denominator of $P(x)$ and at most to the second power in the denominator of $Q(x)$, then $x = x_0$ is a **regular singular point**.*

Example 1

$x = \pm 2$ are singular points of the equation

$$(x^2 - 4)^2 y'' + (x - 2)y' + y = 0$$

Dividing the equation by $(x^2 - 4)^2 = (x - 2)^2 (x + 2)^2$, we find that

$$P(x) = \frac{1}{(x-2)(x+2)^2} \text{ and } Q(x) = \frac{1}{(x-2)^2(x+2)^2}$$

1. $x = 2$ is a regular singular point because power of $x-2$ in $P(x)$ is 1 and in $Q(x)$ is 2.
2. $x = -2$ is an irregular singular point because power of $x+2$ in $P(x)$ is 2.

   The 1st condition is violated.

Example 2

Both $x = 0$ and $x = -1$ are singular points of the differential equation

$$x^2(x+1)^2 y'' + (x^2-1)y' + 2y = 0$$

Because $x^2(x+1)^2 = 0$ or $x = 0$,-1

Now write the equation in the form

$$y'' + \frac{x^2-1}{x^2(x+1)^2} y' + \frac{2}{x^2(x+1)^2} y = 0$$

or $$y'' + \frac{x-1}{x^2(x+1)} y' + \frac{2}{x^2(x+1)^2} y = 0$$

So $P(x) = \frac{x-1}{x^2(x+1)}$ and $Q(x) = \frac{2}{x^2(x+1)^2}$

Shows that $x = 0$ is a irregular singular point since $(x-0)$ appears to the second powers in the denominator of $P(x)$.

Note, however, $x = -1$ is a regular singular point.

Example 3

a) $x = 1$ and $x = -1$ are singular points of the differential equation

$$(1-x^2)y'' + -2xy' + 30y = 0$$

Because $1 - x^2 = 0$ or $x = \pm 1$.

Now write the equation in the form

$$y'' - \frac{2x}{(1-x^2)} y' + \frac{30}{1-x^2} y = 0$$

or $$y'' - \frac{2x}{(1-x)(1+x)} y' + \frac{30}{(1-x)(1+x)} y = 0$$

$$P(x) = \frac{-2x}{(1-x)(1+x)} \quad \text{and} \quad Q(x) = \frac{30}{(1-x\ )(1+x)}$$

Clearly $x = \pm 1$ are regular singular points.

(b) $x = 0$ is an irregular singular points of the differential equation

$$x^3 y'' - 2xy' + 5y = 0$$

or $y'' - \frac{2}{x^2} y' + \frac{5}{x^3} y = 0$ giving $Q(x) = \frac{5}{x^3}$.

(c) $x = 0$ is a regular singular points of the differential equation

$$x\ \ y'' - 2xy' + 5y = 0$$

Because the equation can be written as $y'' - 2y' + \frac{5}{x} y = 0$ giving $P(x) = -2$ and

$$Q(x) = \frac{5}{x}$$

In part (c) of Example 3 we noticed that ($x-0$) and $(x-0)^2$ do not even appear in the denominators of $P(x)$ and $Q(x)$ respectively. Remember, these factors can appear at most in this fashion. For a singular point $x = x_0$, any nonnegative power of $(x - x_0)$ less than one (namely, zero) and nonnegative power less than two (namely, zero and one) in the denominators of $P(x)$ and $Q(x)$, respectively, imply x_0 is a ***regular singular point.***

Please note that the singular points can also be complex numbers.
For example, $x = \pm\, 3i$ are regular singular points of the equation

$$(x^2 + 9)y'' + -3xy' + (1-x)y = 0$$

Because the equation can be written as

$$y'' - \frac{3x}{x^2+9} y' + \frac{1-x}{x^2+9} y = 0.$$

$$\therefore\ P(x) = \frac{-3x}{(x-3i)(x+3i)}. \quad Q(x) = \frac{1-x}{(x-3i\ \ )(x+3i)}.$$

Method of Frobenius

To solve a differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ about a regular singular point we employ the Frobenius' Theorem.

Frobenius' Theorem

If $x = x_0$ is a regular singular point of equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, then there exists at least one series solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

where the number r is a constant that must be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

Note that the solutions of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ are not guaranteed.

Method of Frobenius

1. Identify regular singular point x_0,
2. Substitute $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ in the given differential equation,
3. Determine the unknown exponent r and the coefficients c_n.
4. For simplicity assume that $x_0 = 0$.

Example 4

As $x = 0$ is regular singular points of the differential equation

$$3xy'' + y' - y = 0.$$

We try a solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$.

Therefore $\quad y' = \sum_{n=0}^{\infty} (n + r) c_n x^{n+r-1}$.

And $\quad y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) c_n x^{n+r-2}$.

$$3xy'' + y' - y = 3\sum_{n=0}^{\infty} (n + r)(n + r - 1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n + r) c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r}.$$

$$= \sum_{n=0}^{\infty} (n + r)(3n + 3r - 2) c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r}.$$

$$= x^r \left[r(3r - 2) c_0 x^{-1} + \sum_{n=1}^{\infty} (n + r)(3n + 3r - 2) c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n \right]$$

$$k = n - 1 \qquad\qquad k = n$$

$$= x^r \left[r(3r - 2) c_0 x^{-1} + \sum_{k=0}^{\infty} [(k + r + 1)(3k + 3r + 1) c_{k+1} - c_k] x^k \right] = 0$$

which implies $r(3r - 2) c_0 = 0$

$(k+r+1)(3k+3r+1)c_{k+1} - c_k = 0$, $k = 0,1,2,...$

Since nothing is gained by taking $c_0 = 0$, we must then have

$r(3r-2) = 0$ [called the indicial equation and its roots $r = \frac{2}{3}, 0$ are called indicial roots or exponents of the singularity.]

and $$c_{k+1} = \frac{c_k}{(k+r+1)(3k+3r+1)}, k = 0,1,2,...$$

Substitute $r_1 = \frac{2}{3}$ and $r_2 = 0$ in the above equation and these values will give two different recurrence relations:

For $r_1 = \frac{2}{3}$, $$c_{k+1} = \frac{c_k}{(3k+5)(k+1)}, \quad k = 0,1,2,... \quad (1)$$

For $r_2 = 0$ $$c_{k+1} = \frac{c_k}{(k+1)(3k+1)}, \quad k = 0,1,2,... \quad (2)$$

Iteration of (1) gives

$$c_1 = \frac{c_0}{5.1}$$

$$c_2 = \frac{c_1}{8.2} = \frac{c_0}{2!5.8}$$

$$c_3 = \frac{c_2}{11.3} = \frac{c_0}{3!5.8.11}$$

$$c_4 = \frac{c_3}{14.4} = \frac{c_0}{4!5.8.11.14}$$

.

In general $$c_n = \frac{c_0}{n!5.8.11.14...(3n+2)}, \quad n = 1,2,...$$

Iteration of (2) gives

$$c_1 = \frac{c_0}{1.1}$$

$$c_2 = \frac{c_1}{2.4} = \frac{c_0}{2!1.4}$$

$$c_3 = \frac{c_2}{3.7} = \frac{c_0}{3!1.4.7}$$

$$c_4 = \frac{c_3}{4.10} = \frac{c_0}{4!1.4.7.10}$$

In general $$c_n = \frac{c_0}{n!1.4.7...(3n-2)}, \quad n = 1,2,...$$

Thus we obtain two series solutions

$$y_1 = c_0 x^{\frac{2}{3}}\left[1+\sum_{n=1}^{\infty}\frac{1}{n!5.8.11.14...(3n+2)}x^n\right] \qquad (3)$$

$$y_2 = c_0 x^0\left[1+\sum_{n=1}^{\infty}\frac{1}{n!1.4.7...(3n-2)}x^n\right]. \qquad (4)$$

By the ratio test it can be demonstrated that both (3) and (4) converge for all finite values of x. Also it should be clear from the form of (3) and (4) that neither series is a constant multiple of the other and therefore, $y_1(x)$ and $y_2(x)$ are linearly independent on the x-axis. Hence by the superposition principle

$$y = C_1 y_1(x) + C_2 y_2(x) = C_1\left[x^{\frac{2}{3}}+\sum_{n=1}^{\infty}\frac{1}{n!5.8.11.14...(3n+2)}x^{n+\frac{2}{3}}\right]$$

$$+C_2\left[1+\sum_{n=1}^{\infty}\frac{1}{n!1.4.7...(3n-2)}x^n\right], \quad |x|<\infty$$

is another solution of the differential equation. On any interval not containing the origin, this combination represents the general solution of the differential equation

Remark: The method of Frobenius may not always provide 2 solutions.

Example 5

The differential equation

$xy'' + 3y' - y = 0$ has regular singular point at $x = 0$

We try a solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$

Therefore $y' = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$.

so that

$$xy'' + 3y' - y = x^r\left[r(r+2)c_0 x^{-1} + \sum_{k=0}^{\infty}[(k+r+1)(k+r+3)c_{k+1} - c_k]x^k\right] = 0$$

so that the indicial equation and exponent are $r(r+2) = 0$ and $r_1 = 0$, $r_2 = -2$, respectively.

Since $(k+r+1)(k+r+3)c_{k+1} - c_k = 0$, $\quad k = 0,1,2,... \qquad (1)$

it follows that when $r_1 = 0$,

$$c_{k+1} = \frac{c_k}{(k+1)(k+3)},$$

$$c_1 = \frac{c_0}{1.3}$$

$$c_2 = \frac{c_1}{2.4} = \frac{2c_0}{2!4!}$$

$$c_3 = \frac{c_2}{3.5} = \frac{2c_0}{3!5!}$$

$$c_4 = \frac{c_3}{4.6} = \frac{2c_0}{4!6!}$$

$$\vdots$$

$$c_n = \frac{2c_0}{n!(n+2)!}, \quad n = 1, 2, \ldots$$

Thus one series solution is

$$y_1 = c_0 x^0 \left[1 + \sum_{n=1}^{\infty} \frac{2}{n!(n+2)!} x^n \right] = c_0 \sum_{n=0}^{\infty} \frac{2}{n!(n+2)!} x^n, |x| < \infty.$$

Now when $r_2 = -2$, (1) becomes

$$(k-1)(k+1)c_{k+1} - c_k = 0 \qquad (2)$$

but note here that we do not divide by $(k-1)(k+1)$ immediately since this term is zero for $k = 1$. However, we use the recurrence relation (2) for the cases $k = 0$ and $k = 1$:

$$-1.1c_1 - c_0 = 0 \qquad \text{and } 0.2c_2 - c_1 = 0$$

The latter equation implies that $c_1 = 0$ and so the former equation implies that $c_0 = 0$. Continuing, we find

$$c_{k+1} = \frac{c_k}{(k-1)(k+1)} \qquad k = 2, 3, \ldots$$

$$c_3 = \frac{c_2}{1.3}$$

$$c_4 = \frac{c_3}{2.4} = \frac{2c_2}{2!.4!}$$

$$c_5 = \frac{c_4}{3.5} = \frac{2c_2}{3!.5!}, \ldots$$

$$\vdots$$

In general $\quad c_n = \dfrac{2c_2}{(n-2)!n!}, \quad n = 3, 4, 5, \ldots$

Thus $$y_2 = c_2 x^{-2}\left[x^2 + \sum_{n=3}^{\infty}\frac{2}{(n-2)!n!}x^n\right]. \qquad (3)$$

However, close inspection of (3) reveals that y_2 is simply constant multiple of y_1. To see this, let $k = n-2$ in (3). We conclude that the method of Frobenius gives only one series solution of the given differential equation.

Cases of Indicial Roots

When using the method of Frobenius, we usually distinguish three cases corresponding to the nature of the indicial roots. For the sake of discussion let us suppose that r_1 and r_2 are the real solutions of the indicial equation and that, when appropriate, r_1 denotes the largest root.

Case I: Roots not Differing by an Integer

If r_1 and r_2 are distinct and do not differ by an integer, then their exist two linearly independent solutions of the differential equation of the form

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1} \,.\, . c_0 \neq 0, \text{ and } y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}, \quad b_0 \neq 0.$$

Example 6

Solve $$2xy'' + (1+x)y' + y = 0.$$

Solution

If $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, then

$$2xy'' + (1+x)y' + y = 2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty}c_n x^{n+r}$$

$$= \sum_{n=0}^{\infty}(n+r)(2n+2r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^{n+r}$$

$$= x^r\left[r(2r-1)c_0 x^{-1} + \sum_{n=1}^{\infty}(n+r)(2n+2r-1)c_n x^{n-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^n\right]$$

$$n = k+1 \qquad\qquad k = n$$

$$= x^r\left[r(2r-1)c_0x^{-1} + \sum_{k=0}^{\infty}[(k+r+1)(2k+2r+1)c_{k+1} + (k+r+1)c_k]x^k\right] = 0$$

which implies $r(2r-1)=0$

$$(k+r+1)(2k+2r+1)c_{k+1} + (k+r+1)c_k = 0,\ k = 0,1,2,... \qquad (1)$$

For $r_1 = \frac{1}{2}$, we can divide by $k+\frac{3}{2}$ in the above equation to obtain

$$c_{k+1} = \frac{-c_k}{2(k+1)},$$

$$c_1 = \frac{-c_0}{2.1}$$

$$c_2 = \frac{-c_1}{2.2} = \frac{c_0}{2^2.2!}$$

$$c_3 = \frac{-c_2}{2.3} = \frac{-c_0}{2^3.3!}$$

$$\vdots$$

In general $c_n = \dfrac{(-1)^n c_0}{2^n n!},\quad n = 1,2,3,...$

Thus we have

$$y_1 = c_0 x^{\frac{1}{2}}\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n!}x^n\right],$$ which converges for $x \geq 0$.

As given, the series is not meaningful for $x < 0$ because of the presence of $x^{\frac{1}{2}}$.

Now for $r_2 = 0$, (1) becomes

$$c_{k+1} = \frac{-c_k}{2k+1}$$

$$c_1 = \frac{-c_0}{1}$$

$$c_2 = \frac{-c_1}{3} = \frac{c_0}{1.3}$$

$$c_3 = \frac{-c_2}{5} = \frac{-c_0}{1.3.5}$$

$$c_4 = \frac{-c_3}{7} = \frac{c_0}{1.3.5.7}$$

$$\vdots$$

In general $c_n = \dfrac{(-1)^n c_0}{1.3.5.7...(2n-1)},\quad n = 1,2,3,...$

Thus second solution is

$$y_2 = c_0\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n}{1.3.5.7...(2n-1)}x^n\right]. \quad |x|<\infty.$$

On the interval $(0,\infty)$, the general solution is

$$y = C_1 y_1(x) + C_2 y_2(x).$$

Lecture 18
Solutions about Singular Points

Method of Frobenius-Cases II and III

When the roots of the indicial equation differ by a positive integer, we may or may not be able to find two solutions of

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (1)$$

having form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r} \qquad (2)$$

If not, then one solution corresponding to the smaller root contains a logarithmic term. When the exponents are equal, a second solution always contains a logarithm. This latter situation is similar to the solution of the Cauchy-Euler differencial equation when the roots of the auxiliary equation are equal. We have the next two cases.

Case II: Roots Differing by a Positive Integer

If $r_1 - r_2 = N$, where N is a positive integer, then there exist two linearly independent solutions of the form

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, c_0 \neq 0 \qquad (3a)$$

$$y_2 = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, b_0 \neq 0 \qquad (3b)$$

Where C is a constant that could be zero.

Case III: Equal Indicial Roots:

If $r_1 = r_2$, there always exist two linearly independent solutions of (1) of the form

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, c_0 \neq 0 \qquad (4a)$$

$$y_2 = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^{n+r_1} \qquad \because r_1 = r_2 \qquad (4b)$$

Example 7: Solve $xy''+(x-6)y'-3y=0$ (1)

Solution: The assumption $y=\sum_{n=0}^{\infty}c_n x^{n+r}$ leads to

$$xy''+(x-6)y'-3y$$

$$=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1}-6\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}-3\sum_{n=0}^{\infty}c_n x^{n+r}$$

$$=x^r\left[r(r-7)c_0x^{-1}+\sum_{n=1}^{\infty}(n+r)(n+r-7)c_nx^{n-1}+\sum_{n=0}^{\infty}(n+r-3)c_nx^n\right]$$

$$x^r\left[r(r-7)c_0x^{-1}+\sum_{k=0}^{\infty}\left[(k+r+1)(k+r-6)c_{k+1}+(k+r-3)c_k\right]x^k\right]=0$$

Thus $\text{r(r - 7)}=0$ so that $r_1=7, r_2=0, r_1-r_2=7, and$

$$(k+r+1)(k+r-6)\ c_{k+1}+(k+r-3)c_k=0, \qquad k=0,1,2,3,... \qquad (2)$$

For smaller root $r_2=0,(2) becomes$

$$(k+1)(k-6)c_{k+1}+(k-3)c_k=0 \qquad (3)$$

$recurrence\ relation\ becomes$

$$c_{k+1}=-\frac{(k-3)}{(k+1)(k-6)}c_k$$

Since k-6=0, when, k=6, we do not divide by this term until k>6.we find

$$1.(-6)c_1+(-3)c_0=0$$
$$2.(-5)c_2+(-2)c_1=0$$
$$3.(-4)c_3+(-1).c_2=0$$
$$4.(\text{-}3)c_4+0.c_3=0$$
$$5.\ (\text{-}2)c_5+1.c_4=0$$
$$6.\ (\text{-}1)c_6+2.c_5=0$$
$$7.0.c_7+3.c_6=0$$

This implies that

$c_4=c_5=c_6=0,$ But c_0 and c_7 can be chosen arbitrarily.

Hence $c_1=-\frac{1}{2}c_0$

$$c_2 = -\frac{1}{5}c_1 \quad = \frac{1}{10}c_0$$

$$c_3 = -\frac{1}{12}c_2 = -\frac{1}{120}c_0 \qquad (4)$$

and for k ≥ 7

$$c_{k+1} = \frac{-(k-3)}{(k+1)(k-6)}c_k$$

$$c_8 = \frac{-4}{8.1}c_7$$

$$c_9 = -\frac{5}{9.2}c_8 = \frac{4.5}{2!8.9}c_7$$

$$c_{10} = \frac{-6}{10.3}c_9 = \frac{-4.5.6}{3!8.9.10}c_7$$

.

.

.

$$c_n = \frac{(-1)^{n+1}4\cdot5\cdot6\cdots(n-4)}{(n-7)!8\cdot9\cdot10\cdots(n)}c_7, \qquad n = 8,9,10,\cdots \qquad (5)$$

If we choose $c_7 = 0 and c_0 \neq 0$ It follows that we obtain the polynomial solution

$$y_1 = \qquad c_0[1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3],$$

But when $c_7 \neq = 0 and c_0 = 0$, It follows that a second, though infinite series solution is

$$y_2 = c_7[x^7 + \sum_{n=8}^{\infty}\frac{(-1)^{n+1}4\cdot5\cdot6\cdots(n-4)}{(n-7)!\,8\cdot9\cdot10\cdots n}x^n]$$

$$= \quad c_7[x^7 + \sum_{k=1}^{\infty}\frac{(-1)^k 4\cdot5\cdot6\ \cdots\ (k+3)}{k!8\cdot9\cdot10\ \cdots\ (k+7)}x^{k+3}], \qquad |x| < \infty \qquad (6)$$

Finally the general solution of equation (1) on the interval $(0,\infty)$ is

Y $= c_1y_1(x) + c_2y_2(x)$

$$= c_1\,[1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3] + c_2[x^7 + \sum_{n=1}^{\infty}\frac{(-1)^k 4\cdot5\cdot6\cdots(k+3)}{k!\cdot8\cdot9\cdot10\cdots(k+7)}x^{k+7}]$$

It is interesting to observe that in example 9 the larger root r_1=7 were not used. Had we done so, we would have obtained a series solution of the form*

$$y = \sum_{n=0}^{\infty}c_n x^{n+7} \qquad (7)$$

Where c_n are given by equation (2) with r_1=7

$$c_{k+1} = \frac{-(k+4)}{(k+8)(k+1)} c_k, \qquad k = 0,1,2...$$

Iteration of this latter recurrence relation then would yield only one solution, namely the solution given by (6) with c_0 playing the role of c_7)

When the roots of indicial equation differ by a positive integer, the second solution may contain a logarithm.

On the other hand if we fail to find second series type solution, we can always use the fact that

$$y_2 = y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1^{\,2}(x)} dx \qquad (8)$$

is a solution of the equation $y'' + P(x)y' + Q(x)y = 0$,whenever y_1 is a known solution.

Note: In case 2 it is always a good idea to work with smaller roots first.

Example8:

Find the general solution of $xy'' + 3y' - y = 0$

Solution The method of frobenius provide only one solution to this equation, namely,

$$y_1 = \sum_{n=0}^{\infty} \frac{2}{n!(n+2)!} x^n = 1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^3 + \cdots \qquad (9)$$

From (8) we obtain a second solution

$$y_2 = y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1^{\,2}(x)} dx = y_1(x) \int \frac{dx}{x^3[1+\frac{1}{3}x+\frac{1}{24}x^2+\frac{1}{360}x^3+\cdots]^2}$$

$$= y_1(x) \int \frac{dx}{x^3[1+\frac{2}{3}x+\frac{7}{36}x^2+\frac{1}{30}x^3+\cdots]}$$

$$= y_1(x) \int \frac{1}{x^3}[1-\frac{2}{3}x+\frac{1}{4}x^2-\frac{19}{270}x^3+\cdots]dx$$

$$= y_1(x)\left[-\frac{1}{2x^2}+\frac{2}{3x}+\frac{1}{4}\ln x-\frac{19}{270}x+...\right]$$

$$= \frac{1}{4}y_1(x)\ln x + y_1(x)\left[-\frac{1}{2x^2}+\frac{2}{3x}-\frac{19}{270}x+...\right] \qquad (*)$$

$$\therefore y = c_1 y_1(x) + c_2\left[\frac{1}{4}y_1(x)\ln x + y_1(x)\left(-\frac{1}{2x^2}+\frac{2}{3x}-\frac{19}{270}x+...\right)\right] \qquad (**)$$

Example 9:
Find the general solution of

$xy'' + 3y' - y = 0$

Solution :

$$y_2 = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n-2} \qquad (10)$$

$$y_1 = \sum_{n=0}^{\infty} \frac{2}{n!(n+2)!} x^n \qquad (11)$$

differentiate (10) gives

$$y_2{}' = \frac{y_1}{x} + y_1{}' \ln x + \sum_{n=0}^{\infty} (n-2) b_n x^{n-3}$$

$$y_2{}'' = -\frac{y_1}{x^2} + \frac{2y_1{}'}{x} + y_1{}'' \ln x + \sum_{n=0}^{\infty} (n-2)(n-3) b_n x^{n-4}$$

so that

$$xy_2'' + 3y_2' - y_2 = \ln x \left[xy_1{}'' + 3y_1{}' - y_1 \right] + 2y_1{}' + \frac{2y_1}{x} + \sum_{n=0}^{\infty} (n-2)(n-3) b_n x^{n-3}$$
$$+3\sum_{n=0}^{\infty} (n-2) b_n x^{n-3} - \sum_{n=0}^{\infty} b_n x^{n-2}$$
$$= 2y_1{}' + \frac{2y_1}{x} + \sum_{n=0}^{\infty} (n-2) n b_n x^{n-3} - \sum_{n=0}^{\infty} b_n x^{n-2} \qquad (12)$$

where we have combined the 1st two summations and used the fact that

$xy_1'' + 3y_1' - y_1 = 0$

Differentiate (11) we can write (12) as

$$\sum_{n=0}^{\infty} \frac{4n}{n!(n+2)!} x^{n-1} + \sum_{n=0}^{\infty} \frac{4}{n!(n+2)!} x^{n-1} + \sum_{n=0}^{\infty} (n-2) n b_n x^{n-3} - \sum_{n=0}^{\infty} b_n x^{n-2}$$

$$= 0(-2) b_0 x^{-3} + (-b_0 - b_1) x^{-2} + \sum_{n=0}^{\infty} \frac{4(n+1)}{n!(n+2)!} x^{n-1} + \sum_{n=2}^{\infty} (n-2) n b_n x^{n-3} - \sum_{n=1}^{\infty} b_n x^{n-2}$$

$$-(b_0 + b_1) x^{-2} + \sum_{k=0}^{\infty} \left[\frac{4(k+1)}{k!(k+2)!} + k(k+2) b_{k+2} - b_{k+1} \right] x^{k-1}. \qquad \textbf{(13)}$$

Setting (13) equal to zero then gives $b_1 = -b_0$ **and**

$$\frac{4(k+1)}{k!(k+1)!} + k(k+2) b_{k+2} - b_{k+1} = 0, \qquad \textbf{For k=0, 1, 2, …} \qquad \textbf{(14)}$$

When k=0 in equation (14) we have $2+0\cdot 2b_2 - b_1 = 0$ so that but
$b_1 = 2, b_0 = -2,$ but b_2 is arbitrary
Rewriting equation (14) as

$$b_{k+2} = \frac{b_{k+1}}{k(k+2)} - \frac{4(k+1)}{k!(k+2)!k(k+2)} \tag{15}$$

and evaluating for k=1,2,... gives

$$b_3 = \frac{b_2}{3} - \frac{4}{9}$$

$$b_4 = \frac{1}{8}b_3 - \frac{1}{32} = \frac{1}{24}b_2 - \frac{25}{288}$$

and so on. Thus we can finally write

$$y_2 = y_1 \ln x + b_0 x^{-2} + b_1 x^{-1} + b_2 + b_3 x + \cdots$$

$$= y_1 \ln x - 2x^{-2} + 2x^{-1} + b_2 + \left(\frac{b_2}{3} - \frac{4}{9}\right)x + \cdots \tag{16}$$

Where b_2 is arbitrary.

Equivalent Solution

At this point you may be wondering whether (*) and (16) are really equivalent. If we choose $c_2 = 4$ in equation (**), then

$$y_2 = y_1 \ln x + \left(-\frac{2}{x^2} + \frac{8}{3x} - \frac{38}{135}x + \cdots\right)$$

$$y_2 = y_1 \ln x + \left(1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^3 + \cdots\right)\left(-\frac{2}{x^2} + \frac{8}{3x} - \frac{38}{135}x + \cdots\right) \tag{17}$$

$$= y_1 \ln x - 2x^{-2} + 2x^{-1} + \frac{29}{36} - \frac{19}{108}x + \ldots$$

Which is precisely obtained what we obtained from (16). If b_2 is chosen as $\frac{29}{36}$

The next example illustrates the case when the indicial roots are equal.

Example :10

Find the general solution of $xy'' + y' - 4y = 0$ (18)

Solution :*The* assumption $y=\sum_{n=0}^{\infty} c_n x^{n+r}$ leads to

$$xy'' + y' - 4y = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} - 4\sum_{n=0}^{\infty} c_n x^{n+r}$$

$$=\sum_{n=0}^{\infty} (n+r)^2 c_n x^{n+r-1} - 4\sum_{n=0}^{\infty} c_n x^{n+r}$$

$$= x^r \left[r^2 c_0 x^{-1} + \sum_{n=1}^{\infty} (n+r)^2 c_n x^{n-1} - 4\sum_{n=0}^{\infty} c_n x^n \right]$$

$$= x^r \left[r^2 c_0 x^{-1} + \sum_{k=0}^{\infty} (k+r+1)^2 c_{k+1} - 4c_k \right] x^k = 0$$

Therefore $r^2=0$, and so the indicial roots are equal: $r_1 = r_2 = 0$. Moreover we have

$$(k+r+1)^2 c_{k+1} - 4c_k = 0, k=0,1,2,\ldots \qquad (19)$$

Clearly the roots $r_1 = 0$ will yield one solution corresponding to the coefficients defined by the iteration of

$$c_{k+1} = \frac{4c_k}{(k+1)^2} \qquad k=0,1,2,\ldots$$

The result is

$$y_1 = c_0 \sum_{n=0}^{\infty} \frac{4^n}{(n!)^2} x^n, |x| < \infty \qquad (20)$$

$$y_2 = y_1(x)\int \frac{e^{-\int \frac{1}{x})dx}}{y_1^2(x)} dx = y_1(x)\int \frac{dx}{x\left[1 + 4x + 4x^2 + \frac{16}{9}x^3 + \cdots\right]^2}$$

$$= y_1(x)\int \frac{1}{x}\left[1 - 8x + 40x^2 - \frac{1472}{9}x^3 + \cdots\right]dx$$

$$= y_1(x)\int \left[\frac{1}{x} - 8 + 40x - \frac{1472}{9}x^2 + \ldots\right]dx$$

$$= y_1(x)\left[\ln x - 8x + 20x^2 - \frac{1472}{27}x^3 + \cdots\right]$$

Thus on the interval $(0, \infty)$ the general solution of (18) is

$$y = c_1 y_1(x) + c_2\left[y_1(x)\ln x + y_1(x)\left(-8x + 20x^2 - \frac{1472}{27}x^3 + ...\right)\right]$$

where $y_1(x)$ is defined by (20)

In caseII we can also determine $y_2(x)$ of example9 directly from assumption (4b)

Exercises

In problem 1-10 determine the singular points of each differential equation. Classify each the singular point as regular or irregular.

1 $x^3y'' + 4x^2y' + 3y = 0$

2 $xy'' - (x+3)^{-2}y = 0$

3 $(x^2 - 9)y'' + (x+3) + 2y = 0$

4 $y'' - \frac{1}{x}y' + \frac{1}{(x-1)^3}y = 0$

5 $(x^3 + 4x)y'' - 2xy' + 6y = 0$)

6 $x^2(x-5)^2y'' + 4xy' + (x-2)y = 0$

7 $(x^2 + x - 6)^2y'' + (x+3)y' + (x-2)y = 0$

8 $x(x^2+1)^2y'' + y = 0$

9 $x^3(x^2 - 25)(x-2)^2y'' + 3x(x-2)y' + 7(x+5)y = 0$

10 $(x^3 - 2x^2 - 3x)^2y'' + x(x+3)^2y' + (x+1)y = 0$

In problem 11-22 show that the indicial roots do not differ by an integer. Use the method of frobenius to obtain two linearly independent series solutions about the regular singular point $x_0 = 0$ Form the general solution on $(0, \infty)$

11. $2xy'' - y' + 2y = 0$

12. $2xy'' + 5y' + xy = 0$

13. $4xy'' + \frac{1}{2}y' + y = 0$

14. $2x^2y'' - xy' + (x^2+1)y = 0$

15. $3xy'' + (2-x)y' + y = 0$

16. $x^2y'' - \left(x - \frac{2}{9}\right)y' + xy = 0$

17. $2xy'' + (3 + 2x)y' + y = 0$

18. $x^2y'' + xy' + \left(x^2 - \frac{4}{9}\right)y = 0$

19. $9x^2y'' + 9x^2y' + 2y = 0$

20. $2x^2y'' + 3xy' + (2x - 1)y = 0$

21. $2x^2y'' - x(x - 1)y' - y = 0$

22. $x(x - 2)y'' - y' - 2y = 0$

In problem 23-34 show that the indicial roots differ by an integer. Use the method of frobenius to obtain two linearly independent series solutions about the regular singular point $x_0 = 0$ Form the general solution on $(0, \infty)$

23. $xy'' + 2y' - xy = 0$

24. $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$

25. $x(x - 1)y'' + 3y' - 2y = 0$

26. $y'' + \frac{3}{x}y' - 2y = 0$

27. $xy'' + (1 - x)y' - y = 0$

28. $xy'' + y = 0$

29. $xy'' + y' + y = 0$

30. $xy'' - y' + y = 0$

31. $x^2y'' + x(x - 1)y' + y = 0$

32. $xy'' + y' - 4xy = 0$

33. $x^2y'' + (x - 1)y' - 2y = 0$

34. $xy'' - y' + x^3y = 0$

Lecture 19

Bessel's Differential Equation

A second order linear differential equation of the form

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-v^2\right)y=0$$

is called Bessel's differential equation.

Solution of this equation is usually denoted by $J_v(x)$ and is known as Bessel's function. This equation occurs frequently in advanced studies in applied mathematics, physics and engineering.

Series Solution of Bessel's Differential Equation

Bessel's differential equation is

$$x^2y''+xy'+\left(x^2-v^2\right)y=0 \qquad (1)$$

If we assume that

$$y=\sum_{n=0}^{\infty}C_nx^{n+r}$$

Then

$$y'=\sum_{n=0}^{\infty}C_n(n+r)x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}C_n(n+r)(n+r-1)x^{n+r-2}$$

So that

$$x^2y''+xy'+\left(x^2-v^2\right)y=\sum_{n=0}^{\infty}C_n(n+r)(n+r-1)x^{n+r}+\sum_{n=0}^{\infty}C_n(n+r)x^{n+r}$$
$$+\sum_{n=0}^{\infty}C_nx^{n+r+2}-v^2\sum_{n=0}^{\infty}C_nx^{n+r}=0$$

$$C_o\left(r^2-v^2\right)x^r+x^r\sum_{n=1}^{\infty}C_n\left[(n+r)(n+r-1)+(n+r)-v^2\right]x^n+x^r\sum_{n=0}^{\infty}C_nx^{n+2}=0 \quad \ldots (2)$$

From (2) we see that the indicial equation is $r^2-v^2=0$, so the indicial roots are $r_1=v$, $r_2=-v$. When $r_1=v$ then (2) becomes

$$x^v\sum_{n=1}^{\infty}C_n n(n+2v)x^n + x^v\sum_{n=0}^{\infty}C_n x^{n+2} = 0$$

$$x^v\left[(1+2v)C_1x + \underbrace{\sum_{n=2}^{\infty}C_n n(n+2v)x^n}_{k=n-2} + \underbrace{\sum_{n=0}^{\infty}C_n x^{n+2}}_{k=n}\right] = 0$$

$$x^v\left[(1+2v)C_1x + \sum_{k=0}^{\infty}\left[(k+2)(k+2+2v)C_{k+2} + C_k\right]x^{k+2}\right] = 0$$

We can write

$$(1+2v)C_1 = 0$$

$$(k+2)(k+2+2v)C_{k+2} + C_k = 0$$

$$C_{k+2} = \frac{-C_k}{(k+2)(k+2+2v)} \qquad (3)$$

$$k = 0,1,2,\dots$$

The choice $C_1 = 0$ in (3) implies

$$C_1 = C_3 = C_5 = \dots = 0$$

so for $k = 0,2,4,\dots$ we find, after letting $k+2 = 2n$, $n = 1,2,3,\dots$ that

$$C_{2n} = \frac{-C_{2n-2}}{2^2 n(n+v)} \qquad (4)$$

Thus

$$\begin{aligned}
C_2 &= -\frac{C_0}{2^2\cdot 1\cdot(1+v)} \\
C_4 &= -\frac{C_2}{2^2\cdot 2\cdot(2+v)} = \frac{C_0}{2^4\cdot 1\cdot 2\cdot(1+v)(2+v)} \\
C_6 &= -\frac{C_4}{2^2\cdot 3\cdot(3+v)} = -\frac{C_0}{2^6\cdot 1\cdot 2\cdot 3\cdot(1+v)(2+v)(3+v)} \qquad (5) \\
&\dots \qquad \dots \qquad \dots \\
C_{2n} &= \frac{(-1)^n C_0}{2^{2n}\cdot n!(1+v)(2+v)\cdots(n+v)} \qquad n = 1,2,3,\dots
\end{aligned}$$

It is standard practice to choose C_0 to be a specific value namely

$$C_0 = \frac{1}{2^v \Gamma(1+v)}$$

where $\Gamma(1+v)$ the Gamma function. Also

$$\Gamma(1+\alpha) = \alpha\Gamma(\alpha).$$

Using this property, we can reduce the indicated product in the denominator of (5) to one term. For example

$$\Gamma(1+v+1) = (1+v)\Gamma(1+v)$$

$$\begin{aligned}\Gamma(1+v+2) &= (2+v)\Gamma(2+v)\\ &= (2+v)(1+v)\Gamma(1+v)\end{aligned}$$

Hence we can write (5) as

$$\begin{aligned}C_{2n} &= \frac{(-1)^n}{2^{2n+v} n!(1+v)(2+v)\cdots(n+v)\Gamma(1+v)}\\ &= \frac{(-1)^n}{2^{2n+v} n!\Gamma(1+v+n)}, \qquad n = 0,1,2,\ldots\end{aligned}$$

So the solution is

$$y = \sum_{n=0}^{\infty} C_{2n} x^{2n+v} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+v+n)} \left(\frac{x}{2}\right)^{2n+v}$$

If $v \geq 0$, the series converges at least on the interval $[0 \quad \infty)$.

Bessel's Function of the First Kind

As for $r_1 = v$, we have

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\Gamma(1+v+n)} \left(\frac{x}{2}\right)^{2n+v} \qquad (6)$$

Also for the second exponent $r_2 = -v$, we have

$$J_{-v}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\Gamma(1-v+n)} \left(\frac{x}{2}\right)^{2n-v} \qquad (7)$$

The function $J_v(x)$ and $J_{-v}(x)$ are called Bessel function of the first kind of order v and $-v$ respectively.

Now some care must be taken in writing the general solution of (1). When $v = 0$, it is clear that (6) and (7) are the same. If $v > 0$ and $r_1 - r_2 = v - (-v) = 2v$ is not a positive integer, then $J_v(x)$ and $J_{-v}(x)$ are linearly independent solutions of (1) on $(0, \infty)$ and so the general solution of the interval would be

$$y = C_1 J_v(x) + C_2 J_{-v}(x)$$

If $r_1 - r_2 = 2v$ is a positive integer, a second series solution of (1) may exists.

Example 1

Find the general solution of the equation

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \text{ on } (0, \infty)$$

Solution

The Bessel differential equation is

$$x^2 y'' + xy' + \left(x^2 - v^2\right)y = 0 \qquad (1)$$

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \qquad (2)$$

Comparing (1) and (2), we get $v^2 = \frac{1}{4}$, therefore $v = \pm\frac{1}{2}$

So general solution of (1) is $y = C_1 J_{1/2}(x) + C_2 J_{-1/2}(x)$

Example 2

Find the general solution of the equation

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$$

Solution

We identify $v^2 = \frac{1}{9}$, therefore $v = \pm\frac{1}{3}$

So general solution is $y = C_1 J_{1/3}(x) + C_2 J_{-1/3}(x)$

Example 3

Derive the formula $xJ'_v(x) = vJ_v(x) - xJ_{v+1}(x)$

Solution

As $J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+v+n)}\left(\frac{x}{2}\right)^{2n+v}$

$$xJ'_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+v)}{n!\Gamma(1+v+n)}\left(\frac{x}{2}\right)^{2n+v}$$

$$= v\cdot\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+v+n)}\left(\frac{x}{2}\right)^{2n+v} + 2\cdot\sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+v+n)}\left(\frac{x}{2}\right)^{2n+v}$$

$$= vJ_v(x) + x\cdot\underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(n-1)!\Gamma(1+v+n)}\left(\frac{x}{2}\right)^{2n+v-1}}_{k=n-1}$$

$$= vJ_v(x) - x\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(2+v+k)}\left(\frac{x}{2}\right)^{2k+v+1}$$

$$= vJ_v(x) - xJ_{v+1}(x)$$

So $xJ'_v(x) = vJ_v(x) - xJ_{v+1}(x)$

Example 4

Derive the recurrence relation $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$

Solution:

As $J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$

$$J'_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}(n+2s)\left(\frac{x}{2}\right)^{n+2s-1}\left(\frac{1}{2}\right)$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}(n+s+s)\left(\frac{x}{2}\right)^{n+2s-1}\left(\frac{1}{2}\right)$$

$$=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\overline{(n+s)}\left(\frac{x}{2}\right)^{n+2s-1}\left(\frac{1}{2}\right)+\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}s\cdot\left(\frac{x}{2}\right)^{n+2s-1}\left(\frac{1}{2}\right)$$

$$=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)(n+s-1)!}(n+s)\left(\frac{x}{2}\right)^{n+2s-1}\left(\frac{1}{2}\right)$$

$$+\sum_{s=0}^{\infty}\frac{(-1)^s s}{s(s-1)!(n+s)!}\cdot\left(\frac{x}{2}\right)^{n+2s-1}\left(\frac{1}{2}\right)$$

$$=\frac{1}{2}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n-1+s)!}\left(\frac{x}{2}\right)^{n-1+2s}+\frac{1}{2}\sum_{s=1}^{\infty}\frac{(-1)^s}{(s-1)!(n+s)!}\cdot\left(\frac{x}{2}\right)^{n+2s-1}$$

$$=\frac{1}{2}J_{n-1}(x)+\frac{1}{2}\sum_{s=1}^{\infty}\frac{(-1)^s}{(s-1)!(n+s)!}\left(\frac{x}{2}\right)^{n+2s-1}$$

Put $s-1=p$ in 2nd term $\Rightarrow s=p+1$

$$=\frac{1}{2}J_{n-1}(x)+\frac{1}{2}\sum_{p=0}^{\infty}\frac{(-1)^{p+1}}{p!(n+p+1)!}\left(\frac{x}{2}\right)^{n+2(p+1)-1}$$

$$=\frac{1}{2}J_{n-1}(x)+\frac{1}{2}\sum_{p=0}^{\infty}\frac{-1(-1)^p}{p!(\overline{n+1}+p)!}\left(\frac{x}{2}\right)^{\overline{n+1}+2p}$$

$$J_n'(x)=\frac{1}{2}J_{n-1}(x)-\frac{1}{2}J_{n+1}(x)$$

$$\Rightarrow 2J_n'(x)=J_{n-1}(x)-J_{n+1}(x)$$

Example 5

Derive the expression of $J_n(x)$ for $n=\pm\frac{1}{2}$

Solution:

Consider $$J_n(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$$

As $n!=\Gamma(n+1)$

$$\Rightarrow J_n(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{\Gamma(s+1)\Gamma(n+s+1)}\left(\frac{x}{2}\right)^{n+2s}$$

put $n=1/2$

$$J_{1/2}(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{\Gamma(s+1)\Gamma(1/2+s+1)}\left(\frac{x}{2}\right)^{\frac{1}{2}+2s}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+1)\Gamma(s+3/2)}\left(\frac{x}{2}\right)^{\frac{1}{2}+2s}$$

Expanding R.H.S of above

$$J_{1/2}(x) = \frac{(-1)^0}{\Gamma(0+1)\Gamma(0+3/2)}\left(\frac{x}{2}\right)^{\frac{1}{2}+0} + \frac{(-1)^1}{\Gamma(1+1)\Gamma(1+3/2)}\left(\frac{x}{2}\right)^{\frac{1}{2}+2(1)}$$

$$+ \frac{(-1)^2}{\Gamma(2+1)\Gamma(2+3/2)}\left(\frac{x}{2}\right)^{\frac{1}{2}+2(2)} + \frac{(-1)^3}{\Gamma(3+1)\Gamma(3+3/2)}\left(\frac{x}{2}\right)^{\frac{1}{2}+2(3)} + \cdots$$

$$= \frac{2}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{\frac{1}{2}} - \frac{2\cdot 2}{3\sqrt{\pi}}\left(\frac{x}{2}\right)^{\frac{5}{2}} + \frac{2\cdot 2\cdot 2}{2\cdot 5\cdot 3\sqrt{\pi}}\left(\frac{x}{2}\right)^{\frac{9}{2}} - \cdots$$

$$= \frac{1}{\sqrt{\pi}}\left[2\cdot\frac{\sqrt{x}}{\sqrt{2}} - \frac{4}{3}\frac{\sqrt{x}\cdot x^2}{2^{5/2}} + \frac{4}{15}\frac{\sqrt{x}\cdot x^4}{2^{9/2}} - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{\pi}}\left[\frac{2}{\sqrt{2}} - \frac{4x^2}{3\cdot 2^{5/2}} + \frac{4}{15}\frac{x^4}{2^{9/2}} - \cdots\right]$$

$$= \frac{\sqrt{2}\cdot\sqrt{x}}{\sqrt{\pi}}\left[\frac{2}{\sqrt{2}\sqrt{2}} - \frac{4x^2}{3\cdot\sqrt{2}\cdot 2^{5/2}} + \frac{4x^4}{15\cdot\sqrt{2}\cdot 2^{9/2}} - \cdots\right]$$

$$= \frac{\sqrt{2}\cdot\sqrt{x}}{\sqrt{\pi}}\left[1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots\right]$$

$$= \frac{\sqrt{2}\cdot\sqrt{x}}{\sqrt{\pi}}\cdot\frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= \frac{\sqrt{2}\cdot\sqrt{x}}{\sqrt{\pi}}\sin x$$

$$\Rightarrow J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

Similarly for $n=-1/2$, we proceed further as before,

$$J_n(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} \quad \text{where } n!=\Gamma(n+1)$$

$$\Rightarrow J_n(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{\Gamma(s+1)\Gamma(n+s+1)}\left(\frac{x}{2}\right)^{n+2s}$$

put $n=-\frac{1}{2}$

$$J_{-1/2}(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{\Gamma(s+1)\Gamma(-1/2+s+1)}\left(\frac{x}{2}\right)^{-\frac{1}{2}+2s}$$

$$J_{-1/2}(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{\Gamma(s+1)\Gamma(s+1/2)}\left(\frac{x}{2}\right)^{-\frac{1}{2}+2s}$$

Expanding the R.H.S of above we get

$$J_{-1/2}(x)=\frac{(-1)^0}{\Gamma(0+1)\Gamma(0+1/2)}\left(\frac{x}{2}\right)^{-\frac{1}{2}}+\frac{(-1)^1}{\Gamma(1+1)\Gamma(1+1/2)}\left(\frac{x}{2}\right)^{-\frac{1}{2}+2(1)}$$

$$+\frac{(-1)^2}{\Gamma(2+1)\Gamma(2+1/2)}\left(\frac{x}{2}\right)^{-\frac{1}{2}+2(2)}+\cdots$$

$$J_{-1/2}(x)=\frac{1}{\Gamma(1)\Gamma(1/2)}\sqrt{\frac{2}{x}}-\frac{1}{\Gamma(2)\Gamma(3/2)}\left(\frac{x}{2}\right)^{\frac{3}{2}}+\frac{1}{\Gamma(3)\Gamma(5/2)}\left(\frac{x}{2}\right)^{\frac{7}{2}}-\cdots$$

$$=\frac{1}{(1)\Gamma(1/2)}\sqrt{\frac{2}{x}}-\frac{1}{1\cdot\frac{1}{2}\cdot\Gamma(1/2)}\left(\frac{x}{2}\right)^{\frac{3}{2}}+\frac{1}{2\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}\left(\frac{x}{2}\right)^{\frac{7}{2}}-\cdots$$

$$=\frac{1}{\Gamma(1/2)}\left[\frac{\sqrt{2}}{\sqrt{x}}-\frac{2x^{3/2}}{2^{3/2}}+\frac{2\cdot 2}{2\cdot 3}\frac{x^{7/2}}{2^{7/2}}-\cdots\right]$$

$$=\frac{1}{\sqrt{\pi}}\left[\frac{\sqrt{2}}{\sqrt{x}}-\frac{2x^{3/2}}{2^{3/2}}+\frac{2}{3}\frac{x^{7/2}}{2^{7/2}}-\cdots\right]$$

$$=\frac{\sqrt{2}}{\sqrt{\pi}}\left[\frac{\sqrt{2}}{\sqrt{x}\sqrt{2}}-\frac{2x^{3/2}}{4}+\frac{2}{3}\frac{x^{7/2}}{16}-\cdots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}}\left[\frac{\sqrt{2}}{\sqrt{x}\sqrt{2}} - \frac{x^{3/2}}{2} + \frac{1}{3}\frac{x^{7/2}}{8} - \cdots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}}\left[\frac{1}{\sqrt{x}} - \frac{x^{3/2}}{2} + \frac{x^{7/2}}{24} - \cdots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{x}}\left[\frac{\sqrt{x}}{\sqrt{x}} - \frac{x^{3/2}}{2} + \frac{x^{7/2}}{24} - \cdots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\cos x \qquad \because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\Rightarrow J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

Remarks:

Bessel functions of index half an odd integer are called Spherical Bessel functions. Like other Bessel functions spherical Bessel functions are used in many physical problems.

Exercise

Find the general solution of the given differential equation on $(0, \infty)$.

1. $x^2 y'' + xy' + \left(x^2 - \frac{1}{9}\right) y = 0$
2. $x^2 y'' + xy' + \left(x^2 - 1\right) y = 0$
3. $4x^2 y'' + 4xy' + \left(4x^2 - 25\right) y = 0$
4. $16x^2 y'' + 16xy' + \left(16x^2 - 1\right) y = 0$

Express the given Bessel function in terms of $\sin x$ and $\cos x$, and power of x.

5. $J_{3/2}(x)$
6. $J_{5/2}(x)$
7. $J_{7/2}(x)$

Lecture 20

Legendre's Differential Equation

A second order linear differential equation of the form

$$(1-x^2)y''-2xy'+n(n+1)y=0$$

is called Legendre's differential equation and any of its solution is called Legendre's function. If n is positive integer then the solution of Legendre's differential equation is called a Legendere's polynomial of degree n and is denoted by $P_n(x)$.

We assume a solution of the form $\quad y=\sum_{k=0}^{\infty}C_k x_k$

$$\therefore (1-x^2)y''-2xy'+n(n+1)y=$$

$$(1-x^2)\sum_{k=2}^{\infty}C_k k(k-1)x^{k-2}-2\sum_{k=1}^{\infty}C_k kx^k+n(n+1)\sum_{k=0}^{\infty}C_k x^k$$

$$=\sum_{k=2}^{\infty}C_k k(k-1)x^{k-2}-\sum_{k=2}^{\infty}C_k k(k-1)x^k-2\sum_{k=1}^{\infty}C_k kx^k+n(n+1)\sum_{k=0}^{\infty}C_k x^k$$

$$=[n(n+1)C_0+2C_2]x^0+[n(n+1)C_1-2C_1+6C_3]x+\underbrace{\sum_{k=4}^{\infty}C_k k(k-1)x^{k-2}}_{j=k-2}$$

$$-\underbrace{\sum_{k=2}^{\infty}C_k k(k-1)x^k}_{j=k}-2\underbrace{\sum_{k=2}^{\infty}C_k kx^k}_{j=k}+n(n+1)\underbrace{\sum_{k=2}^{\infty}C_k x^k}_{j=k}$$

$$=\left[n(n+1)C_0+2C_2\right]+\left[(n-1)(n+2)C_1+6C_3\right]x$$

$$+\sum_{j=2}^{\infty}\left[(j+2)(j+1)C_{j+2}+(n-j)(n+j+1)C_j\right]x^j=0$$

$$\Rightarrow \qquad n(n+1)C_0+2C_2=0$$

$$(n-1)(n+2)C_1+6C_3=0$$

$$(j+2)(j+1)C_{j+2}+(n-j)(n+j+1)C_j=0,\ \ j=2,3,4,...$$

or

$$C_2=-\frac{n(n+1)}{2!}C_0$$

$$C_3 = -\frac{(n-1)(n+2)}{3!} C_1$$

$$C_{j+2} = -\frac{(n-j)(n+j+1)}{(j+2)(j+1)} C_j; \quad j = 2,3,\ldots \quad (1)$$

From Iteration formula (1)

$$C_4 = -\frac{(n-2)(n+3)}{4 \cdot 3} C_2 = \frac{(n-2)(n)(n+1)(n+3)}{4!} C_0$$

$$C_5 = -\frac{(n-3)(n+4)}{5 \cdot 4} C_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} C_1$$

$$C_6 = -\frac{(n-4)(n+5)}{5 \cdot 6} C_4 = -\frac{(n-4)(n-2)\ n(n+\ )(n+3)(n+5)}{6!} C_0$$

$$C_7 = -\frac{(n-5)(n+6)}{7 \cdot 6} C_5 = -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!} C_1$$

and so on. Thus at least $|x| < 1$, we obtain two linearly independent power series solutions.

$$y_1(x) = C_0 \left[1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)n(n+1)(n+3)}{4!} x^4 \right.$$
$$\left. - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!} x^6 + \cdots \right]$$

$$y_2(x) = C_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 \right.$$
$$\left. - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!} x^7 + \cdots \right]$$

Note that if n is even integer, the first series terminates, where $y_2(x)$ is an infinite series. For example if $n = 4$, then

$$y_1(x) = C_0 \left[1 - \frac{4 \cdot 5}{2!} x^2 + \frac{2 \cdot 4 \cdot 5 \cdot 7}{4!} x^4 \right] = C_0 \left[1 - 10x^2 + \frac{35}{3} x^4 \right]$$

Similarly, when n is an odd integer, the series for $y_2(x)$ terminates with x^n .i.e when n is a non-negative integer, we obtain an nth-degree polynomial solution of Legendre's equation. Since we know that a constant multiple of a solution of Legendre's equation is also a solution, it is traditional to choose specific values for C_0 and C_1 depending on whether n is even or odd positive integer, respectively.

For $n = 0$, we choose $C_0 = 1$ and for $n = 2,4,6,\ldots$

$$C_0 = (-1)^{n/2} \frac{1\cdot 3\cdot \ldots (n-1)}{2\cdot 4\cdot \ldots (n)}$$

Whereas for $n = 1$, we choose $C_1 = 1$ and for $n = 3,5,7,\ldots$

$$C_1 = (-1)^{(n-1)/2} \frac{1\cdot 3\cdot \ldots n}{2\cdot 4\cdot \ldots (n-1)}$$

For example, when $n = 4$, we have

$$y_1(x) = (-1)^{4/2} \frac{1\cdot 3}{2\cdot 4}\left[1 - 10x^2 + \frac{35}{3}x^4\right]$$

$$= \frac{3}{8} - \frac{30}{8}x^2 + \frac{35}{8}x^4$$

$$y_1(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

Legendre's Polynomials are specific n[th] degree polynomials and are denoted by $P_n(x)$. From the series for $y_1(x)$ and $y_2(x)$ and from the above choices of C_0 and C_1, we find that the first several Legendre's polynomials are

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$

Note that $P_0(x), P_1(x), P_2(x), P_3(x), \ldots$ are, in turn particular solution of the differential equations

$$n = 0 \qquad \left(1 - x^2\right)y'' - 2xy' = 0$$
$$n = 1 \qquad \left(1 - x^2\right)y'' - 2xy' - 2y = 0$$
$$n = 2 \qquad \left(1 - x^2\right)y'' - 2xy' + 6y = 0$$
$$n = 3 \qquad \left(1 - x^2\right)y'' - 2xy' + 12y = 0$$
$$\cdots \qquad \cdots \quad \cdots \quad \cdots \quad \cdots$$

Rodrigues Formula for Legendre's Polynomials

The Legendre Polynomials are also generated by Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2 - 1\right)^n$$

Generating Function For Legendre's Polynomials

The Legendre's polynomials are the coefficient of z^n in the expansion of

$$\phi = \left(1 - 2xz + z^2\right)^{-\frac{1}{2}}$$

in ascending powers of z.

Now $\qquad \phi = \left(1 - 2xz + z^2\right)^{-\frac{1}{2}} = \left\{1 - z\left(2x - z\right)\right\}^{-\frac{1}{2}}$

Therefore by Binomial Series

$$\phi = 1 + \frac{1}{2}z\left(2x - z\right) + \frac{-\frac{1}{2}\left(\frac{-3}{2}\right)}{2!}\left\{-z\left(2x - z\right)\right\}^2 + \frac{-\frac{1}{2}\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!}\left\{-z\left(2x - z\right)\right\}^3 + \cdots$$

$$= 1 + \frac{1}{2}z\left(2x - z\right) + \frac{3}{8}z^2\left(4x^2 + z^2 - 4xz\right) + \frac{5}{16}z^3\left(8x^3 - z^3 - 12x^2z + 6xz^2\right) + \cdots$$

$$= 1 + zx - \frac{1}{2}z^2 + \frac{3}{2}x^2z^2 + \frac{3}{8}z^4 - \frac{3}{2}xz^3 - \frac{5}{2}x^3z^3 - \frac{5}{16}z^6 - \frac{15}{4}x^2z^4 + \frac{15}{8}xz^5 + \cdots$$

$$= 1 + xz + \frac{1}{2}\left(3x^2 - 1\right)z^2 + \frac{1}{2}\left(5x^3 - 3x\right)z^3 + \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)z^4 + \cdots \qquad (1)$$

Also

$$\sum_{n=0}^{\infty} P_n(x)z^n = P_0(x) + P_1(x)z + P_2(x)z^2 + P_3(x)z^3 + \cdots$$

Equating Coefficients of (1) and (2)

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

Which are Legendre's Polynomials

Recurrence Relation

Recurrence relations that relate Legendre's polynomials of different degrees are also very important in some aspects of their application. We shall derive one such relation using the formula

$$\left(1 - 2xt + t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) \cdot t^n \qquad (1)$$

Differentiating both sides of (1) with respect to t gives

$$\left(1 - 2xt + t^2\right)^{-\frac{3}{2}}(x - t) = \sum_{n=0}^{\infty} nP_n(x)t^{n-1} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1}$$

so that after multiplying by $1 - 2xt + t^2$, we have

$$(x - t)\left(1 - 2xt + t^2\right)^{-\frac{1}{2}} = \left(1 - 2xt + t^2\right)\sum_{n=1}^{\infty} nP_n(x)t^{n-1}$$

$$(x - t)\sum_{n=0}^{\infty} P_n(x)t^n = \left(1 - 2xt + t^2\right)\sum_{n=1}^{\infty} nP_n(x)t^{n-1}$$

$$\sum_{n=0}^{\infty} xP_n(x)t^n - \sum_{n=0}^{\infty} P_n(x)t^{n+1} - \sum_{n=1}^{\infty} nP_n(x)t^{n-1} + 2x\sum_{n=1}^{\infty} nP_n(x)t^n$$
$$-\sum_{n=1}^{\infty} nP_n(x)t^{n+1} = 0$$

$$x + x^2t + \sum_{n=2}^{\infty} xP_n(x)t^n - t - \sum_{n=1}^{\infty} P_n(x)t^{n+1} - x - 2\left(\frac{3x^2 - 1}{2}\right)t$$
$$-\sum_{n=3}^{\infty} nP_n(x)t^{n-1} + 2x^2t + 2x\sum_{n=2}^{\infty} nP_n(x)t^n - \sum_{n=1}^{\infty} nP_n(x)t^{n+1} = 0$$

Observing the appropriate cancellations, simplifying and changing the summation indices

$$\sum_{k=2}^{\infty}\left[-(k+1)P_{k+1}(x)+(2k+1)xP_k(x)-kP_{k-1}(x)\right]t^k=0$$

Equating the total coefficient of t^k to zero gives the three-term recurrence relation

$$(k+1)P_{k+1}(x)-(2k+1)xP_k(x)+kP_{k-1}(x)=0, \qquad k=2,3,4,\ldots$$

Legendre's Polynomials are orthogonal

Proof:

Legendre's Differential Equation is $\left(1-x^2\right)y''-2xy'+n(n+1)y=0$

Let $P_n(x)$ and $P_m(x)$ are two solutions of Legendre's differential equation then

$$\left(1-x^2\right)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0\text{, and}$$
$$\left(1-x^2\right)P_m''(x)-2xP_m'(x)+m(m+1)P_m(x)=0$$

which we can write

$$\left[\left(1-x^2\right)P_n'(x)\right]'+n(n+1)P_n(x)=0 \qquad (1)$$

$$\left[\left(1-x^2\right)P_m'(x)\right]'+m(m+1)P_m(x)=0 \qquad (2)$$

Multiplying (1) by $P_m(x)$ and (2) by $P_n(x)$ and subtracting, we get

$$\begin{aligned}&P_m(x)\left\{\left(1-x^2\right)P_n'\right\}'-P_n(x)\left\{\left(1-x^2\right)P_m'(x)\right\}'\\&+\left\{n(n+1)-m(m+1)\right\}P_m(x)P_n(x)=0\end{aligned} \qquad (3)$$

Now

$$Add\ and\ subtract\,(1-x^2)P'_mP'_n\ to\ formulize\,the\,above$$

$$P_m(x)\left\{(1-x^2)P'_n\right\}'-P_n(x)\left\{(1-x^2)P'_m\right\}'$$

$$=\left(1-x^2\right)P_m'(x)P_n'(x)+P_m(x)\left[\left(1-x^2\right)P_n'(x)\right]'$$

$$-\left(1-x^2\right)P_m'(x)P_n'(x)+P_n(x)\left[\left(1-x^2\right)P_m'(x)\right]'$$

$$=\left(1-x^2\right)\left[P_n(x)P_n'(x)-P_m'(x)P_n(x)\right]'$$

Which shows that (3) can be written as

$$\left[\left(1-x^2\right)\left\{P_m(x)P_n'(x)-P_m'(x)P_n(x)\right\}\right]'$$
$$+\left[n(n+1)-m(m+1)\right]P_m(x)P_n(x)=0$$

$$\left(\left(1-x^2\right)\left\{P_m(x)P_n'(x)-P_m'(x)P_n\right\}\right)'+(n-m)(n+m+1)P_m(x)P_n(x)=0$$

$$(n-m)(m+n+1)P_m(x)P_n(x)=\left(\left(1-x^2\right)\left\{P_m'(x)P_n(x)-P_m(x)P_n'(x)\right\}\right)'$$

$$(n-m)(m+n+1)\int_a^b P_m(x)P_n(x)dx=\int_a^b\left(\left(1-x^2\right)\left\{P_m'(x)P_n(x)-P_m(x)P_n'(x)\right\}\right)'dx$$

$$(n-m)(m+n+1)\int_a^b P_m(x)P_n(x)dx=\left(1-x^2\right)\left\{P_m'(x)P_n(x)-P_m(x)P_n'(x)\right\}\Big|_a^b$$

As $1-x^2=0$ for $x=\pm 1$ so

$$(n-m)(n+m+1)\int_{-1}^{1}P_m(x)P_n(x)\,dx=0 \qquad \text{for } x=\pm 1$$

Since m & n are non-negative

$$\Rightarrow \int_{-1}^{1}P_m(x)P_n(x)\,dx=0 \quad \text{for } m\neq n$$

which shows that Legendre's Polynomials are orthogonal w.r.to the weight function $w(x)=1$ over the interval $[-1 \quad 1]$

Normality condition for Legendre' Polynomials

Consider the generating function

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{m=0}^{\infty} P_m(x)t^m \qquad (1)$$

Also

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n \qquad (2)$$

Multiplying (1) and (2)

$$\left(1-2xt+t^2\right)^{-1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x)P_n(x)t^{m+n}$$

Integrating from -1 to 1

$$\int_{-1}^{1}\frac{1}{\left(1-2xt+t^2\right)}dx = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{1} P_m(x)P_n(x)t^{m+n}dx$$

$$-\frac{1}{2t}\int_{-1}^{1}\frac{-2t}{\left(1-2xt+t^2\right)}dx = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{1} P_m(x)P_n(x)t^{m+n}dx$$

$$-\frac{1}{2t}\ln\left(1-2xt+t^2\right)\Big|_{-1}^{1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{1} P_m(x)P_n(x)t^{m+n}dx$$

$$\Rightarrow \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{1} P_m(x)P_n(x)t^{m+n}dx = -\frac{1}{2t}\left[\ln\left(1-2t+t^2\right)-\ln\left(1+2t+t^2\right)\right]$$

$$= -\frac{1}{2t}\left[\ln(1-t)^2-\ln\left(1+t^2\right)\right]$$

$$= -\frac{1}{2t}\left\{\ln(1+t)^2-\ln\left(1-t^2\right)\right\}$$

$$=-\frac{1}{t}\left[\ln(1+t)-\ln(1-t)\right]$$

$$=\frac{1}{t}\left[\left(t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\cdots\right)-\left(-t-\frac{t^2}{2}-\frac{t^3}{3}-\frac{t^4}{4}-\cdots\right)\right]$$

$$=\frac{1}{t}\left\{2t+\frac{2t^3}{3}+\frac{2t^5}{5}+\cdots\right\}$$

$$=\frac{2}{t}\left\{t+\frac{t^3}{3}+\frac{t^5}{5}+\cdots\right\}$$

$$=2\left\{1+\frac{t^2}{3}+\frac{t^4}{5}+\cdots\right\}$$

$$\Rightarrow\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{1}P_m(x)P_n(x)t^{m+n}dx=2\left\{1+\frac{t^2}{3}+\frac{t^4}{5}+\cdots\right\}$$

for $m=n$

$$\Rightarrow\sum_{n=0}^{\infty}\int_{-1}^{1}P_n(x)P_n(x)t^{n+n}dx=2\left\{1+\frac{t^2}{3}+\frac{t^4}{5}+\cdots\right\}$$

$$\Rightarrow\sum_{n=0}^{\infty}\int_{-1}^{1}\left[P_n(x)\right]^2t^{2n}dx=2\left\{1+\frac{t^{2(1)}}{2(1)+1}+\frac{t^{2(2)}}{2(2)+1}+\cdots+\frac{t^{2n}}{2(n)+1}\right\}$$

Equating coefficient of t^{2n} on both sides

$$\Rightarrow\int_{-1}^{1}\left[P_n(x)\right]^2dx=\frac{2}{2n+1}$$

$$\Rightarrow\int_{-1}^{1}P_n(x)P_n(x)dx=\frac{2}{2n+1}$$

$$\Rightarrow\int_{-1}^{1}P_n(x)P_n(x)\frac{2n+1}{2}dx=1$$

which shows that Legender polynomials are normal with respect to the weight function $w(x) = \frac{2n+1}{2}$ over the interval $-1 < x < 1$.

Remark:

Orthognality condition for $P_n(x)$ can also be written as

$$\int_{-1}^{1} P_n(x) P_n(x) dx = \left(\frac{2}{2n+1}\right)\delta_{m,n}$$

where $\delta_{m,n} = \begin{cases} 0 & , \text{ if } m \neq n \\ 1 & ,\text{otherwise} \end{cases}$

Exercise

1. Show that the Legendre's equation has an alternative form
$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right] + n(n+1)y = 0$$
2. Show that the equation
$$\sin\theta \frac{d^2y}{d\theta^2} + \cos\theta \frac{dy}{d\theta} + n(n+1)(\sin\theta)y = 0$$ can be transformed into Legendre's equation by means of the substitution $x = \cos\theta$
3. Use the explicit Legendre's polynomials $P_1(x)$, $P_2(x)$ $P_2(x)$ and $P_3(x)$ to evaluate $\int_{-1}^{1} P_n^2 dx$ for $n = 0, 1, 2, 3$. Generalize the results.
4. Use the explicit Legendre polynomials $P_1(x)$, $P_2(x)$ $P_2(x)$ and $P_3(x)$ to evaluate $\int_{-1}^{1} P_n(x) P_m(x) dx$ for $n \neq m$. Generalize the results.
5. The Legendre's polynomials are also generated by **Rodrigues' formula**
$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2 - 1\right)^n$$
verify the results for $n = 0, 1, 2, 3$.

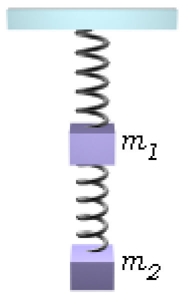
Lecture 21

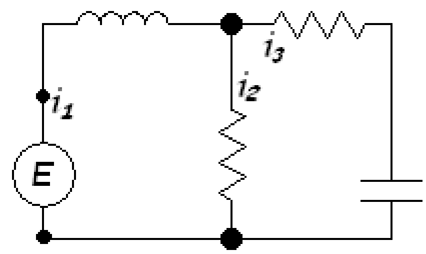
Systems of Linear Differential Equations

- Recall that the mathematical model for the motion of a mass attached to a spring or for the response of a series electrical circuit is a differential equation.

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

- However, we can attach two or more springs together to hold two masses m_1 and m_2. Similarly a network of parallel circuits can be formed.





- To model these latter situations, we would need two or more coupled or simultaneous equations to describe the motion of the masses or the response of the network.
- Therefore, in this lecture we will discuss the theory and solution of the systems of simultaneous linear differential equations with constant coefficients.

Note that

An nth order linear differential equation with constant coefficients $a_0,\ a_1,\ \ldots,\ a_n$ is an equation of the form

$$a_n\frac{d^ny}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0y=g(x)$$

If we write $D=\frac{d}{dx},D^2=\frac{d^2}{dx^2},\cdots,D^n=\frac{d^n}{dx^n}$ then this equation can be written as follows

$$\left(a_nD^n+a_{n-1}D^{(n-1)}+\cdots+a_1D+a_0\right)y=g\left(t\right)$$

Simultaneous Differential Equations

The simultaneous ordinary differential equations involve two or more equations that contain derivatives of two or more unknown functions of a single independent variable.

Example 1
If x, y and z are functions of the variable t, then

$$4\frac{d^2x}{dt^2} = -5x + y$$
$$2\frac{d^2y}{dt^2} = 3x - y$$

and

$$x' - 3x + y' + z' = 5$$
$$x + y' - 6z' = t - 1$$

are systems of simultaneous differential equations.

Solution of a System
A solution of a system of differential equations is a set of differentiable functions

$$x = f(t), \quad y = g(t), \quad x = h(t), \ldots$$

those satisfy each equation of the system on some interval I.

Systematic Elimination: Operator Method

- This method of solution of a system of linear homogeneous or linear non-homogeneous differential equations is based on the process of systematic elimination of the dependent variables.
- This elimination provides us a single differential equation in one of the dependent variables that has not been eliminated.
- This equation would be a linear homogeneous or a linear non-homogeneous differential equation and can be solved by employing one of the methods discussed earlier to obtain one of the dependent variables.

Notice that the analogue of multiplying an algebraic equation by a constant is operating on a differential equation with some combination of derivatives.

The Method

Step 1 First write the differential equations of the system in a form that involves the differential operator D.

Step 2 We retain first of the dependent variables and eliminate the rest from the differential equations of the system.

Step 3 The result of this elimination is to be a single linear differential equation with constant coefficients in the retained variable. We solve this equation to obtain the value of this variable.

Step 4 Next, we retain second of the dependent variables and eliminate all others variables

Step 5 The result of the elimination performed in step 4 is to be again a single linear differential equation with constant coefficients in the retained 2^{nd} variable. We again solve this equation and obtain the value of the second dependent variable. This process of elimination is continued untill all the variables are taken care of.

Step 6 The computed values of the dependent variables don't satisfy the given system for every choice of all the arbitrary constants. By substituting the values of the dependent variables computed in step 5 into an equation of the original system, we can reduce the number of constant from the solution set.

Step 7 We use the work done in step number 6 to write the solution set of the given system of linear differential equations.

Example 1

Solve the system of differential equations

$$\frac{dy}{dt} = 2x, \qquad \frac{dx}{dt} = 3y$$

Solution:

Step 1 The given system of linear differential equations can be written in the differential operator form as

$$Dy = 2x, \qquad Dx = 3y$$

or

$$2x - Dy = 0, \qquad Dx - 3y = 0$$

Step 2 Next we eliminate one of the two variables, say x, from the two differential equations. Operating on the first equation by D while multiplying the second by 2 and then subtracting eliminates x from the system. It follows that

$$-D^2y + 6y = 0 \quad \text{or} \quad D^2y - 6y = 0.$$

Step 3 Clearly, the result is a single linear differential equation with constant coefficients in the retained variable y. The roots of the auxiliary equation are real and distinct

$$m_1 = \sqrt{6} \quad \text{and} \quad m_2 = -\sqrt{6},$$

Therefore,

$$y(t) = c_1e^{\sqrt{6}\,t} + c_2e^{-\sqrt{\ }\,t}$$

Step 4 We now eliminate the variable y that was retained in the previous step. Multiplying the first equation by -3, while operating on the second by D and then adding gives the differential equation for x,

$$D^2x - 6x = 0.$$

Step 5 Again, the result is a single linear differential equation with constant coefficients in the retained variable x. We now solve this equation and obtain the value of the second dependent variable. The roots of the auxiliary equation are $m = \pm\sqrt{6}$. It follows that

$$x(t) = c_3e^{\sqrt{6}\,t} + c_4e^{-\sqrt{\ }\,t}$$

Hence the values of the dependent variables $x(t)$, $y(t)$ are.

$$x(t) = c_3e^{\sqrt{6}\,t} + c_4e^{-\sqrt{6}\,t}$$

$$y(t) = c_1e^{\sqrt{6}\,t} + c_2e^{-\sqrt{\ }\,t}$$

Step 6 Substituting the values of $x(t)$ and $y(t)$ from step 5 into first equation of the given system, we have

$$\left(\sqrt{6}c_1 - 2c_3\right)e^{\sqrt{6}t} + \left(-\sqrt{6}c_2 - 2c_4\right)e^{-\sqrt{6}t} = 0.$$

Since this expression is to be zero for all values of t, we must have

$$\sqrt{6}c_1 - 2c_3 = 0, \quad -\sqrt{6}c_2 - 2c_4 = 0$$

or
$$c_3 = \frac{\sqrt{6}}{2}c_1, \qquad c_4 = -\frac{\sqrt{6}}{2}c_2$$

Notice that if we substitute the computed values of $x(t)$ and $y(t)$ into the second equation of the system, we shall find that the same relationship holds between the constants.

Step 7 Hence, by using the above values of c_1 and c_2, we write the solution of the given system as

$$x(t) = \frac{\sqrt{6}}{2}c_1 e^{\sqrt{6}t} - \frac{\sqrt{6}}{2}c_2 c^{-\sqrt{6}t}$$

$$y(t) = c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t}$$

Example 2
Solve the following system of differential equations

$$Dx + (D+2)y = 0$$
$$(D-3)\ x - 2y = 0$$

Solution:

Step 1 The differential equations of the given system are already in the operator form.
Step 2 We eliminate the variable x from the two equations of the system. Thus operating on the first equation by $D-3$ and on the second by D and then subtracting eliminates x from the system. The resulting differential equation for the retained variable y is

$$[(D-3)(D+2) + 2D]\,y = 0$$
$$\left(D^2 + D - 6\right)y = 0$$

Step 3 The auxiliary equation of the differential equation for y obtained in the last step is

$$m^2 + m - 6 = 0 \Rightarrow (m-2)(m+3) = 0$$

Since the roots of the auxiliary equation are

$$m_1 = 2, \quad m_2 = -3$$

Therefore, the solution for the dependent variable y is

$$y(t) = c_1 e^{2t} + c_2 e^{-3t}$$

Step 4 Multiplying the first equation by 2 while operating on the second by $(D+2)$ and then adding yields the differential equation for x

$$(D^2 + D - 6)x = 0,$$

Step 5 The auxiliary equation for this equation for x is

$$m^2 + m - 6 = 0 = (m-2)(m+3)$$

The roots of this auxiliary equation are

$$m_1 = 2, \quad m_2 = -3$$

Thus, the solution for the retained variable x is

$$x(t) = c_3 e^{2t} + c_4 e^{-3t}$$

Writing two solutions together, we have

$$x(t) = c_3 e^{2t} + c_4 e^{-3t}$$
$$y(t) = c_1 e^{2t} + c_2 e^{-3t}$$

Step 6 To reduce the number of constants, we substitute the last two equations into the first equation of the given system to obtain

$$(4c_1 + 2c_3)e^{2t} + (-c_2 - 3c_4)e^{-3t} = 0$$

Since this relation is to hold for all values of the independent variable t. Therefore, we must have

$$4c_1 + 2c_3 = 0, \qquad -c_2 - 3c_4 = 0.$$

or

$$c_3 = -2c_1, \qquad c_4 = -\frac{1}{3}c_2$$

Step 7 Hence, a solution of the given system of differential equations is

$$x(t) = -2c_1 e^{2t} - \frac{1}{3}c_2 e^{-3t}$$
$$y(t) = c_1 e^{2t} + c_2 e^{-3t}$$

Example 3

Solve the system

$$\frac{dx}{dt} - 4x + \frac{d^2y}{dt^2} = t^2$$
$$\frac{dx}{dt} + x + \frac{dy}{dt} = 0$$

Solution:

Step 1 First we write the differential equations of the system in the differential operator form:

$$(D-4)x + D^2 y = t^2$$
$$(D+1)x + Dy = 0$$

Step 2 Then we eliminate one of the dependent variables, say x. Operating on the first equation with the operator $D+1$, on the second equation with the operator $D-4$ and then subtracting, we obtain

$$[(D+1)D^2 - (D-4)D]\, y = (D+1)t^2$$

or $$\left(D^3 + 4D\right)y = t^2 + 2t.$$

Step 3 The auxiliary equation of the differential equation found in the previous step is

$$m^3 + 4m = 0 = m\,(m^2 + 4)$$

Therefore, roots of the auxiliary equation are

$$m_1 = 0, \quad m_2 = 2\,i, \quad m_3 = -2\,i$$

So that the complementary function for the retained variable y is

$$y_c = c_1 + c_2 \cos 2t + c_3 \sin 2t.$$

To determine the particular solution y_p we use undetermined coefficients. Therefore, we assume

$$y_p = At^3 + Bt^2 + Ct.$$

So that $$y_p' = 3At^2 + 2Bt + C,$$
$$y_p'' = 6At + 2B, \quad y_p''' = 6A$$

Thus $$y_p''' + 4y_p' = 12At^2 + 8Bt + 6A + 4C$$

Substituting in the differential equation found in step, we obtain

$$12At^2 + 8Bt + 6A + 4C = t^2 + 2t$$

Equating coefficients of t^2, t and constant terms yields

$$12A = 1, \quad 8B = 2, \quad 6A + 4C = 0,$$

Solving these equations give

$$A = 1/12, \; 1B = \; /4, \; C = -1/8.$$

Hence, the solution for the variable y is given by

$$y = y_c + y_p$$

or $$y = c_1 + c_2 \cos 2t + c_3 \sin 2t + \frac{1}{12}t^3 + \frac{1}{4}t^2 - \frac{1}{8}t.$$

Step 4 Next we eliminate the variable y from the given system. For this purpose we multiply first equation with 1 while operate on the second equation with the operator D and then subtracting, we obtain

$$[(D-4) - D(D+1)]x = \quad t^2$$

or $$(D^2 + 4)x = -t^2$$

Step 5 The auxiliary equation of the differential equation for x is

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

The roots of the auxiliary equation are complex. Therefore, the complementary function for x

$$x_c = c_4 \cos 2t + c_5 \sin 2t$$

The method of undetermined coefficients can be applied to obtain a particular solution. We assume that

$$x_p = At^2 + Bt + C.$$

Then $$x'_p = 2At + B,\ x''_p = 2A$$

Therefore $$x''_p + 4x_p = 2A + 4At^2 + 4Bt + 4C$$

Substituting in the differential equation for x, we obtain

$$4At^2 + 4Bt + 2A + 4C = -t^2$$

Equating the coefficients of t^2, t and constant terms, we have

$$4A = -1,\quad 4B = 0,\quad 2A + 4C = 0$$

Solving these equations we obtain

$$A = -1/4,\ B = 0,\ C = 1/8$$

Thus $$x_p = -\frac{1}{4}t^2 + \frac{1}{8}$$

So that $$x = x_c + x_p = c_4 \cos 2t + c_5 \sin 2t - \frac{1}{4}t^2 + \frac{1}{8}$$

Hence, we have

$$x = x_c + x_p = c_4 \cos 2t + c_5 \sin 2t - \frac{1}{4}t^2 + \frac{1}{8}$$

$$y = c_1 + c_2 \cos 2t + c_3 \sin 2t + \frac{1}{12}t^3 + \frac{1}{4}t^2 - \frac{1}{8}t.$$

Step 6 Now c_4 and c_5 can be expressed in terms of c_2 and c_3 by substituting these values of x and y into the second equation of the given system and we find, after combining the terms,

$$(c_5 - 2c_4 - 2c_2)\sin 2t + (2c_5 + c_4 + 2c_3)\cos 2t = 0$$

So that $$c_5 - 2c_4 - 2c_2 = 0,\qquad 2c_5 + c_4 + 2c_3 = 0$$

Solving the last two equations for c_4 and c_5 in terms of c_2 and c_3 gives

$$c_4 = -\frac{1}{5}(4c_2 + 2c_3), \qquad c_5 = \frac{1}{5}(2c_2 - 4c_3).$$

Step 7 Finally, a solution of the given system is found to be

$$x(t) = -\frac{1}{5}(4c_2 + 2c_3)\cos 2t + \frac{1}{5}(2c_2 - 4c_3)\sin 2t - \frac{1}{4}t^2 + \frac{1}{8}t$$

$$y(t) = c_1 + c_2\cos 2t + c_3\sin 2t + \frac{1}{12}t^3 + \frac{1}{4}t^2 - \frac{1}{8}t.$$

Exercise

Solve, if p ossible, th e g iven s ystem o f d ifferential e quations b y either s ystematic elimination.

1. $\frac{dx}{dt} = x + 7y, \quad \frac{dy}{dt} = x - 2y$
2. $\frac{dx}{dt} - 4y = 1, \quad x + \frac{dy}{dt} = 2$
3. $(D+1)x + (D-1)y = 2, \quad 3x + (D+2)y = -1$
4. $\frac{d^2x}{dt^2} + \frac{dy}{dt} = -5x, \quad \frac{dx}{dt} + \frac{dy}{dt} = -x + 4y$
5. $D^2x - Dy = t, \quad (D+3)x + (D+3)y = 2$
6. $\frac{dx}{dt} + \frac{dy}{dt} = e^t, \quad -\frac{d^2x}{dt^2} + \frac{dx}{dt} + x + y = 0$
7. $(D-1)x + (D^2+1)y = 1, \quad (D^2-1)x + (D+1)y = 2$
8. $Dx = y, \quad Dy = z, \quad Dz = x$
9. $\frac{dx}{dt} = -x + z, \quad \frac{dy}{dt} = -y + z, \quad \frac{dz}{dt} = -x + y$
10. $Dx - 2Dy = t^2, \quad (D+1)x - 2(D+1)y = 1$

Lecture 22
Systems of Linear Differential Equations

Solution of Using Determinants

If L_1, L_2, L_3 and L_4 denote linear differential operators with constant coefficients, then a system of linear differential equations in two variables x and y can be written as

$$L_1 x + L_2 y = g_1(t)$$
$$L_3 x + L_4 y = g_2(t)$$

To eliminate y, we operate on the first equation with L_4 and on the second equation with L_2 and then subtracting, we obtain

$$(L_1 L_4 - L_2 L_3)x = L_4 g_1 - L_2 g_2$$

Similarly, operating on the first equation with L_3 and second equation with L_1 and then subtracting, we obtain

$$(L_1 L_4 - L_2 L_3)y = L_1 g_2 - L_3 g_1$$

Since $$L_1 L_4 - L_2 L_3 = \begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix}$$

Therefore $$L_4 g_1 - L_2 g_2 = \begin{vmatrix} g_1 & L_2 \\ g_2 & L_4 \end{vmatrix}$$

and $$L_1 g_2 - L_3 g_1 = \begin{vmatrix} L_1 & g_1 \\ L_3 & g_2 \end{vmatrix}$$

Hence, the given system of differential equations can be decoupled into *nth* order differential equations. These equations use determinants similar to those used in Cramer's rule:

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} x = \begin{vmatrix} g_1 & L_2 \\ g_2 & L_4 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} y = \begin{vmatrix} L_1 & g_1 \\ L_3 & g_2 \end{vmatrix}$$

The uncoupled differential equations can be solved in the usual manner.

Note that

- The determinant on left hand side in each of these equations can be expanded in the usual algebraic sense. This means that the symbol D occurring in L_i is to be treated as an algebraic quantity. The result of this expansion is a differential operator of order n, which is operated on x and y.

- However, some care should be exercised in the expansion of the determinant on the right hand side. We must expand these determinants in the sense of the internal differential operators actually operating on the functions g_1 and g_2. Therefore, the symbol D occurring in L_i is to be treated as an algebraic quantity.

The Method

The steps involved in application of the method of detailed above can be summarized as follows:

Step 1 First we have to write the differential equations of the given system in the differential operator form

$$L_1 x + L_2 y = g_1(t)$$
$$L_3 x + L_4 y = g_2(t)$$

Step 2 We find the determinants

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix}, \begin{vmatrix} g_1 & L_2 \\ g_2 & L_4 \end{vmatrix}, \begin{vmatrix} L_1 & g_1 \\ L_3 & g_2 \end{vmatrix}$$

Step 3 If the first determinant is non-zero, then it represents an n^{th} order differential operator and we decoupled the given system by writing the differential equations

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} \cdot x = \begin{vmatrix} g_1 & L_2 \\ g_2 & L_4 \end{vmatrix}$$
$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} \cdot y = \begin{vmatrix} L_1 & g_1 \\ L_3 & g_2 \end{vmatrix}$$

Step 4 Find the complementary functions for the two equations. Remember that the auxiliary equation and hence the complementary function of each of these differential equations is the same.

Step 5 Find the particular integrals x_p and y_p using method of undetermined coefficients or the method of variation of parameters.

Step 6 Finally, we write the general solutions for both the dependent variables x and y

$$x = x_c + x_p, \; y = y_c + y_p.$$

Step 7 Reduce the number of constants by substituting in one of the differential equations of the given system

Note that

If the determinant found in step 2 is zero, then the system may have a solution containing any number of independent constants or the system may have no solution at all. Similar remarks hold for systems larger than system indicated in the previous discussion.

Example 1

Solve the following homogeneous system of differential equations

$$2\frac{dx}{dt}-5x+\frac{dy}{dt}=e^t$$
$$\frac{dx}{dt}-x+\frac{dy}{dt}=5e^t$$

Solution:

Step 1 First we write the differential equations of the system in terms of the differential operator D

$$(2D-5)x+Dy=e^t$$
$$(D-1)x+Dy=5e^t$$

Step 2 We form the determinant

$$\begin{vmatrix}2D-5 & D\\ D-1 & D\end{vmatrix},\quad \begin{vmatrix}e^t & D\\ 5e^t & D\end{vmatrix},\quad \begin{vmatrix}2D-5 & e^t\\ D-1 & 5e^t\end{vmatrix}$$

Step 3 Since the 1st determinant is non-zero

$$\begin{vmatrix}2D-5 & D\\ D-1 & D\end{vmatrix}=(2D-5)D-(D-1)D$$

or
$$\begin{vmatrix}2D-5 & D\\ D-1 & D\end{vmatrix}=D^2-4D\neq 0$$

Therefore, we write the decoupled equations

$$\begin{vmatrix}2D-5 & D\\ D-1 & D\end{vmatrix}x=\begin{vmatrix}e^t & D\\ 5e^t & D\end{vmatrix}$$

$$\begin{vmatrix}2D-5 & D\\ D-1 & D\end{vmatrix}y=\begin{vmatrix}2D-5 & e^t\\ D-1 & 5e^t\end{vmatrix}$$

After expanding we find that

$$\left(D^2-4D\right)x = De^t - D(5)e^t = -4e^t$$

$$\left(D^2-4D\right)y = (2D-5)(5e^t) - (D-1)e^t = -15e^t$$

Step 4 We find the complementary function for the two equations. The auxiliary equation for both of the differential equations is:

$$m^2 - 4m = 0 \Rightarrow m = 0, 4$$

The auxiliary equation has real and distinct roots

$$x_c = c_1 + c_2 e^{4t}$$

$$y_c = c_3 + c_4 e^{4t}$$

Step 5 We now use the method of undetermined coefficients to find the particular integrals x_p and y_p.

Since $g_1(t) = -4\ e^t,\quad g_2(t) = -15\ e^t$

We assume that

$$x_p = Ae^t, \quad y_p = Be^t$$

Then $D\ x_p = Ae^t, \quad D^2 x_p = Ae^t$

And $D\ y_p = Be^t, \quad D^2 y_p = Be^t$

Substituting in the differential equations, we have

$$Ae^t - 4Ae^t = -4e^t$$

$$Be^t - 4Be^t = -15e^t$$

or $-3Ae^t = -4e^t, \quad -3Be^t = -15e^t$

Equating coefficients of e^t and constant terms, we obtain

$$A = \frac{4}{3}, \quad B = 5$$

So that $x_p = \frac{4}{3}e^t, \quad y_p = 5e^t$

Step 6 Hence, the general solution of the two decoupled equations

$$x = x_c + x_p = c_1 + c_2 e^{4t} + \frac{4}{3}e^t$$

$$y = y_c + y_p = c_3 + c_4 e^{4t} + 5e^t$$

Step 7 Substituting these solutions for x and y into the second equation of the given system, we obtain

$$-c_1 + \left(3c_2 + 4c_4\right)e^{4t} = 0$$

or
$$c_1 = 0, \quad c_4 = -\frac{3}{4}c_2.$$

Hence, the general solution of the given system of differential equations is

$$x(t) = c_2 e^{4t} + \frac{4}{3}e^t$$

$$y(t) = c_3 - \frac{3}{4}c_2 e^{4t} + 5e^t$$

If we re-notate the constants c_2 and c_3 as c_1 and c_2, respectively. Then the solution of the system can be written as:

$$x(t) = c_1 e^{4t} + \frac{4}{3}e^t$$

$$y(t) = -\frac{3}{4}c_1 e^{4t} + c_2 + 5e^t$$

Example 2

Solve

$$x' = 3x - y - 1$$

$$y' = x + y + 4e^t$$

Solution:

Step 1 First we write the differential equations of the system in terms of the differential operator D

$$(D-3)x + y = -1$$

$$-x + (D-1)y = 4e^t$$

Step 2 We form the determinant

$$\begin{vmatrix} D-3 & 1 \\ -1 & D-1 \end{vmatrix}, \begin{vmatrix} -1 & 1 \\ 4e^t & D-1 \end{vmatrix}, \begin{vmatrix} D-3 & 1 \\ -1 & 4e^t \end{vmatrix}$$

Step 3 Since the 1st determinant is non-zero

$$\begin{vmatrix} D-3 & 1 \\ -1 & D-1 \end{vmatrix} = D^2 - 4D + 4 \neq 0$$

Therefore, we write the decoupled equations

$$\begin{vmatrix} D-3 & 1 \\ -1 & D-1 \end{vmatrix} x = \begin{vmatrix} -1 & 1 \\ 4e^t & D-1 \end{vmatrix}$$

$$\begin{vmatrix} D-3 & 1 \\ -1 & D-1 \end{vmatrix} y = \begin{vmatrix} D-3 & -1 \\ -1 & 4e^t \end{vmatrix}$$

After expanding we find that

$$(D-2)^2 x = 1 - 4\,e^t$$

$$(D-2)^2 y = -1 - 8\,e^t.$$

Step 4 We find the complementary function for the two equations. The auxiliary equation for both of the differential equations is:

$$(m-2)^2 = 0 \Rightarrow m = 2,2$$

The auxiliary equation has real and equal roots

$$x_c = c_1 e^{2t} + c_2 t e^{2t}$$

$$y_c = c_3 e^{2t} + c_4 t e^{2t}$$

Step 5 We now use the method of undetermined coefficients to find the particular integrals x_p and y_p.

Since $\quad g_1(t) = 1 - 4\,e^t, \quad g_2(t) = -1 - 8\,e^t$

We assume that

$$x_p = A + Be^t, \quad y_p = C + Ee^t$$

Then $\quad D\,x_p = Be^t, \quad D^2 x_p = Be^t$

And $\quad D\,y_p = Ee^t, \quad D^2 y_p = Ee^t$

Substituting in the differential equations

$$(D-2)^2 x_p = D^2 x_p - 4Dx_p + 4x_p = 1 - 4e^t$$

$$(D-2)^2 y_p = D^2 y_p - 4Dy_p + 4y_p = -1 - 8e^t$$

Therefore, we have

$$Be^t - 4Be^t + 4A + 4Be^t = 1 - 4e^t$$

$$Ee^t - 4Ee^t + 4C + 4Ee^t = -1 - 8e^t$$

or $\quad Be^t + 4A = 1 - 4e^t, \quad Ee^t + 4C = -1 - 8e^t$

Equating coefficients of e^t and constant terms, we obtain

$$B = -4, \quad A = \frac{1}{4}$$

$$C = -\frac{1}{4}, \qquad E = -8$$

So that $$x_p = \frac{1}{4} - 4e^t, \quad y_p = -\frac{1}{4} - 8e^t$$

Step 6 Hence, the general solution of the two decoupled equations

$$x = x_c + x_p = c_1 e^{2t} + c_2 te^{2t} + \frac{1}{4} - 4e^t$$

$$y = y_c + y_p = c_3 e^{2t} + c_4 te^{2t} - \frac{1}{4} - 8e^t$$

Step 7 Substituting these solutions for x and y into the second equation of the given system, we obtain

$$(c_3 - c_1 + c_4)e^{2t} + (c_4 - c_2)te^{2t} = 0$$

or $$c_4 = c_2, \quad c_3 = c_1 - c_4 = c_1 - c_2.$$

Hence, a solution of the given system of differential equations is

$$x(t) = c_1 e^{2t} + c_2 te^{2t} + \frac{1}{4} - 4e^t$$

$$y(t) = (c_1 - c_2)e^{2t} + c_2 te^{2t} - \frac{1}{4} - 8e^t$$

Example 3

Given the system

$$Dx + Dz = t^2$$

$$2x + D^2 y = e^t$$

$$-2Dx - 2y + (D+1)z = 0$$

Find the differential equation for the dependent variables x, y and z.

Solution:

Step1 The d ifferential e quations o f th e s ystem a re a lready w ritten in th e d ifferential operator form.

Step 2 We form the determinant

$$\begin{vmatrix} D & 0 & D \\ 2 & D^2 & 0 \\ -2D & -2 & D+1 \end{vmatrix}, \begin{vmatrix} t^2 & 0 & D \\ e^t & D^2 & 0 \\ 0 & -2 & D+1 \end{vmatrix}, \begin{vmatrix} D & t^2 & D \\ 2 & e^t & 0 \\ -2D & 0 & D+1 \end{vmatrix}, \begin{vmatrix} D & 0 & t^2 \\ 2 & D^2 & e^t \\ -2D & -2 & 0 \end{vmatrix}$$

Step 3 Since the first determinant is non-zero.

$$\begin{vmatrix} D & 0 & D \\ 2 & D^2 & 0 \\ -2D & -2 & D+1 \end{vmatrix} = D\begin{vmatrix} D^2 & 0 \\ -2 & D+1 \end{vmatrix} + D\begin{vmatrix} 2 & D^2 \\ -2D & -2 \end{vmatrix}$$

or
$$\begin{vmatrix} D & 0 & D \\ 2 & D^2 & 0 \\ -2D & -2 & D+1 \end{vmatrix} = D\left(3D^3 + D^2 - 4\right) \neq 0$$

Therefore, we can write the decoupled equations

$$\begin{vmatrix} D & 0 & D \\ 2 & D^2 & 0 \\ -2D & -2 & D+1 \end{vmatrix} \cdot x = \begin{vmatrix} t^2 & 0 & D \\ e^t & D^2 & 0 \\ 0 & -2 & D+1 \end{vmatrix}$$

$$\begin{vmatrix} D & 0 & D \\ 2 & D^2 & 0 \\ -2D & -2 & D+1 \end{vmatrix} \cdot y = \begin{vmatrix} D & t^2 & D \\ 2 & e^t & 0 \\ -2D & 0 & D+1 \end{vmatrix}$$

$$\begin{vmatrix} D & 0 & D \\ 2 & D^2 & 0 \\ -2D & -2 & D+1 \end{vmatrix} \cdot z = \begin{vmatrix} D & 0 & t^2 \\ 2 & D^2 & e^t \\ -2D & -2 & 0 \end{vmatrix}$$

The determinant on the left hand side in these equations has already been expanded. Now we expand the determinants on the right hand side by the cofactors of an appropriate row.

$$\begin{vmatrix} t^2 & 0 & D \\ e^t & D^2 & 0 \\ 0 & -2 & D+1 \end{vmatrix} = \begin{vmatrix} D^2 & 0 \\ -2 & D+1 \end{vmatrix} t^2 + D\begin{vmatrix} e^t & D^2 \\ 0 & -2 \end{vmatrix}$$

$$= D^2(D+1)t^2 + D(-2e^t) = (D^3 + D^2)t^2 - 2e^t$$

$$= 2 - 2e^t$$

$$\begin{vmatrix} D & t^2 & D \\ 2 & e^t & 0 \\ -2D & 0 & D+1 \end{vmatrix} = D\begin{vmatrix} e^t & 0 \\ 0 & D+1 \end{vmatrix} - \begin{vmatrix} 2 & 0 \\ -2D & D+1 \end{vmatrix} t^2 + D\begin{vmatrix} 2 & e^t \\ -2D & 0 \end{vmatrix}$$

$$= D[(D+1)e^t] - [(D+1)(2t^2)] + D[2De^t]$$

$$= 2e^t - 4t - 2t^2 + 2e^t = 4e^t - 2t^2 - 4t.$$

$$\begin{vmatrix} D & 0 & t^2 \\ 2 & D^2 & e^t \\ -2D & -2 & 0 \end{vmatrix} = D\begin{vmatrix} D^2 & e^t \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} 2 & D^2 \\ -2D & -2 \end{vmatrix} t^2$$

$$= D(2e^t) + (-4 + 2D^3)t^2 = 2e^t - 4t^2 + 0$$

$$= 2e^t - 4t^2$$

Hence the differential equations for the dependent variables x, y and z can be written as

$$D\left(3D^3 + D^2 - 4y\right)x = 2 - 2e^t$$

or

$$D\left(3D^3 + D^2 - 4y\right)y = 4e^t - 2t^2 - 4t.$$

$$D\left(3D^3 + D^2 - 4y\right)z = 2e^t - 4t^2$$

Again we remind that the D symbol on the left-hand side is to be treated as an algebraic quantity, but this is not the case on the right-hand side.

Exercise

Solve, if possible, the given system of differential equations by use of determinants.

1. $\dfrac{dx}{dt} = 2x - y, \qquad \dfrac{dy}{dt} = x - 2y$

2. $\dfrac{dx}{dt} = -y + t, \qquad \dfrac{dy}{dt} = x - t$

3. $\left(D^2 + 5\right)x - 2y = 0, \qquad -2x + \left(D^2 + 2\right)y = 0$

4. $\dfrac{d^2x}{dt^2} = 4y + e^t, \qquad \dfrac{d^2y}{dt^2} = 4x - e^t$

5. $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} = -5x, \qquad \dfrac{dx}{dt} + \dfrac{dy}{dt} = -x + 4y$

6. $Dx + D^2y = e^{3t}, \quad (D+1)x + (D-1)y = 4e^{3t}$

7. $\left(D^2 - 1\right)x - y = 0, \qquad \left(D - 1\right)x + Dy = 0$

8. $(2D^2 - D - 1)x - (2D+1)y = 1, \qquad (D-1)x + Dy = -1$

9. $\dfrac{dx}{dt} + \dfrac{dy}{dt} = e^t, \qquad -\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x + y = 0$

10. $2Dx + (D-1)y = t, \qquad Dx + Dy = t^2$

Lecture 23
Systems of Linear First-Order Equation

In Previous Lecture

In the preceding lectures we dealt with linear systems of the form

$$\begin{aligned}
P_{11}(D)x_1 + P_{12}(D)x_2 + \cdots + P_{1n}(D)x_n &= b_1(t) \\
P_{21}(D)x_1 + P_{22}(D)x_2 + \cdots + P_{2n}(D)x_n &= b_2(t) \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad &\vdots \\
P_{n1}(D)x_1 + P_{n2}(D)x_2 + \cdots + P_{nn}(D)x_n &= b_n(t)
\end{aligned}$$

where the P_{ij} were polynomials in the differential operator D.

The nth **Order System**

1. The study of systems of first-order differential equations

$$\begin{aligned}
\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= g_2(t, x_1, x_2, \ldots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= g_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}$$

is also particularly important in advanced mathematics. This system of n first-order equations is called and ***n*th-order system.**

2. Every *n*th-order differential equation

$$y^{(n)} = F\left(t, y, y', \ldots, y^{(n-1)}\right)$$

as well as ***most systems*** of differential equations, can be reduced to the ***nth-order system***.

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Linear Normal Form

A particularly, but important, case of the *nth-order system* is of those systems having the linear normal or canonical form:

$$\frac{dx_1}{dt}=a_{11}(t)x_1+a_{12}(t)x_2+\cdots+a_{1n}(t)x_n+f_1(t)$$
$$\frac{dx_2}{dt}=a_{21}(t)x_1+a_{22}(t)x_2+\cdots+a_{2n}(t)x_n+f_2(t)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\frac{dx_n}{dt}=a_{n1}(t)x_1+a_{n2}(t)x_2+\cdots+a_{nn}(t)x_n+f_n(t)$$

where the coefficients a_{ij} and the f_i are the continuous functions on a common interval I**.** When $f_i(t)=0, i=1,2,\ldots,n,$ the system is said to be **homogeneous**; otherwise it is called **non-homogeneous**.

Reduction of Equation to a System

Suppose a linear *n*th-order differential equation is first written as

$$\frac{d^n y}{dt^n}=-\frac{a_0}{a_n}y-\frac{a_1}{a_n}y'-\cdots-\frac{a_{n-1}}{a_n}y^{(n-1)}+f(t).$$

If we then introduce the variables

$$y=x_1,\quad y'=x_2,\quad y''=x_3,\ldots,y^{(n-1)}=x_n$$

it follows that

$$y'=x_1'=x_2,\; y''=x_2'=x_3,\ldots,y^{(n-1)}=x_{n-1}'=x_n,\; y^{(n)}=x_n'$$

Hence the given *n*th-order differential equation can be expressed as an *n*th-order system:

$$\begin{aligned}
x_1' &= x_2\\
x_2' &= x_3\\
x_3' &= x_4\\
&\vdots\\
x_{n-1}' &= x_n\\
x_n' &= -\frac{a_0}{a_n}x_1-\frac{a_1}{a_n}x_2-\cdots-\frac{a_{n-1}}{a_n}x_n+f(t).
\end{aligned}$$

Inspection of this system reveals that it is in the form of an *n*th-order system.

Example 1

Reduce the third-order equation

$$2y''' = -y - 4y' + 6y'' + \sin t$$

or $$2y''' - 6y'' + 4y' + y = \sin t$$

to the normal form.

Solution: Write the differential equation as

$$y''' = -\frac{1}{2}y - 2y' + 3y'' + \frac{1}{2}\sin t$$

Now introduce the variables

$$y = x_1, y' = x_2, y'' = x_3.$$

Then

$$x_1' = y' = x_2$$
$$x_2' = y'' = x_3$$
$$x_3' = y'''$$

Hence, we can write the given differential equation in the linear normal form

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = -\frac{1}{2}x_1 - 2x_2 + 3x_3 + \frac{1}{2}\sin t$$

Example 2

Rewrite the given second order differential equation as a system in the normal form

$$2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 5y = 0$$

Solution:

We write the given the differential equation as

$$\frac{d^2y}{dx^2} = -2\frac{dy}{dx} + \frac{5}{2}y$$

Now introduce the variables

$$y = x_1, \;\; y' = x_2$$

Then

$$y' = x_1' = x_2$$
$$y'' = x_2'$$

So that the given differential equation can be written in the form of a system

$$x_1' = x_2$$
$$x_2' = -2x_2 + \frac{5}{2}x_1$$

This is the linear normal or canonical form.

Example 3
Write the following differential equation as an equivalent system in the Canonical form.

$$4\frac{d^3y}{dt^3}+y=e^t$$

Solution:
First write the given differential equation as

$$4\frac{d^3y}{dt^3}=-y+e^t$$

dividing by 4 on both sides

or $$\frac{d^3y}{dt^3}=-\frac{1}{4}y+\frac{1}{4}e^t$$

Now introduce the variables

$$y=x_1,\ y'=x_2\quad y''=x_3$$

Then

$$y'=x_1'=x_2$$
$$y''=x_2'=x_3$$
$$y'''=x_3'$$

Hence, the given differential equation can be written as an equivalent system.

$$x_1'=x_2$$
$$x_2'=x_3$$
$$x_3'=-\frac{1}{4}x_1+\frac{1}{4}e^t$$

Clearly, this system is in the linear normal or the Canonical form.

Example 4
Rewrite the differential equation in the linear normal form

$$t^2y''+ty'+(t^2-4)y=0$$

Solution:
First we write the equation in the form

$$t^2y''=-ty'-\left(t^2-4\right)y$$

or $$y''=-\frac{1}{t}y'-\frac{\left(t^2-4\right)}{t^2}y,\quad t\neq 0$$

or $$y''=-\frac{1}{t}y'-\frac{t^2-4}{t^2}y$$

Then introduce the variables

$$y=x_1,\ y'=x_2$$

Then

$$y' = x_1' = x_2$$
$$y'' = x_2'$$

Hence, the given equation is equivalent to the following system.

$$x_1' = x_2$$
$$x_2' = -\frac{1}{t}x_2 - \frac{t^2-4}{t^2}x_1$$

The system is in the required linear normal or the cnonical form.

Systems Reduced to Normal Form

Using Procedure similar to that used for a single equation, we can reduce most systems of the linear form

$$\begin{aligned}
P_{11}(D)x_1 + P_{12}(D)x_2 + \cdots + P_{1n}(D)x_n &= b_1(t)\\
P_{21}(D)x_1 + P_{22}(D)x_2 + \cdots + P_{2n}(D)x_n &= b_2(t)\\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad &\vdots\\
P_{n1}(D)x_1 + P_{n2}(D)x_2 + \cdots + P_{nn}(D)x_n &= b_n(t)
\end{aligned}$$

to the cano nical form. To accom plish this we need to solv e the system for the hi ghest order derivative of each dependent variable.

Note:
It is not always possible to solve the given system for the highest-order derivative of each dependent variable.

Example 5
Reduce the following system to the normal form.

$$\left(D^2 - D + 5\right)x + 2D^2y = e^t$$
$$-2x + \left(D^2 + 2\right)y = 3t^2$$

Solution:

First write the given system in the differential operator form

$$D^2x + 2D^2y = e^t - 5x + Dx$$
$$D^2y = 3t^2 + 2x - 2y$$

Then eliminate D^2y by multiplying the s econd equation by 2 and subtractin g from first equation to have

$$D^2x = e^t - 6t^2 - 9x + 4y + Dx.$$

Also $$D^2 y = 3t^2 + 2x - 2y$$
We are now in a position to introduce the new variables. Therefore, we suppose that

$$Dx = u, \quad Dy = v$$
Thus, the expressions for $D^2 x$ and $D^2 y$, respectively, become

$$Du = e^t - 6t^2 - 9x + 4y + u$$
$$Dv = 3t^2 + 2x - 2y.$$

Thus the original system can be written as
$$Dx = u$$
$$Dy = v$$
$$Du = -9x + 4y + u + e^t - 6t^2$$
$$Dv = 2x - 2y + 3t^2$$
Clearly, this system is in the canonical form.

Example 6

If possible, re-write the given system in the canonical form
$$x' + 4x - y' \qquad = 7t$$
$$x' + \qquad y' - 2y = 3t$$
Solution:

First we write the differential equations of the system in the differential operator form
$$Dx + 4x - Dy \qquad = 7t$$
$$Dx \qquad + Dy - 2y = 3t$$
To eliminate Dy we add the two equations of the system, to obtain
$$2Dx = 10t - 4x + 2y$$
or $$Dx = -2x + y + 5t$$
Next to solve for the Dy, we eliminate Dx. For this purpose we simply subtract the first equation from second equation of the system, to have
$$-4x + 2Dy - 2y = -4t$$
$$2Dy = 4x + 2y - 4t$$
or $$Dy = 2x + y - 2t$$
Hence the original system is equivalent to the following system
$$Dx = -2x + y + 5t$$
$$Dy = 2x + y - 2t$$
Clearly the system is in the normal form.

Example 7

If possible, re-write the given system in the linear normal form

$$\frac{d^3x}{dt^3} = 4x - 3\frac{d^2x}{dt^2} + 4\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = 10t^2 - 4\frac{dx}{dt} + 3\frac{dy}{dt}$$

Solution:
First write the given system in the differential operator form

$$D^3x = 4x - 3D^2x + 4Dy$$

$$D^2y = 10t^2 - 4Dx + 3Dy$$

No need to elim inate anything as the equations are already expres sing the highest-order derivatives of x and y in terms of the remaining functions and derivatives. Therefore, we are now in a position to introduce new variables. Suppose that

$$Dx = u, \quad Dy = v$$

$$D^2x = Du = w$$

$$D^2y = Dv, \quad D^3x = Dw$$

Then the expressions for D^3x and for D^2y can be written as

$$Dw = 4x + 4v - 3w$$

$$Dv = 10t^2 - 4u + 3v$$

Hence, the given system of differential equations is equivalent to the following system

$$Dx = u$$

$$Dy = v$$

$$Du = w$$

$$Dv = 10t^2 - 4u + 3v$$

$$Dw = 4x + 4v - 3w$$

This new system is clearly in the linear normal form.

Degenerate Systems
The systems of differential equations of the form

$$\begin{array}{l} P_{11}(D)x_1 + P_{12}(D)x_2 + \cdots + P_{1n}(D)x_n = b_1(t) \\ P_{21}(D)x_1 + P_{22}(D)x_2 + \cdots + P_{2n}(D)x_n = b_2(t) \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ P_{n1}(D)x_1 + P_{n2}(D)x_2 + \cdots + P_{nn}(D)x_n = b_n(t) \end{array}$$

those cannot be reduced to a linear system in norm al for m is said to be a degenerate system.

Example 8

If possible, re-write the following system in a linear normal form

$$(D+1)x+(D+1)y=0$$
$$2Dx+(2D+1)y=0$$

Solution:

The given system is already written in the differential operator form. The system can be written in the form

$$Dx+x\ +\ Dy+y=0$$
$$2Dx\quad +2Dy\ +y=0$$

We eliminate Dx to solve for the highest derivative Dy by multiplying the first equation with 2 and then subtracting second equation from the first one. Thus we have

$$\begin{array}{l} 2Dx+2x+2Dy+2y=0 \\ \underline{\pm 2Dx \qquad \pm\ 2Dy\pm\ y=0} \\ \qquad 2x \qquad\qquad +\ y=0 \end{array}$$

Therefore, it is impossible to solve the system for the highest derivative of each dependent variable; the system cannot be reduced to the canonical form. Hence the system is a degenerate.

Example 9

If possible, re-write the following system of differential equations in the canonical form

$$x''+y'=1$$
$$x''+y'=-1$$

Solution:

We write the system in the operator form

$$D^2x+Dy=\ \ 1$$
$$D^2x+Dy=-1$$

To solve for a highest order derivative of y in terms of the remaining functions and derivatives, we subtract the second equation from the first and we obtain

$$\begin{array}{l} D^2x+Dy=\ \ 1 \\ \underline{\pm D^2x\pm Dy=-1} \\ \qquad 0=2 \end{array}$$

This is absurd. Thus the given system cannot be reduced to a canonical form. Hence the system is a degenerate system.

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Example 10

If possible, re-write the given system

$$(2D+1)x-2Dy = 4$$
$$Dx - Dy = e^t$$

Solution:

The given system is already in the operator form and can be written as

$$2Dx+x-2Dy = 4$$
$$Dx - Dy = e^t$$

To solve for the highest derivative Dy, we eliminate the highest derivative Dx. Therefore, multiply the second equation with 2 and then subtract from the first equation to have

$$\begin{aligned} 2Dx+x-2Dy &= 4 \\ \pm 2Dx \quad \mp 2Dy &= \pm 2e^t \\ \hline x &= 4-2e^t \end{aligned}$$

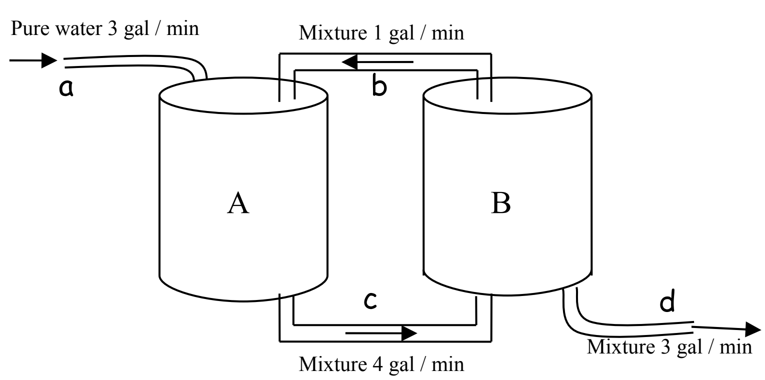
Therefore, it is impossible to solve the system for the highest derivatives of each variable. Thus the system cannot be reduced to the linear normal form. Hence, the system is a degenerate system.

Applications

The systems having the linear normal form arise naturally in some physical applications. The following example provides an application of a homogeneous linear normal system in two dependent variables.

Example 11

Tank A contains 50 gallons of water in which 25 pounds of salt are dissolved. A second tank B contains 50 gallons of pure water. Liquid is pumped in and out of the tank at rates shown in Figure. Derive the differential equations that describe the number of pounds $x_1(t)$ and $x_2(t)$ of salt at any time in tanks A and B, respectively.



Solution:

Tank A

Input through pipe a $= (3 \text{ gal/min}) \cdot (0 \text{ lb/gal}) = 0$

Input through pipe b $= (1 \text{ gal/min}) \cdot \left(\dfrac{x_2}{50} \text{lb/gal} \right) = \dfrac{x_2}{50} \text{ lb/min}$

Thus, total input for the tank $A = 0 + \dfrac{x_2}{50} = \dfrac{x_2}{50}$

Output through pipe $c = (4 \text{ gal/min}) \cdot \left(\dfrac{x_1}{50} \text{lb/gal} \right) = \dfrac{4x_1}{50} \text{lb/min}$

Hence, the net rate of change of $x_1(t)$ in lb / min is given by

$$\frac{dx_1}{dt} = input \text{ - } output$$

or $$\frac{dx_1}{dt} = \frac{x_2}{50} - \frac{4x_1}{50}$$

or $$\frac{dx_1}{dt} = \frac{-2}{25} x_1 + \frac{x_2}{50}$$

Tank B

Input through pipe c is $4 \text{ gal/min} = \dfrac{4x_1}{50} \text{lb/min}$

Output through pipe b is $1 \text{ gal/min} = \dfrac{x_2}{50} \text{lb/min}$

Similarly output through pipe d is $3 \text{ gal/min} = \dfrac{3x_2}{50} \text{lb/min}$

Total output for the tank $b = \dfrac{x_2}{50} + \dfrac{3x_2}{50} = \dfrac{4x_2}{50}$

Hence, the net rate of change of $x_2(t)$ in lb/min

$$\frac{dx_2}{dt} = input - output$$

or $$\frac{dx_2}{dt} = \frac{4x_1}{50} - \frac{4x_2}{50}$$

or $$\frac{dx_2}{dt} = \frac{2x_1}{25} - \frac{2x_2}{25}$$

Thus we obtain the first order system

$$\frac{dx_1}{dt} = \frac{-2}{25}x_1 + \frac{x_2}{50}$$

$$\frac{dx_2}{dt} = \frac{2x_1}{25} - \frac{2x_2}{25}$$

We observe that the foregoing system is accompanied the initial conditions

$$x_1(0) = 25, \ x_2(0) = 0.$$

Exercise

Rewrite the given differential equation as a system in linear normal form.

1. $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 4y = \sin 3t$
2. $y''' - 3y'' + 6y' - 10y = t^2 + 1$
3. $\frac{d^4y}{dt^4} - 2\frac{d^2y}{dt^2} + 4\frac{dy}{dx} + y = t$
4. $2\frac{d^4y}{dt^4} + \frac{d^3y}{dt^3} - 8y = 10$

Rewrite, if possible, the given system in the linear normal form.

5. $(D-1)x - Dy = t^2, \quad x + Dy = 5t - 2$
6. $x'' - 2y'' = \sin t, \quad x'' + y'' = \cos t$
7. $m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1), \qquad m_2 x_2'' = -\ k_2(x_2 - x_1)$
8. $D^2 x + Dy = 4t, \qquad -D^2 x + (D+1)y = 6t^2 + 10$

Lecture 24
Introduction to Matrices

Matrix

A rectangular array of numbers or functions subject to certain rules and conditions is called a matrix. Matrices are denoted by capital letters $A, B, \ldots, Y, Z$. The numbers or functions are called elements or entries of the matrix. The elements of a matrix are denoted by small letters $a, b, \ldots, y, z$.

Rows and Columns

The horizontal and vertical lines in a matrix are, respectively, called the rows and columns of the matrix.

Order of a Matrix

If a matrix has m rows and n columns then we say that the size or order of the matrix is $m \times n$. If A is a matrix having m rows and n columns then the matrix can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

Square Matrix

A matrix having n rows and n columns is said to be a $n \times n$ square matrix or a square matrix of order n. The element, or entry, in the ith row and jth column of a $m \times n$ matrix A is written as a_{ij}. Therefore a 1 x 1 matrix is simply a constant or a function.

Equality of matrix

Any two matrices A and B are said to be equal if and only if they have the same orders and the corresponding elements of the two matrices are equal. Thus if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ then

$$A = B \Leftrightarrow a_{ij} = b_{ij}, \quad \forall\, i, j$$

Column Matrix

A column matrix X is any matrix having n rows and only one column. Thus the column matrix X can be written as

$$X = \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \\ \vdots \\ b_{n1} \end{pmatrix} = [b_{i1}]_{n\times 1}$$

A column matrix is also called a column vector or simply a vector.

Multiple of matrices

A multiple of a matrix A is defined to be

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix} = [ka_{ij}]_{m\times n}$$

Where k is a constant or it is a function. Notice that the product kA is sa me as the product Ak . Therefore, we can write

$$kA = Ak$$

Example 1

(a)
$$5\cdot\begin{bmatrix} 2 & -3 \\ 4 & -1 \\ 1/5 & 6 \end{bmatrix} = \begin{bmatrix} 10 & -15 \\ 20 & -5 \\ 1 & 30 \end{bmatrix}$$

(b)
$$e^t\cdot\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} e^t \\ -2e^t \\ 4e^t \end{bmatrix}$$

Since we know that $kA = Ak$. Therefore, we can write

$$e^{-3t}\cdot\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2e^{-3t} \\ 5e^{-3t} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} e^{-3t}$$

Addition of Matrices

Any two matrices can be added only when they have same orders and the resulting matrix is obtained by adding the correspon ding entries. Therefore, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $m\times n$ matrices then their sum is defined to be the matrix $A + B$ defined by

$$A + B = [a_{ij} + b_{ij}]$$

Example 2

Consider the following two matrices of order 3×3

$$A=\begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \\ -6 & 10 & -5 \end{pmatrix},\quad B=\begin{pmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{pmatrix}$$

Since the given m atrices have sam e orders. Therefore, these m atrices can be added and their sum is given by

$$A+B=\begin{pmatrix} 2+4 & -1+7 & 3+(-8) \\ 0+9 & 4+3 & 6+5 \\ -6+1 & 10+(-1) & -5+2 \end{pmatrix}=\begin{pmatrix} 6 & 6 & -5 \\ 9 & 7 & 11 \\ -5 & 9 & -3 \end{pmatrix}$$

Example 3

Write the following single column matrix as the sum of three column vectors

$$\begin{pmatrix} 3t^2-2e^t \\ t^2+7t \\ 5t \end{pmatrix}$$

Solution

$$\begin{pmatrix} 3t^2-2e^t \\ t^2+7t \\ 5t \end{pmatrix}=\begin{pmatrix} 3t^2 \\ t^2 \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ 7t \\ 5t \end{pmatrix}+\begin{pmatrix} -2e^t \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}t^2+\begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix}t+\begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}e^t$$

Difference of Matrices

The difference of two matrices A and B of same order $m\times n$ is defined to be the matrix

$$A-B=A+(-B)$$

The matrix $-B$ is obtained by multiplying the matrix B with -1. So that

$$-B=(-1)\,B$$

Multiplication of Matrices

Any two matrices A and B are conform able for the product AB, if the number of columns in the f irst m atrix A is equal to the number of rows in the second m atrix B. Thus if the order of the m atrix A is $m\times n$ then to m ake the product AB possible order of the matrix B must be $n\times p$. Then the order of the product matrix AB is $m\times p$. Thus

$$A_{m\times n}\cdot B_{n\times p}=C_{m\times p}$$

If the matrices A and B are given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} \cdots a_{11}b_{1p} + a_{12}b_{2p} + \cdots + a_{1n}b_{np} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1} \cdots a_{21}b_{1p} + a_{22}b_{2p} + \cdots + a_{2n}b_{np} \\ \vdots \\ \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mn}b_{n1} \cdots a_{m1}b_{1p} + a_{m2}b_{2p} + \cdots + a_{mn}b_{np} \end{pmatrix}$$

$$= \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)_{n \times p}$$

Example 4

If possible, find the products AB and BA, when

(a) $A = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 9 & -2 \\ 6 & 8 \end{pmatrix}$

(b) $A = \begin{pmatrix} 5 & 8 \\ 1 & 0 \\ 2 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & -3 \\ 2 & 0 \end{pmatrix}$

Solution

(a) The matrices A and B are square matrices of order 2. Therefore, both of the products AB and BA are possible.

$$AB = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 9 & -2 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 4\cdot 9 + 7\cdot 6 & 4\cdot(-2) + 7\cdot 8 \\ 3\cdot 9 + 5\cdot 6 & 3\cdot(-2) + 5\cdot 8 \end{pmatrix} = \begin{pmatrix} 78 & 48 \\ 57 & 34 \end{pmatrix}$$

Similarly
$$BA = \begin{pmatrix} 9 & -2 \\ 6 & 8 \end{pmatrix}\begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 9\cdot 4 + (-2)\cdot 3 & 9\cdot 7 + (-2)\cdot 5 \\ 6\cdot 4 + 8\cdot 3 & 6\cdot 7 + 8\cdot 5 \end{pmatrix} = \begin{pmatrix} 30 & 53 \\ 48 & 82 \end{pmatrix}$$

(b) The product AB is possible as th e num ber of colum ns in the m atrix A and the number of rows in B is 2. However, the product BA is not possible because the number of rows in the matrix B and the number of rows in A is not same.

$$AB = \begin{pmatrix} 5\cdot(-4) + 8\cdot 2 & 5\cdot(-3) + 8\cdot 0 \\ 1\cdot(-4) + 0\cdot 2 & 1\cdot(-3) + 0\cdot 0 \\ 2\cdot(-4) + 7\cdot 2 & 2\cdot(-3) + 7\cdot 0 \end{pmatrix} = \begin{pmatrix} -4 & -15 \\ -4 & -3 \\ 6 & -6 \end{pmatrix}$$

Note that

In gene ral, m atrix m ultiplication is not com mutative. This m eans tha t $AB \neq BA$. For example, we observe in part (a) of the previous example

$$AB = \begin{pmatrix} 78 & 48 \\ 57 & 34 \end{pmatrix}, \quad BA = \begin{pmatrix} 30 & 53 \\ 48 & 82 \end{pmatrix}$$

Clearly $AB \neq BA$.. Similarly in part (b) of the example, we have

$$AB = \begin{pmatrix} -4 & -15 \\ -4 & -3 \\ 6 & -6 \end{pmatrix}$$

However**,** the product BA is not possible.

Example 5

(a)

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ 1 & -7 & 9 \end{pmatrix}\begin{pmatrix} -3 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\cdot(-3)+(-1)\cdot 6+3\cdot 4 \\ 0\cdot(-3)+4\cdot 6+5\cdot 6 \\ 1\cdot(-3)+(-7)\cdot 6+9\cdot 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 44 \\ -9 \end{pmatrix}$$

(b)

$$\begin{pmatrix} -4 & 2 \\ 3 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4x+2y \\ 3x+8y \end{pmatrix}$$

Multiplicative Identity

For a given positive integer n, the $n\times n$ matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

is called the multiplicative identity matrix. If A is a matrix of order $n\times n$, then it can be verified that

$$I\cdot A = A\cdot I = A$$

Also, it is readily verified that if X is any $n\times 1$ column matrix, then $I\cdot X = X$

Zero Matrix

A matrix consisting of all zero entries is called a zero matrix or null matrix and is denoted by O. For example

$$O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and so on. If A and O are $m\times n$ matrices, then

$$A+O = O+A = A$$

Associative Law

The matrix multiplication is associative. This means that if A, B and C are $m\times p$, $p\times r$ and $r\times n$ matrices, then

$$A(BC) = (AB)C$$

The result is a $m\times n$ matrix.

Distributive Law

If B and C are matrices of order $r\times n$ and A is a matrix of order $m\times r$, then the distributive law states that

$$A(B+C) = AB + AC$$

Furthermore, if the product $(A+B)C$ is defined, then

$$(A+B)C = AC + BC$$

Determinant of a Matrix

Associated with every square matrix A of constants, there is a number called the determinant of the matrix, which is denoted by $\det(A)$ or $|A|$

Example 6

Find the determinant of the following matrix

$$A = \begin{pmatrix} 3 & 6 & 2 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{pmatrix}$$

Solution

The determinant of the matrix A is given by

$$\det(A) = \begin{vmatrix} 3 & 6 & 2 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{vmatrix}$$

We expand the $\det(A)$ by cofactors of the first row, we obtain

$$\det(A) = \begin{vmatrix} 3 & 6 & 2 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{vmatrix} = 3\begin{vmatrix} 5 & 1 \\ 2 & 4 \end{vmatrix} - 6\begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix}$$

or $$\det(A) = 3(20 - 2) - 6(8+1) + 2(4+5) = 18$$

Transpose of a Matrix

The transpose of a $m \times n$ matrix A is obtained by interchanging rows and columns of the matrix and is denoted by A^{tr}. In other words, rows of A become the columns of A^{tr}. If

$$A = \begin{pmatrix} a_{11} & a_{12} & . & . & . & \cdots & a_{1n} \\ a_{21} & a_{22} & . & . & . & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \cdots & . & \cdots & \vdots \\ a_{m1} & a_{m2} & . & . & . & . & a_{mn} \end{pmatrix}$$

Then

$$A^{tr} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Since order of the matrix A is $m \times n$, the order of the transpose matrix A^{tr} is $n \times m$.

Example 7

(a) The transpose of matrix

$$A = \begin{pmatrix} 3 & 6 & 2 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{pmatrix}$$

is

$$A^{tr} = \begin{pmatrix} 3 & 2 & -1 \\ 6 & 5 & 2 \\ 2 & 1 & 4 \end{pmatrix}$$

(b) If X denotes the matrix

$$X = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$$

Then $$X^{tr} = [5 \quad 0 \quad 3]$$

Multiplicative Inverse of a Matrix

Suppose that A is a square matrix of order $n \times n$. If there exists an $n \times n$ matrix B such that

$$AB = BA = I$$

Then B is said to be the multiplicative inverse of the matrix A and is denoted by $B = A^{-1}$.

Non-Singular Matrices

A square matrix A of order $n \times n$ is said to be a non-singular matrix if

$$\det(A) \neq 0$$

Otherwise the square matrix A is said to be singular. Thus for a singular A we must have

$$\det(A) = 0$$

Theorem

If A is a square matrix of order $n \times n$ then the matrix has a multiplicative inverse A^{-1} if and only if the matrix A is non-singular.

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Theorem

Let A be a non singular matrix of order $n \times n$ and let C_{ij} denote the cofactor (signed minor) of the corresponding entry a_{ij} in the matrix A i.e.

$$C_{ij} = (-1)^{i+j} M_{ij}$$

M_{ij} is the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and jth column from A. Then inverse of the matrix A is given by

$$A^{-1} = \frac{1}{\det(A)} (C_{ij})^{tr}$$

Further Explanation

1. For further reference we take $n = 2$ so that A is a 2×2 non-singular matrix given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Therefore $C_{11} = a_{22}$, $C_{12} = -a_{21}$, $C_{21} = -a_{12}$ and $C_{22} = a_{11}$. So that

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}^{tr} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

2. For a 3×3 non-singular matrix

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$C_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, C_{12}=-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad C_{13}=\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

and so on. Therefore, inverse of the matrix A is given by

$$A^{-1}=\frac{1}{\det A}\begin{pmatrix} C_{11} & C_{21} & a_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}.$$

Example 8

Find, if possible, the multiplicative inverse for the matrix

$$A=\begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix}.$$

Solution:

The matrix A is non-singular because

$$\det(A)=\begin{vmatrix} 1 & 4 \\ 2 & 10 \end{vmatrix}=10 - 8 = 2$$

Therefore, A^{-1} exists and is given by

.

$$A^{-1}=\frac{1}{2}\begin{pmatrix} 10 & -4 \\ -2 & 1 \end{pmatrix}=\begin{pmatrix} 5 & -2 \\ -1 & 1/2 \end{pmatrix}$$

Check

$$AA^{-1}=\begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -1 & 1/2 \end{pmatrix}=\begin{pmatrix} 5-4 & -2+2 \\ 10-10 & -4+5 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=I$$

$$AA^{-1}=\begin{pmatrix} 5 & -2 \\ -1 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix}=\begin{pmatrix} 5-4 & 20-20 \\ -1+1 & -4+5 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=I$$

Example 9

Find, if possible, the multiplicative inverse of the following matrix

$$A = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

Solution:

The matrix is singular because

$$\det(A) = \begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} = 2\cdot 3 - 2\cdot 3 = 0$$

Therefore, the multiplicative inverse A^{-1} of the matrix does not exist.

Example 10

Find the multiplicative inverse for the following matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ -2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}.$$

Solution:

Since $$\det(A) = \begin{vmatrix} 2 & 2 & 0 \\ -2 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = 2(1-0) - 2(-2-3) + 0(0-3) = 12 \neq 0$$

Therefore, the given matrix is non singular. So that, the multiplicative inverse A^{-1} of the matrix A exists. The cofactors corresponding to the entries in each row are

$$C_{11} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1, \quad C_{12} = -\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = 5, \quad C_{13} = \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} = -3$$

$$C_{21} = -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = -2, \quad C_{22} = \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = 2, \quad C_{23} = -\begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} = 6$$

$$C_{31} = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2, \quad C_{32} = -\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = -2, \quad C_{33} = \begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} = 6$$

Hence $$A^{-1} = \frac{1}{12}\begin{pmatrix} 1 & -2 & 2 \\ 5 & 2 & -2 \\ -3 & 6 & 6 \end{pmatrix} = \begin{pmatrix} 1/12 & -1/6 & 1/6 \\ 5/12 & 1/6 & -1/6 \\ -1/4 & 1/2 & 1/2 \end{pmatrix}$$

Please verify that $A\cdot A^{-1} = A^{-1}\cdot A = I$

Derivative of a Matrix of functions

Suppose that

$$A(t) = \left[a_{ij}(t) \right]_{m\times n}$$

is a m atrix whose entries are functions t hose are differentiable on a comm on interval, then deriv ative of the m atrix $A(t)$ is a m atrix whose entries are deriv atives of the corresponding entries of the matrix $A(t)$. Thus

$$\frac{dA}{dt} = \left[\frac{da_{ij}}{dt} \right]_{m\times n}$$

The derivative of a matrix is also denoted by $A'(t)$.

Integral of a Matrix of Functions

Suppose that $A(t) = \left(a_{ij}(t)\right)_{m\times n}$ is a m atrix whose entrie s are f unctions those are continuous on a common interval containing t, then in tegral of the m atrix $A(t)$ is a matrix whose entries are integrals of the corresponding entries of the matrix $A(t)$. Thus

$$\int_{t_0}^{t} A(s)ds = \left(\int_{t_0}^{t} a_{ij}(s)ds \right)_{m\times n}$$

Example 11

Find the derivative and the integral of the following matrix

$$X(t) = \begin{pmatrix} \sin 2t \\ e^{3t} \\ 8t-1 \end{pmatrix}$$

Solution:

The derivative and integral of the given matrix are, respectively, given by

$$X'(t) = \begin{pmatrix} \frac{d}{dt}(\sin 2t) \\ \frac{d}{dt}(e^{3t}) \\ \frac{d}{dt}(8t-1) \end{pmatrix} = \begin{pmatrix} 2\cos 2t \\ 3e^{3t} \\ 8 \end{pmatrix}$$

$$\int_0^t X(s)ds = \begin{pmatrix} \int_0^t \sin 2tds \\ \int_0^t e^{3t}ds \\ \int_0^t 8t-1ds \end{pmatrix} = \begin{pmatrix} -1/2\cos 2t + 1/2 \\ 1/3e^{3t} - 1/3 \\ 4t^2 - t \end{pmatrix}$$

Augmented Matrix

Consider an algebraic system of n linear equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Suppose that A denotes the coefficient matrix in the above algebraic system, then

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

It is well known that Cramer's rule can be used to solve the system, whenever $\det(A) \neq 0$. However, it is also well known that a Hercul ean effort is required to solve the system if $n > 3$. Thus for larger systems the Gaussian and Gauss-Jordon elimination methods are preferred and in these methods we apply elementary row operations on augmented matrix. The augm ented matrix of the system of linear e quations is the following $n \times (n+1)$ matrix

$$A_b = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{211} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}$$

If B denotes the colum n matrix of the b_i, $\forall i = 1, 2, \ldots, n$ then the a ugmented matrix of the above mentioned system of linear algebraic equations can be written as $(A \mid B)$.

Elementary Row Operations

The elementary row operations consist of the following three operations

- Multiply a row by a non-zero constant.
- Interchange any row with another row.
- Add a non-zero constant multiple of one row to another row.

These row operations on the augmented matrix of a system are equivalent to, multiplying an equation by a non-zero constant, interchang ing position of any tw o equations of the system and adding a constant multiple of an equation to another equation.

The Gaussian and Gauss-Jordon Methods

In the Gaussian Elim ination m ethod we carry out a succession of elem entary row operations on the augmented matrix of the system of linear equations to be solved until it is transformed into row-echelon form, a matrix that has the following structure:

- The first non-zero entry in a non-zero row is 1.
- In consecutive nonzero rows the first entry 1 in the lower row appears to the right of the first 1 in the higher row.
- Rows consisting of all 0's are at the bottom of the matrix.

In the Gauss-Jordan method the row operations are continued until the augmented matrix is transformed into the reduced row-echelon form**.** A reduced row-echelon matrix has the structure similar to row-echelon, but with an additional property.

- The first non-zero entry in a non-zero row is 1.
- In consecutive nonzero rows the first entry 1 in the lower row appears to the right of the first 1 in the higher row.
- Rows consisting of all 0's are at the bottom of the matrix.
- A column containing a first entry 1 has 0's everywhere else.

Example 1

(a) The following two matrices are in row-echelon form.

$$\left(\begin{array}{ccc|c} 1 & 5 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{ccccc|c} 0 & 0 & 1 & -6 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

Please verify that the three conditions of the structure of the echelon form are satisfied.

(b) The following two matrices are in reduced row-echelon form**.**

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{ccccc|c} 0 & 0 & 1 & -6 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

Please notice that all remaining entries in the columns containing a leading entry 1 are 0.

Notation

To keep track of the row operations on an augmented matrix, we utilized the following notation:

Symbol	Meaning
R_{ij}	Interchange the rows i and j.
cR_i	Multiply the ith row by a nonzero constant c.
$cR_i + R_j$	Multiply the ith row by c and then add to the jth row.

Example 2

Solve the following system of linear algebraic equations by the (a) Gaussian elimination and (b) Gauss-Jordan elimination

$$\begin{aligned} 2x_1 + 6x_2 + x_3 &= 7 \\ x_1 + 2x_1 - x_3 &= -1 \\ 5x_1 + 7x_2 - 4x_3 &= 9 \end{aligned}$$

Solution

(a) The augmented matrix of the system is

$$\left(\begin{array}{ccc|c} 2 & 6 & 1 & 7 \\ 1 & 2 & -1 & -1 \\ 5 & 7 & -4 & 9 \end{array}\right)$$

By interchanging first and second row i.e. by R_{12}, we obtain

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 6 & 1 & 7 \\ 5 & 7 & -4 & 9 \end{array}\right)$$

Multiplying first row with -2 and -5 and then adding to 2nd and 3rd row i.e. by $-R_1 + R_2$ and $-5R_1 + R_3$, we obtain

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 2 & 3 & 9 \\ 0 & -3 & 1 & 14 \end{array}\right)$$

Multiply the second row with $1/2$, i.e. the operation $\frac{1}{2}R_2$, yields

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & -3 & 1 & 14 \end{array}\right)$$

Next add three times the second row to the third row, the operation $3R_2 + R_3$ gives

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & 11/2 & 55/2 \end{array}\right)$$

Finally, multiply the third row with $2/11$. This means the operation $\frac{2}{11}R_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

The last matrix is in row-echelon form and represents the system

$$x_1 + x_2 - x_3 = 1$$
$$x_2 + \frac{3}{2}x_3 = 9/2$$
$$x_3 = 5$$

Now by the backward substitution we obtain the solution set of the given system of linear algebraic equations

$$x_1 = 10, \quad x_2 = -3, \quad x_3 = 5$$

(b) W start with the last matrix in part (a). Since the first in the second and third rows are 1's we must, in turn, making the remaining entries in the second and third columns 0s:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

Adding -2 times the 2nd row to first row, this means the operation $-2R_2 + R_1$, we have

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & -10 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

Finally by 4 times the third row to first and $-1/2$ times the third row to second row, i.e. the operations $4R_3 + R_1$ and $\frac{-1}{2}R_3 + R_2$, yields

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right).$$

The last matrix is now in reduce row-echelon form .Because of what the matrix means in terms of equations, it evident that the solution of the system

$$x_1 = 10, \quad x_2 = -3, \quad x_3 = 5$$

Example 3

Use the G auss-Jordan elim ination to solv e th e following system of linear algebraic equations.

$$x + 3y - 2z = -7$$
$$4x + y + 3z = 5$$
$$2x - 5y + 7z = 19$$

Solution:

The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 4 & 1 & 3 & 5 \\ 2 & -5 & 7 & 19 \end{array}\right)$$

$-4R_1 + R_2$ and $-2R_1 + R_3$ yields

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & -11 & 11 & 33 \\ 0 & -11 & 11 & 33 \end{array}\right)$$

$\frac{-1}{11}R_2$ and $\frac{-1}{11}R_3$ produces

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & 1 & -1 & -3 \\ 0 & 1 & -1 & -3 \end{array}\right)$$

$3R_2 + R_1$ and $-R_2 + R_3$ gives

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

In this case the las t matrix in redu ced row-echelon form implies that th e original system of three equations in three unknowns.

$$x + z = 2, \quad y - z = -3$$

We can assign an arbitrarily value to z. If we let $z = t,\ t \in R$, then we see that the system has infinitely many solutions:

$$x = 2 - t, \ y = -3 + t, \ z = t$$

Geometrically, these equations are the parametric equations for the line of intersection of the planes

$$x + 0y + 0z = 2, \ 0x + y - z = -3$$

Exercise

Write the given sum as a single column matrix

1. $3t\begin{pmatrix} 2 \\ t \\ -1 \end{pmatrix} + (t-1)\begin{pmatrix} -1 \\ -t \\ 3 \end{pmatrix} - 2\begin{pmatrix} 3t \\ 4 \\ -5t \end{pmatrix}$

2. $\begin{pmatrix} 1 & -3 & 4 \\ 2 & 5 & -1 \\ 0 & -4 & -2 \end{pmatrix}\begin{pmatrix} t \\ 2t-1 \\ -t \end{pmatrix} + \begin{pmatrix} -t \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 8 \\ -6 \end{pmatrix}$

Determine whether the given matrix is singular or non-singular. If singular, find A^{-1}.

3. $A = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 0 \\ -2 & 5 & -1 \end{pmatrix}$

4. $A = \begin{pmatrix} 4 & 1 & -1 \\ 6 & 2 & -3 \\ -2 & -1 & 2 \end{pmatrix}$

Find $\dfrac{dX}{dt}$

5. $X = \begin{pmatrix} \frac{1}{2}\sin 2t - 4\cos 2t \\ -3\sin 2t + 5\cos 2t \end{pmatrix}$

6. If $A(t) = \begin{pmatrix} e^{4t} & \cos \pi t \\ 2t & 3t^2 - 1 \end{pmatrix}$ then find (a) $\int_0^2 A(t)dt$, (b) $\int_0^t A(s)ds$.

7. Find the integral $\int_1^2 B(t)dt$ if $B(t) = \begin{pmatrix} 6t & 2 \\ 1/t & 4t \end{pmatrix}$

Solve the given system of equations by either Gaussian elimination or by the Gauss-Jordon elimination.

8. $$\begin{aligned} 5x - 2y + 4z &= 10 \\ x + y + z &= 9 \\ 4x - 3y + 3z &= 1 \end{aligned}$$

9. $$\begin{aligned} x_1 + x_2 - x_3 - x_4 &= -1 \\ x_1 + x_2 + x_3 + x_4 &= 3 \\ x_1 - x_2 + x_3 - x_4 &= 3 \\ 4x_1 + x_2 - 2x_3 + x_4 &= 0 \end{aligned}$$

10. $$\begin{aligned} x_1 + x_2 - x_3 + 3x_4 &= 1 \\ x_2 - x_3 - 4x_4 &= 0 \\ x_1 + 2x_2 - 2x_3 - x_4 &= 6 \\ 4x_1 + 7x_2 - 7x_3 &= 9 \end{aligned}$$

Lecture 25
The Eigenvalue problem

Eigenvalues and Eigenvectors

Let A be a $n \times n$ matrix. A number λ is said to be an eigenvalue of A if there exists a nonzero solution vector K of the system of linear differential equations:

$$AK = \lambda K$$

The solution vector K is said to be an eigenvector corresponding to the eigenvalue λ. Using properties of matrix algebra, we can write the above equation in the following alternative form

$$(A - \lambda I)K = 0$$

where I is the identity matrix.

If we let
$$K = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ \vdots \\ k_n \end{pmatrix}$$

Then the above system is same as the following system of linear algebraic equations

$$\begin{aligned} (a_{11} - \lambda) k_1 + a_{12}k_2 + \cdots + a_{1n}k_n &= 0 \\ a_{21}k_1 + (a_{22} - \lambda) k_2 + \cdots + a_{2n}k_n &= 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\ a_{n1}k_n + a_{n2}k_2 + \cdots + (a_{nn} - \lambda) k_n &= 0 \end{aligned}$$

Clearly, an obvious solution of this system is the trivial solution

$$k_1 = k_2 = \ldots = k_n = 0$$

However, we are seeking only a non-trivial solution of the system.

The Non-trivial solution

The non-trivial solution of the system exists only when

$$\det(A - \lambda I) = 0$$

This equation is called the characteristic equation of the matrix A. Thus the Eigenvalues of the matrix A are given by the roots of the characteristic equation. To find an eigenvector corresponding to an eigenvalue λ we simply solve the system of linear algebraic equations

$$\det(A - \lambda I) K = 0$$

This system of equations can be solved by applying the Gauss-Jordan elim ination to the augmented matrix

$$\left(A-\lambda I \;\middle|\; 0\right).$$

Example 4

Verify that the following column vector is an eigenvector

$$K=\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

is an eigenvector of the following 3×3 matrix

$$A=\begin{pmatrix}0 & -1 & -3\\ 2 & 3 & 3\\ -2 & 1 & 1\end{pmatrix}$$

Solution:

By carrying out the multiplication AK, we see that

$$AK=\begin{pmatrix}0 & -1 & -3\\ 2 & 3 & 3\\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}=(-2)\begin{pmatrix}-2\\2\\1\end{pmatrix}=(-2)K$$

Hence the number $\lambda=-2$ is an eigenvalue of the given matrix A.

Example 5

Find the eigenvalues and eigenvectors of

$$A=\begin{pmatrix}1 & 2 & 1\\ 6 & -1 & 0\\ -1 & -2 & -1\end{pmatrix}$$

Solution:

Eigenvalues

The characteristic equation of the matrix A is

$$\det(A-\lambda I)=\begin{vmatrix}1-\lambda & 2 & 1\\ 6 & -1-\lambda & 0\\ -1 & -2 & -1-\lambda\end{vmatrix}=0$$

Expanding the determinant by the cofactors of the second row, we obtain

$$-\lambda^3-\lambda^2+12\lambda=0$$

This is so much easy given below the explanation of the above kindly see it and let me know if you have any more query
L: STAND FOR LEMDA
(1-L)((-1-L) (-1-L) -0)-2(6(-1-L)-0) +1(6(-2) +1(-1-L) =0
(1-L)(1+L^2+2L)-2(-6-6L) +1(-12 -1-L) =0
(1-L)(1+L^2+2L)+12+12L+1(-13-L) =0
1+L^2+2L-L-L^3-2L^2+12+12L-13-L=0
-L^3-L^2+12L=0

$$\lambda(\lambda+4)(\lambda-3)=0$$

Hence the eigenvalues of the matrix are

$$\lambda_1=0,\ \lambda_2=-4,\ \lambda_3=3.$$

Eigenvectors

For $\lambda_1=0$ we have

$$(A-0\,|\,0)=\left(\begin{array}{ccc|c}1&2&1&0\\6&-1&0&0\\-1&-2&-1&0\end{array}\right)$$

By $-6R_1+R_2,\ R_1+R_3$

$$\left(\begin{array}{ccc|c}1&2&1&0\\0&-13&-6&0\\0&0&0&0\end{array}\right)$$

By $-\dfrac{1}{13}R_2$

$$\left(\begin{array}{ccc|c}1&2&1&0\\0&1&6/13&0\\0&0&0&0\end{array}\right)$$

By $-2R_2+R_1$

$$\left(\begin{array}{ccc|c}1&0&1/13&0\\0&1&6/13&0\\0&0&0&0\end{array}\right)$$

Thus we have the following equations in $k_1,\ k_2$ and k_3. The num ber k_3 can be cho sen arbitrarily

$$k_1 = -(1/13)k_3,\ k_2 = -(6/13)k_3$$

Choosing $k_3 = -13$, we get $k_1 = 1$ and $k_2 = 6$. Hence, the eigenvector correspondin g $\lambda_1 = 0$ is

$$K_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}$$

For $\lambda_2 = -4$, we have

$$(A+4\ |0) = \left(\begin{array}{ccc|c} 5 & 2 & 1 & 0 \\ 6 & 3 & 0 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right)$$

By $(-1)R_3,\ R_{32}$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 6 & 3 & 0 & 0 \\ 5 & 2 & 1 & 0 \end{array}\right)$$

By $-6R_1 + R_2, -5R_1 + R_3$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & -8 & 16 & 0 \end{array}\right)$$

By $-\frac{1}{9}R_2,\ -\frac{1}{8}R_3$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right)$$

By $-2R_2 + R_1,\ -R_2 + R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence we obtain the following two equations involving $k_1,\ k_2$ and k_3.

$$k_1 = -k_3,\ k_2 = 2k_3$$

Choosing $k_3 = 1$, we have $k_1 = -1,\ k_2 = 2$. Hence we have an eigenvector corresponding to the eigenvalue $\lambda_2 = -4$

$$K_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Finally, for $\lambda_3 = 3$, we have

$$\left(A - 3I \mid 0\right) = \left(\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 6 & -4 & 0 & 0 \\ -1 & -2 & -4 & 0 \end{array}\right)$$

By using the Gauss Jordon elimination as used for other values, we obtain (verify**!**)

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So that we obtain the equations

$$k_1 = -k_3, \quad k_2 = (-3/2)k_3$$

The choice $k_3 = -2$ leads to $k_1 = 2, \quad k_2 = 3$. Hence, we have the following eigenvector

$$K_3 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$

Note that

The component k_3 could be chosen as any nonzero number. Therefore, a nonzero constant multiple of an eigenvector is also an eigenvector.

Example 6

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 3 & 4 \\ -1 & 7 \end{pmatrix}$$

Solution:

From the characteristic equation of the given matrix is

$$\det\left(A-\lambda I\right)=\begin{vmatrix}3-\lambda & 4\\ -1 & 7-\lambda\end{vmatrix}=0$$

or $$(3-\lambda)(7-\lambda)+4=0\Rightarrow\left(\lambda-5\right)^2=0$$

Therefore, the characteristic equation has repeated real roots. Thus the matrix has an eigenvalue of multiplicity two.

$$\lambda_1=\lambda_2=5$$

In the case of a 2×2 matrix there is no need to use Gauss-Jordan elimination. To find the eigenvector(s) corresponding to $\lambda_1=5$ we resort to the system of linear equations

$$\left(A-5I\right)K=0$$

or in its equivalent form

$$\begin{aligned}-2k_1+4k_2&=0\\ k_1+2k_2&=0\end{aligned}$$

It is apparent from this system that

$$k_1=2k_2\,.$$

Thus if we choose $k_2=1$, we find the single eigenvector

$$K_1=\begin{pmatrix}2\\1\end{pmatrix}$$

Example 7

Find the eigenvalues and eigenvectors of

$$A=\begin{pmatrix}9&1&1\\1&9&1\\1&1&9\end{pmatrix}$$

Solution

The characteristic equation of the given matrix is

$$\det\left(A-\lambda I\right)=\begin{vmatrix}9-\lambda&1&1\\1&9-\lambda&1\\1&1&9-\lambda\end{vmatrix}=0$$

or $$\left(\lambda-11\right)\left(\lambda-8\right)^2=0\Rightarrow\lambda=11,\ 8,\ 8$$

Thus the eigenvalues of the matrix are $\lambda_1=11, \lambda_2=\lambda_3=8$

For $\lambda_1=11$, we have

$$\left(A-11I\,|\,0\right)=\left(\begin{array}{ccc|c}-2&1&1&0\\1&-2&1&0\\1&1&-2&0\end{array}\right)$$

The Gauss-Jordan elimination gives

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence, $k_1 = k_3$, $k_2 = k_3$. If $k_3 = 1$, then

$$K_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Now for $\lambda_2 = 8$ we have

$$(A - 8I \mid 0) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

Again the Gauss-Jordon elimination gives

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore, $k_1 + k_2 + k_3 = 0$

We are free to select two of the variab les arbitrarily. C hoosing, on the one hand, $k_2 = 1, k_3 = 0$ and, on the other, $k_2 = 0, k_3 = 1$, we obtain two linearly independent eigenvectors corresponding to a single eigenvalue

$$K_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \; K_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Note that

Thus we note th at when a $n \times n$ matrix A possesses n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, a set of n linearly independent eigenvectors $K_1, K_2, \ldots, K_n$ can be found.

However, when the characteris tic equation has repeated roots, it m ay not be possible to find n linearly independent eigenvectors of the matrix.

Exercise

Find the eigenvalues and eigenvectors of the given matrix.

1. $\begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}$
2. $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$

3. $\begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix}$

4. $\begin{pmatrix} 5 & -1 & 0 \\ 0 & -5 & 9 \\ 5 & -1 & 0 \end{pmatrix}$

5. $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 4 & 0 & 1 \end{pmatrix}$

6. $\begin{pmatrix} 0 & 4 & 0 \\ -1 & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

Show that the given matrix has complex eigenvalues.

7. $\begin{pmatrix} -1 & 2 \\ -5 & 1 \end{pmatrix}$

8. $\begin{pmatrix} 2 & -1 & 0 \\ 5 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix}$

Lecture 26
Matrices and Systems of Linear First-Order Equations

Matrix form of a system

Consider the following system of linear first-order differential equations

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)$$
$$\frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t)$$
$$\vdots$$
$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)$$

Suppose that X, $A(t)$ and $F(t)$, respectively, denote the following matrices

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \quad \mathrm{F}(\mathrm{t}) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

Then the system of differential equations can be written as

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

or simply

$$\frac{dX}{dt} = A(t)X + F(t)$$

If the system of differential equations is homogenous, then $F(t) = 0$ and we can write

$$\frac{dX}{dt} = A(t)X$$

Both the non-homogeneous and the homogeneous systems can also be written as

$$X' = AX + F, \quad X' = AX$$

Example 1
Write the following non-homogeneous system of differential equations in the matrix form

$$\frac{dx}{dt} = -2x + 5y + e^t - 2t$$

$$\frac{dy}{dt} = 4x - 3y + 10t$$

Solution:
If we suppose that

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

Then, the given non-homogeneous differential equations can be written as

$$\frac{dX}{dt} = \begin{pmatrix} -2 & 5 \\ 4 & -3 \end{pmatrix} X + \begin{pmatrix} e^t - 2t \\ 10t \end{pmatrix}$$

or
$$X' = \begin{pmatrix} -2 & 5 \\ 4 & -3 \end{pmatrix} X + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} -2 \\ 10 \end{pmatrix} t$$

Solution Vector
Consider a homogeneous system of differential equations

$$\frac{dX}{dt} = AX$$

A solution vector on an interval I of the homogeneous system is any column matrix

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

The entr ies of the solu tion ve ctor have to be differentia ble f unctions satisf ying e ach equation of the system on the interval I.

Example 2

Verify that

$$X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}, \quad X_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t} = \begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix}$$

are solution of the following system of the homogeneous differential equations

$$X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$$

on the interval $(-\infty, \infty)$

Solution:

Since

$$X_1 = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix} \Rightarrow X_1^{/} = \begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix}$$

Further

$$AX_1 = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix} = \begin{pmatrix} e^{-2t} - 3e^{-2t} \\ 5e^{-2t} - 3e^{-2t} \end{pmatrix}$$

or

$$AX_1 = \begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix} = X_1^{/}$$

Similarly

$$X_2 = \begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix} \Rightarrow X_2^{/} = \begin{pmatrix} 18e^{6t} \\ 30e^{6t} \end{pmatrix}$$

and

$$AX_2 = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix} = \begin{pmatrix} 3e^{6t} + 15e^{6t} \\ 15e^{6t} + 15e^{6t} \end{pmatrix}$$

or

$$AX_2 = \begin{pmatrix} 18e^{6t} \\ 30e^{6t} \end{pmatrix} = X_2^{/}$$

Thus, the vectors X_1 and X_2 satisfy the homogeneous linear system

$$X^{/} = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$$

Hence, the given vecto rs are solu tions of the given hom ogeneous system of di fferential equations.

Note that

Much of the theory of the system s of n linear first-order differential equations is sim ilar to that of the linear nth-order differential equations.

Initial –Value Problem

Let t_0 denote any point in some interval denoted by I and

$$X(t_o)=\begin{pmatrix} x_1(t_o) \\ x_2(t_o) \\ \vdots \\ x_n(t_o) \end{pmatrix}, \quad X_o=\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}$$

$\gamma_i; i=1,2,\ldots,n$ are given constants. Then the problem of solving the system of differential equations

$$\frac{dX}{dt}=A(t)X+F(t)$$

Subject to the initial conditions

$$X(t_0)=X_0$$

is called an initial value problem on the interval I.

Theorem: *Existence of a unique Solution*

Suppose that the entries of the matrices $A(t)$ and $F(t)$ in the system of differential equations

$$\frac{dX}{dt}=A(t)X+F(t)$$

being considered in the above mentioned initial value problem, are continuous functions on a common interval I that contains the point t_0. Then there exist a unique solution of the initial–value problem on the interval I.

Superposition Principle

Suppose that $X_1, X_2, \ldots, X_n$ be a set of solution vectors of the homogenous system

$$\frac{dX}{dt}=A(t)X$$

on an interval I. Then the principle of superposition states that linear combination

$$X=c_1X_1+c_2X_2+\cdots+c_kX_k$$

$c_i; i=1,2,\ldots,k$ being arbitrary constants, is also a solution of the system on the same interval I.

Note that
An immediate consequence of the principle of superposition is that a constant multiple of any solution vector of a homogenous system of first order differential equation is also a solution of the system.

Example 3

Consider the following homogeneous system of differential equations

$$X' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} X$$

Also consider a solution vector X_1 of the system that is given by

$$X_1 = \begin{pmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{pmatrix}$$

For any constant c_1 the vector $X = c_1 X_1$ is also a solution of the hom ogeneous system. To verify this we differentiae the vector X with respect to t

$$\frac{dX}{dt} = c_1 \frac{dX}{dt} = c_1 \begin{pmatrix} -\sin t \\ \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t + \sin t \end{pmatrix}$$

Also

$$AX = c_1 \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{pmatrix}$$

$$AX = c_1 \begin{pmatrix} -\sin t \\ \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t + \sin t \end{pmatrix}$$

Thus, we have verified that**:**

$$\frac{dX}{dt} = AX$$

Hence the vector $c_1 X_1$ is also a solution vector of the homogeneous system of differential equations.

Example 4

Consider the following system considered in the previous example 4

$$X' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} X$$

We know from the previous example that the vector X_1 is a solution of the system

$$X_1 = \begin{pmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{pmatrix}$$

If $$X_2 = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$$

Then $$X'_2 = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$$

and $$AX_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$$

Therefore

$$AX_2 = X'_2$$

Hence the vector X_2 is a solution vector of the hom ogeneous system. We can verify that the following vector is also a solution of the homogeneous system.

$$X = c_1 X_1 + c_2 X_2$$

or $$X = c_1 \begin{pmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$$

Linear Dependence of Solution Vectors

Let $X_1, X_2, X_3, \ldots, X_k$ be a set of solution vectors, on an interval I, of the homogenous system of differential equations

$$\frac{dX}{dt} = AX$$

We say that the set is linearly dependent on I if there exist constants $c_1, c_2, c_3 \ldots, c_k$ not all zero such that

$$X(t) = c_1 X_1(t) + c_2 X_2(t) + \cdots + c_k X_k(t) = 0, \qquad \forall\ t \in I$$

Note that

- Any two solution vectors X_1 and X_2 are linearly dependent if and only if one of the two vectors is a constant multiple of the other.
- For $k > 2$ if the set of k solution vectors is linearly dependent then we can express at least one of the solution vectors as a linear combination of the remaining vectors.

Linear Independence of Solution Vectors

Suppose that $X_1, X_2, \ldots, X_k$ is a set of solution vectors, on an interval I, of the homogenous system of differential equations

$$\frac{dX}{dt} = AX$$

Then the set of solution vectors is said to be linearly independent if it is not linearly dependent on the interval I. This means that

$$X(t) = c_1 X_1(t) + c_2 X_2(t) + \cdots + c_k X_k(t) = 0$$

only when each $c_i = 0$.

Example 5

Consider the following two column vectors

$$X_1 = \begin{pmatrix} 3e^t \\ e^t \end{pmatrix}, \quad X_2 = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

Since
$$\frac{dX_1}{dt} = \begin{pmatrix} 3e^t \\ e^t \end{pmatrix}, \quad \frac{dX_2}{dt} = \begin{pmatrix} -e^{-t} \\ -e^{-t} \end{pmatrix}$$

and
$$\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 3e^t \\ e^t \end{pmatrix} = \begin{pmatrix} 6e^t - 3e^t \\ 3e^t - 2e^t \end{pmatrix} = \begin{pmatrix} 3e^t \\ e^t \end{pmatrix} = \frac{dX_1}{dt}$$

Similarly

$$\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} = \begin{pmatrix} 2e^{-t} - 3e^{-t} \\ e^{-t} - 2e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ -e^{-t} \end{pmatrix} = \frac{dX_2}{dt}$$

Hence both the vectors X_1 and X_2 are solutions of the homogeneous system

$$X' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} X$$

Now suppose that c_1, c_2 are any two arbitrary real constants such that

$$c_1 X_1 + c_2 X_2 = 0$$

or $$c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This means that

$$3c_1 e^t + c_2 e^{-t} = 0$$

$$c_1 e^t + c_2 e^{-t} = 0$$

The only solution of these equations for the arbitrary constants c_1 and c_2 is

$$c_1 = c_2 = 0$$

Hence, the solution vectors X_1 and X_2 are linearly independent on $(-\infty, \infty)$.

Example 6

Again consider the same homogeneous system as considered in the previous example

$$X' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} X$$

We have already seen that the vectors X_1, X_2 i.e.

$$X_1 = \begin{pmatrix} 3e^t \\ e^t \end{pmatrix}, \quad X_2 = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

are solutions of the homogeneous system. We can verify that the following vector X_3

$$X_3 = \begin{pmatrix} e^t + \cosh t \\ \cosh t \end{pmatrix}$$

is also a solution of the hom ogeneous system However, the set of solutions that consists of X_1, X_2 and X_3 is linearly dependent because X_3 is a lin ear com bination of the other two vectors

$$X_3 = \frac{1}{2} X_1 + \frac{1}{2} X_2$$

Exercise

Write the given system in matrix form.

1. $$\frac{dx}{dt} = x - y + z + t - 1$$
$$\frac{dy}{dt} = 2x + y - z - 3t^2$$
$$\frac{dz}{dt} = x + y + z + t^2 - t + 2$$

2. $$\frac{dx}{dt} = -3x + 4y + e^{-t}\sin 2t$$
$$\frac{dx}{dt} = 5x + 9y + 4e^{-t}\cos 2t$$

3. $$\frac{dx}{dt} = -3x + 4y - 9z$$
$$\frac{dy}{dt} = 6x - y$$
$$\frac{dz}{dt} = 10x + 4y + 3z$$

4. $$\frac{dx}{dt} = -3x + 4y + e^{-t}\sin 2t$$
$$\frac{dy}{dt} = 5x + 9y + 4e^{-t}\cos 2t$$

Write the given system without of use of matrices

5. $$X' = \begin{pmatrix} 7 & 5 & -9 \\ 4 & 1 & 1 \\ 0 & -2 & 3 \end{pmatrix} X + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} e^{5t} - \begin{pmatrix} 8 \\ 0 \\ 3 \end{pmatrix} e^{-2t}$$

6. $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 8 \end{pmatrix}\sin t + \begin{pmatrix} t-4 \\ 2t+1 \end{pmatrix} e^{4t}$$

7. $$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & -4 & 1 \\ -2 & 5 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} e^{-t} - \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} t$$

Verify that the vector X is the solution of the given system

8. $$\frac{dx}{dt} = -2x + 5y$$
$$\frac{dx}{dt} = -2x + 4y, \qquad X = \begin{pmatrix} 5\cos t \\ 3\cos t - \sin t \end{pmatrix} e^t$$

9. $X^{/} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} X$, $X = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{t} + \begin{pmatrix} 4 \\ -4 \end{pmatrix} te^{t}$

10. $\dfrac{dX}{dt} = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} X$; $X = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}$

Lecture 27
Matrices and Systems of Linear 1st-Order Equations (Continued)

Theorem: A necessary and sufficient condition that the set of solutions, on an interval I., consisting of the vectors

$$X_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}, X_2 = \begin{pmatrix} x_{12} \\ x_{21} \\ \vdots \\ x_{n2} \end{pmatrix}, \dots, X_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$$

of the homogenous system $X^{/} = AX$ to be linearly independent is that the Wronskian of these solutions is non-zero for every $t \in I$. Thus

$$W(X_1, X_2, \dots, X_n) = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} \neq 0, \quad \forall t \in I$$

Note that

- It can be shown that if $X_1, X_2, ..., X_n$ are solution vectors of the system, then either

  $$W(X_1, X_2, \dots, X_n) \neq 0, \ \forall t \in I$$

  or $$W(X_1, X_2, \dots, X_n) = 0, \ \forall t \in I$$

  Thus if we can show that $W \neq 0$ for some $t_0 \in I$, then $W \neq 0, \ \forall t \in I$ and hence the solutions are linearly independent on I

- Unlike our previous definition of the Wronskian, the determinant does not involve any differentiation.

Example 1

As verified earlier that the vectors

$$X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}, \quad X_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$$

are solutions of the following homogeneous system.

$$X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$$

Clearly, X_1 and X_2 are linearly independent on $(-\infty, \infty)$ as neither of the vectors is a constant multiple of the other. We now compute Wronskian of the solution vectors X_1 and X_2.

$$W(X_1, X_2) = \begin{vmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{vmatrix} = 8e^{4t} \neq 0, \quad \forall t \in (-\infty, \infty)$$

Fundamental set of solution

Suppose that $\{X_1, X_2, \ldots, X_n\}$ is a set of n solution vectors, on an interval I, of a homogenous system $X' = AX$. The set is said to be a fundamental set of solutions of the system on the interval I if the solution vectors $X_1, X_2, \ldots, X_n$ are linearly independent.

Theorem: *Existence of a Fundamental Set*

There exist a fundamental set of solution for the homogenous system $X' = AX$ on an interval I

General solution

Suppose that $X_1, X_2, \ldots, X_n$ is a fundamental set of solution of the homogenous system $X' = AX$ on an interval I. Then any linear combination of the solution vectors $X_1, X_2, \ldots, X_n$ of the form

$$X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

$c_i; i = 1, 2, \ldots, n$ being arbitrary constants is said to be the general solution of the system on the interval I.

Note that

For appropriate choices of the arbitrary constants $c_1, c_2, \ldots, c_n$ any solution, on the interval I, of the homogeneous system $X' = AX$ can be obtained from the general solution.

Example 2

As discussed in the Example 1, the following vectors are linearly independent solutions

$$X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}, \quad X_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$$

of the following homogeneous system of differential equations on $(-\infty, \infty)$

$$X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$$

Hence X_1 and X_2 form a fundamental set of solution of the system on the interval $(-\infty,\infty)$. Hence, the general solution of the system on $(-\infty,\infty)$ is

$$X = c_1 X_1 + c_2 X_2 = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$$

Example 3

Consider the vectors X_1, X_2 and X_3 these vectors are given by

$$X_1 = \begin{pmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^t, \; X_3 = \begin{pmatrix} \sin t \\ -\frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\sin t + \cos t \end{pmatrix}$$

It has been verified in the last lecture that the vectors X_1 and X_2 are solutions of the homogeneous system

$$X' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} X$$

It can be easily verified that the vector X_3 is also a solution of the system. We now compute the Wronskian of the solution vectors X_1, X_2 and X_3

$$W(X_1, X_2, X_3) = \begin{vmatrix} \cos t & 0 & \sin t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t & e^t & -\frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\cos t - \sin t & 0 & -\sin t + \cos t \end{vmatrix}$$

Expand from 2nd column

or $$W(X_1, X_2, X_3) = e^t \begin{vmatrix} \cos t & \sin t \\ -\cos t - \sin t & -\sin t + \cos t \end{vmatrix}$$

or $$W(X_1, X_2, X_3) = e^t \neq 0, \quad \forall t \in R$$

Thus, we conclude that X_1, X_2 and X_3 form a fundamental set of solution on $(-\infty, \infty)$. Hence, the general solution of the system on $(-\infty, \infty)$ is

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3$$

or

$$X = c_1\begin{pmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}e^t + c_3\begin{pmatrix} \sin t \\ -\frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\sin t + \cos t \end{pmatrix}$$

Non-homogeneous Systems

As stated earlier in this lecture that a system of differential equations such as

$$\frac{dX}{dt} = A(t)X + F(t)$$

is non-homogeneous if $F(t) \neq 0, \ \forall t$. The general solution of such a system consists of a complementary function and a particular integral.

Particular Integral

A particular solution, on an interval I, of a non-hom ogeneous system is any vector X_p free of arbitrary param eters, whose entries ar e functions that satisfy each equation of the system.

Example 4

Show that the vector

$$X_p = \begin{pmatrix} 3t-4 \\ -5t+6 \end{pmatrix}$$

is a particular solution of the following non-homogeneous system on the interval $(-\infty,\infty)$

$$X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}X + \begin{pmatrix} 12t-11 \\ -3 \end{pmatrix}$$

Solution:

Differentiating the given vector with respect to t, we obtain

$$X'_p = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

Further

$$\begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}X_p + \begin{pmatrix} 12t-11 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3t-4 \\ -5t+6 \end{pmatrix} + \begin{pmatrix} 12t-11 \\ -3 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}X_p + \begin{pmatrix} 12t-11 \\ -3 \end{pmatrix} = \begin{pmatrix} (3t-4)+3(-5t+6) \\ 5(3t-4)+3(-5t+6) \end{pmatrix} + \begin{pmatrix} 12t-11 \\ -3 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}X_p + \begin{pmatrix} 12t-11 \\ -3 \end{pmatrix} = \begin{pmatrix} -12t+14 \\ -2 \end{pmatrix} + \begin{pmatrix} 12t-11 \\ -3 \end{pmatrix}$$

or $$\begin{pmatrix}1 & 3\\5 & 3\end{pmatrix}X_p+\begin{pmatrix}12t-11\\-3\end{pmatrix}=\begin{pmatrix}3\\-5\end{pmatrix}=X_P'$$

Thus the given vector X_p satisfies the non-homogeneous system of differential equations. Hence, the given vector X_p is a particular solutio n of the non-hom ogeneous system.

Theorem

Let $X_1, X_2, \ldots, X_k$ be a set of solution vectors of the hom ogenous system $X' = AX$ on an interval I and le t X_p be any solution vector of the non-hom ogenous system $X' = AX + F(t)$ on the same interval I. Then $\exists$ constants $c_1, c_2, \ldots, c_k$ such that

$$X_p = c_1X_1 + c_2X_2 + \ldots + c_kX_k + X_p$$

is also a solution of the non-homogenous system on the interval.

Complementary function

Let $X_1, X_2, \ldots, X_n$ be solution vectors of the homogenous system $X' = AX$ on an interval I, then the general solution

$$X = c_1X_1 + c_2X_2 + \ldots + c_nX_n$$

of the homogeneous system is called the com plementary function of the non-homogeneous system $X' = AX + F(t)$ on the same interval I.

General solution-Non homogenous systems

Let X_p be a particular integral and X_c the complementary function, on an interval I, of the non-homogenous system

$$X' = A(t)X + F(t).$$

The general solution of the non-homogenous system on the interval I is defined to be

$$X = X_c + X_p$$

Example 5

In Example 4 it was verified that

$$X_p = \begin{pmatrix}3t-4\\-5t+6\end{pmatrix}$$

is a particular solution, on $(-\infty, \infty)$, of the non-homogenous system

$$X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X + \begin{pmatrix} 12t - 11 \\ -3 \end{pmatrix}$$

As we have seen earlier, the general soluti on of the associated hom ogeneous system i.e. the complementary function of the given non-homogeneous system is

$$X_c = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$$

Hence the general solution, on $(-\infty, \infty)$, of the non-homogeneous system is

$$X = X_c + X_p$$

$$X = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t} + \begin{pmatrix} 3t - 4 \\ -5t + 6 \end{pmatrix}$$

Fundamental Matrix

Suppose that the a f undamental set of n solution vectors of a hom ogeneous system $X' = AX$, on an interval I, consists of the vectors

$$X_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}, X_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \ldots, X_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$$

Then a fundamental matrix of the system on the interval I is given by

$$\phi(t) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

Example 6

As verified earlier, the following vectors

$$X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t} = \begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix}$$

form a fundamental set of solutions of the system on $(-\infty, \infty)$

$$X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$$

So that the general solution of the system is

$$X = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$$

Hence, a fundamental matrix of the system on the interval is

$$\phi(t) = \begin{pmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{pmatrix}$$

Note that

- The general solution of the system can be written as

$$X = \begin{pmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Or $\qquad X = \phi(t)C, \qquad €\begin{pmatrix} c_1 & c_2 \end{pmatrix}^{tr}$

- Since $X = \phi(t)C$ is a solution of the system $X' = A(t)X$. Therefore

$$\phi'(t)C = A(t)\phi(t)C$$

Or $\qquad [\phi'(t) - A(t)\phi(t)]C = 0$

Since the last equation is to hold for every t in the interval I for every possible column matrix of constants C , we must have

$$\phi'(t) - A(t)\phi(t) = 0$$

Or $\qquad \phi'(t) = A(t)\phi(t)$

Note that

- The funda mental m atrix $\phi(t)$ of a hom ogenous system $X' = A(t)X$ is non-singular because the determ inant $\det(\phi(t))$ coincides with the W ronskian of the solution vectors of the system and linea r independence of the solution vectors guarantees that $\det(\phi(t)) \neq 0$.

- Let $\phi(t)$ be a fundamental matrix of the homogenous system $X' = A(t)X$ on an interval I. Then, in view of the above mentioned observation, the inverse of the matrix $\phi^{-1}(t)$ exists for every value of t in the interval I.

Exercise

The given vectors are the solutions of a system $X' = AX$. Determine whether the vectors form a fundamental set on $(-\infty, \infty)$.

1. $X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t, X_2 = \begin{pmatrix} 2 \\ 6 \end{pmatrix} e^t + \begin{pmatrix} 8 \\ -8 \end{pmatrix} te^t$
2. $X_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}, X_2 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} e^{-4t}, \; X_3 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} e^{3t}$
3. $X' = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} X - \begin{pmatrix} 1 \\ 7 \end{pmatrix} e^t; \quad X_p = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^t$

Verify that vector X_p is a particular solution of the given systems

4. $\dfrac{dx}{dt} = x + 4y + 2t - 7, \quad \dfrac{dy}{dt} = 3x + 2y - 4t - 18$

   $X_p = \begin{pmatrix} 2 \\ -1 \end{pmatrix} t + \begin{pmatrix} 5 \\ 1 \end{pmatrix}$
5. $X' = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} X + \begin{pmatrix} -5 \\ 2 \end{pmatrix}; \quad X_p = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
6. $X' = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ -6 & 1 & 0 \end{pmatrix} X + \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \sin 3t; \quad X_p \begin{pmatrix} \sin 3t \\ 0 \\ \cos 3t \end{pmatrix}$
7. $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}, \quad X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6t}$
8. $X_1 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 3 \\ -6 \\ 12 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$

9. Prove that the general solution of the homogeneous system

$$X' = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} X$$

on the interval $(-\infty, \infty)$ is

$$X = c_1 \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{3t}$$

Lecture 28
Homogeneous Linear Systems

Most of the theory developed for a single linear differential equation can be extended to a system of such differential equations. The extension is not entirely obvious. However, using the notation and some ideas of matrix algebra discussed in a previous lecture most effectively carry it out. Therefore, in the present and in the next lecture we will learn to solve the homogeneous linear systems of linear differential equations with real constant coefficients.

Example 1

Consider the homogeneous system of differential equations

$$\frac{dx}{dt} = x + 3y$$

$$\frac{dy}{dt} = 5x + 3y$$

In matrix form the system can be written as

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

If we suppose that

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

Then the system can again be re-written as

$$X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$$

Now suppose that X_1 and X_2 denote the vectors

$$X_1 = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}, \quad X_2 = \begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix}$$

Then

$$X_1' = \begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix}, \quad X_2' = \begin{pmatrix} 18e^{6t} \\ 30e^{6t} \end{pmatrix}$$

Now $$AX_1 = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix} = \begin{pmatrix} e^{-2t} - 3e^{-2t} \\ 5e^{-2t} - 3e^{-2t} \end{pmatrix}$$

or $$AX_1 = \begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix} = X_1'$$

Similarly

$$AX_2 = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix} = \begin{pmatrix} 3e^{6t} + 15e^{6t} \\ 15e^{6t} + 15e^{6t} \end{pmatrix}$$

or $$AX_2 = \begin{pmatrix} 18\ e^{6t} \\ 30\ e^{6t} \end{pmatrix} = X_2'$$

Hence, X_1 and X_2 are solutions of the homogeneous system of differential equations $X' = AX$. Further

$$W(X_1, X_2) = \begin{vmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{vmatrix} = 8e^{4t} \neq o, \quad \forall t \in R$$

Thus, the solutions vectors X_1 and X_2 are linearly independent. Hence, these vectors form a fundamental set of solutions on $(-\infty, \infty)$. Therefore, the general solution of the system on $(-\infty, \infty)$ is

$$X = c_1 X_1 + c_2 X_2$$

$$X = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$$

Note that

- Each of the solution vectors X_1 and X_2 are of the form

$$X = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda t}$$

Where k_1 and k_2 are constants.

- The question arises whether we can always find a solution of the homogeneous system $X' = AX$, A is $n \times n$ matrix of constants, of the form

$$X = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = Ke^{\lambda t}$$

for the homogenous linear 1st order system.

Eigenvalues and Eigenvectors

Suppose that

$$X = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = Ke^{\lambda t}$$

is a solution of the system

$$\frac{dX}{dt} = AX$$

where A is an $n \times n$ matrix of constants then

$$\frac{dX}{dt} = K \ \lambda \ e^{\lambda t}$$

Substituting this last equation in the homogeneous system $X' = AX$, we have

$$K\lambda e^{\lambda t} = AKe^{\lambda t} \Rightarrow AK = \lambda K$$

or $$(A - \lambda \ I \ ) \ K = 0$$

This represents a system of linear algebraic equations. The linear 1 st order homogenous system of differential equations

$$\frac{dX}{dt} = AX$$

has a non-trivial solution X if there exist a non-trivial solution K of the system of algebraic equations

$$\det(A - \lambda I) = 0$$

This equation is called characteristic equation of the matrix A and represents an nth degree polynomial in λ .

Case 1 *Distinct real eigenvalues*

Suppose that the coefficient matrix A in the homogeneous system of differential equations

$$\frac{dX}{dt} = AX$$

has n distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ and $K_1, K_2, \ldots, K_n$ be the corresponding eigenvectors. Then the general solution of the system on $(-\infty, \infty)$ is given by

$$X = c_1 k_1 e^{\lambda_1 t} + c_2 k_2 e^{\lambda_2 t} + c_3 k_3 e^{\lambda_3 t} + \ldots\ldots + c_n k_n e^{\lambda_n t}$$

Example 2

Solve the following homogeneous system of differential equations

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = 2x + y$$

Solution

The given system can be written in the matrix form as

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Therefore, the coefficient matrix

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$$

Now we find the eigenvalues and eigenvectors of the coefficient A. The characteristics equation is

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 3\lambda - 4$$

Therefore, the characteristic equation is

$$\det(A - \lambda I) = 0 = \lambda^2 - 3\lambda - 4$$

or $(\lambda + 1)(\lambda - 4) = 0 \Rightarrow \lambda = -1, \quad 4$

Therefore, roots of the characteristic equation are real and distinct and so are the eigenvalues.

For $\lambda = -1$**,** we have

$$(A - \lambda I)K = \begin{pmatrix} 2+1 & 3 \\ 2 & 1+1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

or

$$(A - \lambda I)K = \begin{pmatrix} 3k_1 + 3k_2 \\ 2k_1 + 2k_2 \end{pmatrix}$$

Hence

$$(A - \lambda I)K = 0 \Rightarrow \begin{cases} 3k_1 + 3k_2 = 0 \\ 2k_1 + 2k_2 = 0 \end{cases}$$

These two equations are no different and represent the equation

$$k_1 + k_2 = 0 \Rightarrow k_1 = -k_2$$

Thus we can choose value of the constant k_2 arbitrarily. If we choose $k_2 = -1$ then $k_1 = 1$. Hence the corresponding eigenvector is

$$K_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda = 4$ we have

$$(A - \lambda I)K = \begin{pmatrix} 2-4 & 3 \\ 2 & 1-4 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

or $$(A - \lambda I)K = \begin{pmatrix} -2k_1 + 3k_2 \\ 2k_1 - 3k_2 \end{pmatrix}$$

Hence $$(A - \lambda I)K = 0 \Rightarrow \begin{cases} -2k_1 + 3k_2 = 0 \\ 2k_1 - 3k_2 = 0 \end{cases}$$

Again the above two equations are not different and represent the equation

$$2k_1 - 3k_2 = 0 \Rightarrow k_1 = \frac{3k_2}{2}$$

Again, the constan t k_2 can be chosen arbitrarily. Let us choose $k_2 = 2$ then $k_1 = 3$. Thus the corresponding eigenvector is

$$K_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Therefore, we obtain two linearly independent solution vectors of the given homogeneous system.

$$X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}, \quad X_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$$

Hence the general solution of the system is the following

$$X = c_1 X_1 + c_2 X_2$$

or $$X = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$$

or $$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} + 3c_2 e^{4t} \\ -c_1 e^{-t} + 2c_2 e^{4t} \end{pmatrix}$$

This means that the solution of the system is

$$x(t) = c_1 e^{-t} + 3c_2 e^{4t}$$

$$y(t) = -c_1 e^{-t} + 2c_2 e^{4t}$$

Example 3

Solve the homogeneous system

$$\frac{dx}{dt} = -4x + y + z$$
$$\frac{dy}{dt} = x + 5y - z$$
$$\frac{dz}{dt} = y - 3z$$

Solution:

The given system can be written as

$$\begin{pmatrix} dx/dt \\ dy/dt \\ dz/dt \end{pmatrix} = \begin{pmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Therefore the coefficient matrix of the system of differential equations is

$$A = \begin{pmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{pmatrix}$$

Therefore $$A - \lambda I = \begin{pmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{pmatrix}$$

Thus the characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{vmatrix} = 0$$

Expanding the determinant using cofactors of third row, we obtain

$$-(\lambda + 3)(\lambda + 4)(\lambda - 5) = 0$$
$$\lambda = -3,\ 4- ,\ 5$$

Thus the characteristic equation has real and distinct roots and so are the eigenvalues of the coefficient matrix A. To find the eigenvectors corresponding to these computed eigenvalues, we need to solve the following system of linear algebraic equations for k_1, k_2 and k_3 when $\lambda = -3,\ 4-\ ,\ 5$, successively.

$$\det(A - \lambda I)K = 0 \Rightarrow \begin{pmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For solving this system we use Gauss-Jor don elim ination technique, which consists of reducing the augm ented m atrix to th e reduced echelon form by applying the elem entary row operations. The augmented matrix of the system of linear algebraic equations is

$$\begin{pmatrix} -4-\lambda & 1 & 1 & 0 \\ 1 & 5-\lambda & -1 & 0 \\ 0 & 1 & -3-\lambda & 0 \end{pmatrix}$$

For $\lambda = -3$**,** the augmented matrix becomes:

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 8 & -1 & 0 \\ 0 & 1 & & 0 \ 0 \end{pmatrix}$$

Appling the row operation R_{12} , $R_2 + R_1$, R_{23} , $R_3 - 9R_2$, $R_1 - 8R_2$ in success ion reduces the augmented matrix in the reduced echelon form.

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So that we have the following equivalent system

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or $$k_1 = k_3, \ k_2 = 0$$

Therefore, the constant k_3 can be chosen arbitrarily. If we choose $k_3 = 1$, then $k_1 = 1$, So that the corresponding eigenvector is

$$K_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -4$, the augmented matrix becomes

$$((A + 4I) \mid 0) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 9 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

We apply elem entary row operation s to transf orm the m atrix to the f ollowing reduc ed echelon form:

$$\begin{pmatrix} 1 & 0 & -10 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus $$k_1 = 10k_3, \ k_2 = -k_3$$

Again k_3 can be chosen arbitrarily, therefore choosing $k_3 = 1$ we get $k_1 = 10,\ k_2 = -$
Hence, the second eigenvector is

$$K_2 = \begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix}$$

Finally, when $\lambda_3 = 5$ the augmented matrix becomes

$$((A - 5\ I) \mid 0) = \begin{pmatrix} -9 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -8 & 0 \end{pmatrix}$$

The application of the elementary row operation transforms the augmented matrix to the reduced echelon form

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus $$k_1 = k_3, \quad k_2 = 8k_3$$

If we choose $k_3 = 1$, then $k_1 = 1$ and $k_2 = 8$. Thus the eigenvector corresponding to $\lambda_3 = 5$ is

$$K_3 = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}$$

Thus we obtain three linearly independent solution vectors

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-3t},\ X_2 = \begin{pmatrix} 10 \\ - \\ 1 \end{pmatrix} e^{-4t},\ X_3 = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix} e^{5t}$$

Hence, the general solution of the given homogeneous system is

$$X = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix} e^{-4t} + c_3 \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix} e^{5t}$$

Case 2 *Complex eigenvalues*

Suppose that the coef ficient m atrix A in the hom ogeneous system of differential equations

$$\frac{dX}{dt} = AX$$

has complex eigenvalues. This means that roots of the characteristic equation

$$\det(A - \lambda I) = 0$$

are imaginary.

Theorem: *Solutions corresponding to complex eigenvalues*

Suppose that K is an eigenvector corresponding to the complex eigenvalue

$$\lambda_1 = \alpha + i\beta; \quad \alpha, \beta \in R$$

of the coefficient matrix A with real entries, then the vectors X_1 and X_2 given by

$$X_1 = K_1 e^{\lambda_1 t}, \quad X_2 = \overline{K}_1 e^{\overline{\lambda}_1 t}$$

are solution of the homogeneous system.

$$\frac{dX}{dt} = AX$$

Example 4

Consider the following homogeneous system of differential equations

$$\frac{dx}{dt} = 6x - y$$

$$\frac{dy}{dt} = 5x + 4y$$

The system can be written as

or $$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Therefore the coefficient matrix of the system is

$$A = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}$$

So that the characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = 0$$

or $$(6-\lambda)(4-\lambda) + 5 = 0 = \lambda^2 - 10\lambda + 29$$

Now using the quadratic formula we have

$$\lambda_1 = 5 + 2i, \quad \lambda_2 = 5 - 2i$$

For, $\lambda_1 = 5 + 2i$, we must solve the system of linear algebraic equations

$$\left.\begin{array}{l}(1-2i)k_1 - k_2 = 0 \\ 5k_1 - (1+2i)k_2 = 0\end{array}\right\} \Rightarrow (1-2i)k_1 - k_2 = 0$$

or $\quad k_2 = (1-2i)k_1$

Therefore, it follows that after we choose $k_1 = 1$ then $k_2 = 1 - 2i.$ So that one eigenvector is given by

$$K_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}$$

Similarly for $\lambda_2 = 5 - 2i$ we must solve the system of linear algebraic equations

$$\left.\begin{array}{l}(1+2i)k_1 - k_2 = 0 \\ 5k_1 - (1-2i)k_2 = 0\end{array}\right\} \Rightarrow (1+2i)k_1 - k_2 = 0$$

or $\quad k_2 = (1+2i)k_1$

Therefore, it follows that after we choose $k_1 = 1$ then $k_2 = 1 + 2i.$ So that secon d eigenvector is given by

$$K_2 = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix}$$

Consequently, two solution of the homogeneous system are

$$X_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(5+2i)t}, \quad X_2 = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(5-2i)t}$$

By the superposition principle another solution of the system is

$$X = c_1 \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(5+2i)t} + c_2 \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(5-2i)t}$$

Note that

The entrie s in K_2 corresponding to λ_2 are the co njugates of the entries in K_1 corresponding to λ_1. Further, λ_2 is conjugate of λ_1. Therefore, we can write this as

$$\lambda_2 = \overline{\lambda}_1, \quad K_2 = \overline{K}_1$$

Theorem *Real solutions corresponding to a complex eigenvalue*

Suppose that

- $\lambda_1 = \alpha + i\beta$ is a complex eigenvalue of the matrix A in the system $\frac{dX}{dt} = AX$
- K_1 is an eigenvector corresponding to the eigen value λ_1
- $B_1 = \frac{1}{2}(K_1 + \overline{K}_1) = \text{Re}(K_1), B_2 = \frac{i}{2}(-K_1 + \overline{K}_1) = \text{Im}(K_1)$

Then two linearly independent solutions of the system on $(-\infty, \infty)$ are given by

$$X_1 = (B_1 \cos\beta t - B_2 \sin\beta t)e^{\alpha t}$$
$$X_2 = (B_2 \cos\beta t + B_1 \sin\beta t)e^{\alpha t}$$

Example 5

Solve the system

$$X' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} X$$

The coefficient matrix of the system is

$$A = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}$$

Therefore

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{pmatrix}$$

Thus, the characteristic equation is

$$\det(A - \lambda I) = 0 = \begin{vmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{vmatrix}$$
$$-(2-\lambda)(2+\lambda) + 8 = 0 = \lambda^2 + 4$$

Thus the Eigenvalues are of the coefficient matrix are $\lambda_1 = 2i$ and $\lambda_2 = \overline{\lambda}_1 = -2i$.

For λ_1 we see that the system of linear algebraic equations $(A - \lambda I)K = 0$

$$(2-2i)k_1 + 8k_2 = 0$$
$$-k_1 - (2+2i)k_2 = 0$$

Solving these equations, we obtain

$$k_1 = -(2+2i)k_2$$

Choosing $k_2 = -1$ gives $k_1 = (2+2i)k_2$. Thus the corresponding eigenvector is

$$K_1 = \begin{pmatrix} 2+2i \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

So that $$B_1 = \mathrm{Re}(K_1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, B_2 = \mathrm{Im}(K_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Since $\alpha = 0$, the general solution of the given system of differential equations is

$$X = c_1\left[\begin{pmatrix} 2 \\ -1 \end{pmatrix}\cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\sin 2t\right] + c_2\left[\begin{pmatrix} 2 \\ 0 \end{pmatrix}\cos 2t + \begin{pmatrix} 2 \\ -1 \end{pmatrix}\sin 2t\right]$$

$$X = c_1\begin{pmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix} + c_2\begin{pmatrix} 2\cos 2t + 2\sin 2t \\ -\sin 2t \end{pmatrix}$$

Example 6

Solve the following system of differential equations

$$X' = \begin{pmatrix} 1 & 2 \\ -1/2 & 1 \end{pmatrix} X$$

Solution:

The coefficient matrix of the given system is

$$A = \begin{pmatrix} 1 & 2 \\ -1/2 & 1 \end{pmatrix}$$

Thus $$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ -1/2 & 1-\lambda \end{pmatrix}$$

So that the characteristic equation is

$$\det(A - \lambda I) = 0 = \begin{vmatrix} 1-\lambda & 2 \\ -1/2 & 1-\lambda \end{vmatrix}$$

or $$\lambda^2 - 2\lambda + 2 = 0$$

Therefore, by the quadratic formula we obtain

$$\lambda = \left(2 \pm \sqrt{4-8}\right)/2$$

Thus the eigenvalues of the coefficient matrix are

$$\lambda_1 = 1+i,\ \lambda_2 1= \bar{\lambda}_1 = \ -i$$

Now an eigenvector associated with the eigenvalue λ_1 is

$$K_1 = \begin{pmatrix} 2 \\ i \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

From $$B_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix},\quad B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So that we have the following two linearly independent solutions of the system

$$X_1 = \left[\begin{pmatrix} 2 \\ 0 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t\right]e^t, \quad X_2 = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t + \begin{pmatrix} 2 \\ 0 \end{pmatrix}\sin t\right]e^t$$

Hence, the general solution of the system is

$$X = c_1\left[\begin{pmatrix} 2 \\ 0 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t\right]e^t + c_2\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t + \begin{pmatrix} 2 \\ 0 \end{pmatrix}\sin t\right]e^t$$

or

$$X = c_1\begin{pmatrix} 2\cos t \\ -\sin t \end{pmatrix}e^t + c_2\begin{pmatrix} 2\sin t \\ \cos t \end{pmatrix}e^t$$

Exercise

Find the general solution of the given system

1. $\dfrac{dx}{dt} = x + 2y$

   $\dfrac{dy}{dt} = 4x + 3y$

2. $\dfrac{dx}{dt} = \dfrac{1}{2}x + 9y$

   $\dfrac{dy}{dt} = \dfrac{1}{2}x + 2y$

3. $X' = \begin{pmatrix} -6 & 2 \\ -3 & 1 \end{pmatrix}X$

4. $\dfrac{dx}{dt} = 2y$

   $\dfrac{dy}{dt} = 8x$

5. $X' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}X$

6. $\dfrac{dx}{dt} = 6x - 9y$

   $\dfrac{dy}{dt} = 5x + 2y$

7. $\dfrac{dx}{dt} = x + y$

   $\dfrac{dy}{dt} = -2x - y$

8. $\dfrac{dx}{dt} = 4x + 5y$

   $\dfrac{dy}{dt} = -2x + 6y$

9. $X' = \begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix} X$

10. $X' = \begin{pmatrix} 1 & -8 \\ 1 & -3 \end{pmatrix} X$

Lecture 29

Real and Repeated Eigenvalues

In the previous lecture we tried to learn how to solve a system of linear differential equations having a coefficient matrix that has real distinct and complex eigenvalues. In this lecture, we consider the systems

$$X' = AX$$

in which some of the n eigenvalue $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ of the $n \times n$ coefficient matrix A are repeated.

Eigenvalue of multiplicity m

Suppose that m is a positive integer and $(\lambda - \lambda_1)^m$ is a factor of the characteristic equation

$$\det(A - \lambda I) = 0$$

Further, suppose that $(\lambda - \lambda_1)^{m+1}$ is not a factor of the characteristic equation. Then the number λ_1 is said to be an eigenvalue of the coefficient matrix of multiplicity m.

Method of solution:

Consider the following system of n linear differential equations in n unknowns

$$X' = AX$$

Suppose that the coefficient matrix has an eigenvalue of multiplicity of m. There are two possibilities of the existence of the eigenvectors corresponding to this repeated eigenvalue:

- For the $n \times n$ coefficient matrix A, it may be possible to find m linearly independent eigenvectors $K_1, K_2, \ldots, K_m$ corresponding to the eigenvalue λ_1 of multiplicity $m \le n$. In this case the general solution of the system contains the linear combination

$$c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_1 t} + \cdots + c_n K_n e^{\lambda_1 t}$$

- If there is only one eigenvector corresponding to the eigenvalue λ_1 of multiplicity m, then m linearly independent solutions of the form

$$X_1 = K_{11} e^{\lambda_1 t}$$
$$X_2 = K_{21} e^{\lambda_1 t} + K_{22} e^{\lambda_1 t}$$
$$\vdots$$
$$X_m = K_{m1} \frac{t^{m-1}}{(m-1)!} e^{\lambda_1 t} + K_{m2} \frac{t^{m-2}}{(m-2)!} e^{\lambda_1 t} + \cdots + K_{mm} e^{\lambda_1 t}$$

where the column vectors K_{ij} can always be found.

Eigenvalue of Multiplicity Two

We begin by considering the systems of differential equations $X' = AX$ in which the coefficient matrix A has an eigenvalue λ_1 of multiplicity two. Then there are two possibilities;

- Whether we can find two linearly independent eigenvectors corresponding to eigenvalue λ_1 or
- We cannot find two linearly independent eigenvectors corresponding to eigenvalue λ_1.

The case of the possibility of us being able to find two linearly independent eigenvectors K_1, K_2 corresponding to the eigenvalue λ_1 is clear. In this case the general solution of the system contains the linear combination

$$c_1 K_1 t e^{\lambda_1 t} + c_2 K_2 e^{\lambda_1 t}$$

Therefore, we suppose that there is only one eigenvector K_1 associated with this eigenvalue and hence only one solution vector X_1. Then, a second solution can be found of the following form:

$$X_2 = Kte^{\lambda_1 t} + Pe^{\lambda_1 t}$$

In this expression for a second solution, K and P are column vectors

$$K = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}, \quad P = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$$

We substitute the expression for X_2 into the system $X' = AX$ and simplify to obtain

$$\left(AK - \lambda_1 K\right) t\, e^{\lambda_1 t} + \left(AP - \lambda_1 P - K\right) e^{\lambda_1 t} = 0$$

Since this last equation is to hold for all values of t, we must have:

$$(A - \lambda_1 I)K = 0, \quad (A - \lambda_1 I)P = K$$

First equation does not tell anything new and simply states that K must be an eigenvector of the coefficient matrix A associated with the eigenvalue λ_1. Therefore, by solving this equation we find one solution

$$X_1 = Ke^{\lambda_1 t}$$

To find the second solution X_2, we only need to solve, for the vector P, the additional system

$$\left(A-\lambda_1 I\right)P = K$$

First we solve a hom ogeneous system of diffe rential equations having c oefficient matrix for which we can find two distinct eigenvectors corresponding to a double eigenvalue and then in the second example we consider the case when cannot find two eigenvectors.

Example 1

Find general solution of the following system of linear differential equations

$$X' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} X$$

Solution:
The coefficient matrix of the system is

$$A = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}$$

Thus $$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix}$$

Therefore, the characteristic equation of the coefficient matrix A is

$$\det(A-\lambda I) = 0 = \begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix}$$

or $$-(3-\lambda)(9+\lambda)+36 = 0$$

or $$\left(\lambda+3\right)^2 = 0 \Rightarrow \lambda = -3, -3$$

Therefore, the coefficient matrix A of the given system has an eigenvalue of multiplicity two. This means that

$$\lambda_1 = \lambda_2 = -3$$

Now $$(A-\lambda I)K = 0 \Rightarrow \begin{pmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $\lambda = -3$**,** this system of linear algebraic equations becomes

$$\begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 6k_1 - 18k_2 = 0 \\ 2k_1 - 6k_2 = 0 \end{cases}$$

However

$$\left.\begin{aligned}6k_1 - 18k_2 &= 0\\ 2k_1 - 6k_2 &= 0\end{aligned}\right\} \Rightarrow k_1 - 3k_2 = 0$$

Thus $\quad k_1 = 3k_2$

This means that the value of the constant k_2 can be chosen arbitrarily. If we choose $k_2 = 1$, we find the following single eigenvector for the eigenvalue $\lambda = -3$.

$$K = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

The corresponding one solution of the system of differential equations is given by

$$X_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t}$$

But since we are interested in forming the general solution of the system, we need to pursue the question of finding a second solution. We identify the column vectors K and P as:

$$K = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad P = \begin{pmatrix} p1 \\ p2 \end{pmatrix}$$

Then $\quad (A + 3I)P = K \Rightarrow \begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} p1 \\ p2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Therefore, we need to solve the following system of linear algebraic equations to find P

$$\left.\begin{aligned}6p_1 - 18p_2 &= 3\\ 2p_1 - 6p_2 &= 1\end{aligned}\right\} \Rightarrow 2p_1 - 6p_2 = 1$$

or $\quad p_2 = -(1 - 2p_1)/6$

Therefore, the number p_1 can be chosen arbitrarily. So we have an infinite number of choices for p_1 and p_2. However, if we choose $p_1 = 1$, we find $p_2 = 1/6$. Similarly, if we choose the value of $p_1 = 1/2$ then $p_2 = 0$. Hence the column vector P is given by

$$P = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

Consequently, the second solution is given by

$$X_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} t\,e^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{-3t}$$

Hence the general solution of the given system of linear differential equations is then

$$X = c_1 X_1 + c_2 X_2$$

$$X = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{-3t} \right]$$

Example 2

Solve the homogeneous system

$$X' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} X$$

Solution:

The coefficient matrix of the system is:

$$A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

To write the characteristic we find the expansion of the determinant:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix}$$

The value of the determinant is

$$\det(A - \lambda I) = 5 + 9\lambda + 3\lambda^2 - \lambda^3$$

Therefore, the characteristic equation is

$$5 + 9\lambda + 3\lambda^2 - \lambda^3 = 0$$

or $$-(\lambda + 1)^2 (\lambda - 5) = 0$$

or $$\lambda = -1,\ -1,\ 5$$

Therefore, the eigenvalues of the coefficient matrix A are

$$\lambda_1 = \lambda_2 = -1,\ \lambda_3 = 5$$

Clearly -1 is a double root of the coefficient matrix A.

Now $$(A-\lambda I)K = 0 \Rightarrow \begin{pmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_1 = -1$, this system of the algebraic equations become

$$\begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix of the system is

$$\left(A + \mathrm{I} | 0 \right) = \left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right)$$

By applying the Gauss-Jordon method, the augmented matrix reduces to the reduced echelon form

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus $$k_1 - k_2 + k_3 = 0 \Rightarrow k_1 = k_2 - k_3$$

By choosing $k_2 = 1$ and $k_3 = 0$ in $k_1 = k_2 - k_3$, we obtain $k_1 = 1$ and so one eigenvector is

$$K_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

But the choice $k_2 = 1, k_3 = 1$ implies $k_1 = 0$. Hence, a second eigenvector is given by

$$K_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Since neither eigenvector is a constant multiple of the other, we have found, corresponding to the same eigenvalue, two linearly independent solutions

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t}, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t}$$

Last for $\lambda_3 = 5$ we obtain the system of algebraic equations

$$\begin{pmatrix} -4 & -2 & 2 \\ -2 & -4 & -2 \\ 2 & -2 & -4 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix of the algebraic system is

$$\left(A - 5\mathrm{I}|0\right) = \left(\begin{array}{ccc|c} -4 & -2 & 2 & 0 \\ -2 & -4 & -2 & 0 \\ 2 & -2 & -4 & 0 \end{array}\right)$$

By the elementary row operation w e can tran sform the augm ented matrix to the red uced echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

or $$k_1 = k_3, \; k_2 = -k_3$$

Picking $k_3 = 1$, we obtain $k_1 = 1, \; k_2 = -1$. Thus a third eigenvector is the following

$$K_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Hence, we conclude that the general solution of the system is

$$X = c_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}e^{-t} + c_3\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}e^{5t}$$

Eigenvalues of Multiplicity Three

When a matrix A has only one eigenvector associated w ith an eigenvalue λ_1 of multiplicity three of the coef ficient m atrix A, we can find a second solution X_2 and a third solution X_3 of the following forms

$$X_2 = Kte^{\lambda_1 t} + Pe^{\lambda_1 t}$$

$$X_3 = K\frac{t^2}{2}e^{\lambda_1 t} + Pte^{\lambda_1 t} + Qe^{\lambda_1 t}$$

The K, P and Q are vectors given by

$$K = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}, \quad P = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

By substituting X_3 into the system $X' = AX$, we find the column vectors K, P and Q must satisfy the equations

$$(A - \lambda_1 I)K = 0$$
$$(A - \lambda_1 I)P = K$$
$$(A - \lambda_1 I)Q = P$$

The solutions of first and second equations ca n be utilized in the formulation of the solution X_1 and X_2.

Example

Find the general solution of the following homogeneous system

$$X' = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix} X$$

Solution

The coefficient matrix of the system is

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

Then

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{vmatrix}$$

Therefore, the characteristic equation is

$$\det(A - \lambda I) = 0 = \begin{vmatrix} 4-\lambda & 1 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{vmatrix}$$

Expanding the determinant in the last equation *w.r.to* the 3rd row to obtain

$$(-1)^{3+3}(4-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ 0 & 4-\lambda \end{vmatrix} = 0$$

or $$(4-\lambda)\ \left[(\ \ 4\lambda)(4-\lambda)-0\right]=0$$

or $$(4-\lambda)^3=0 \Rightarrow \lambda=4, \quad , \qquad 4 \quad 4$$

Thus, $\lambda=4$ is an eigenvalue of the coefficient matrix A of multiplicity three. For $\lambda=4$, we solve the following system of algebraic equations

$$(A-\lambda \mathrm{I})K=0$$

or $$\begin{pmatrix} 4-\lambda & 1 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or $$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or $$\left.\begin{matrix} 0k_1+1k_2+0k_3=0 \\ 0k_1+0k_2+1k_3=0 \\ 0k_1+0k_2+0k_3=0 \end{matrix}\right\} \Rightarrow \begin{matrix} k_2=0 \\ k_3=0 \end{matrix}$$

Therefore, the value of k_1 is arbitrary. If we choose $k_1=1$, then the eigen vector K is

$$K=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Hence the first solution vector

$$X_1=Ke^{\lambda t}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}e^{4t}$$

Now for the second solution we solve the system

$$(A-\lambda I)P=K$$

or $$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} 0p_1+1p_2+0p_3=1 \\ 0p_1+0p_2+1p_3=0 \\ 0p_1+0p_2+0p_3=0 \end{array}\right\} \Rightarrow \begin{array}{l} p_1=1 \\ p_2=1 \\ p_3=0 \end{array}$$

Hence, the vector P is given by

$$P=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

Therefore, a second solution is

$$X_2=Kte^{\lambda t}+Pe^{\lambda t}$$

$$X_2=\begin{pmatrix}1\\0\\0\end{pmatrix}te^{4t}+\begin{pmatrix}1\\1\\0\end{pmatrix}e^{4t}$$

$$X_2=\left[\begin{pmatrix}1\\0\\0\end{pmatrix}t+\begin{pmatrix}1\\1\\0\end{pmatrix}\right]e^{4t}$$

Finally for the third solution we solve

$$(A-\lambda I)Q=P$$

or
$$\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}\begin{pmatrix}q_1\\q_2\\q_3\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

or
$$\left.\begin{array}{l} 0q_1+1q_2+0q_3=1 \\ 0q_1+0q_2+1q_3=0 \\ 0q_1+0q_2+0q_3=0 \end{array}\right\} \Rightarrow \begin{array}{l} q_1=1 \\ q_2=1 \\ q_3=1 \end{array}$$

Hence, the vector Q is given by

$$Q=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Therefore, third solution vector is

$$X_3 = K\frac{t^2}{2}e^{\lambda t} + Pte^{\lambda t} + Qe^{\lambda t}$$

$$X_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}\frac{t^2}{2}e^{4t} + \begin{pmatrix}1\\1\\0\end{pmatrix}te^{4t} + \begin{pmatrix}1\\1\\1\end{pmatrix}e^{4t}$$

$$X_3 = \left[\begin{pmatrix}1\\0\\0\end{pmatrix}\frac{t^2}{2} + \begin{pmatrix}1\\1\\0\end{pmatrix}t + \begin{pmatrix}1\\1\\1\end{pmatrix}\right]e^{4t}$$

The general solution of the given system is

$$X = c_1X_1 + c_2X_2 + c_3X_3$$

$$X = c_1\begin{pmatrix}1\\0\\0\end{pmatrix}e^{4t} + c_2\left[\begin{pmatrix}1\\0\\0\end{pmatrix}t + \begin{pmatrix}1\\1\\0\end{pmatrix}\right]e^{4t} + \left[\begin{pmatrix}1\\0\\0\end{pmatrix}\frac{t^2}{2} + \begin{pmatrix}1\\1\\0\end{pmatrix}t + \begin{pmatrix}1\\1\\1\end{pmatrix}\right]e^{4t}$$

Exercise

Find the general solution of the give systems

1. $$\frac{dx}{dt} = -6x + 5y$$
$$\frac{dy}{dt} = -5x + 4y$$

2. $$\frac{dx}{dt} = -x + 3y$$
$$\frac{dy}{dt} = -3x + 5y$$

3. $$\frac{dx}{dt} = 3x - y - z$$
$$\frac{dy}{dt} = x + y - z$$
$$\frac{dz}{dt} = x - y + z$$

4. $$X' = \begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix} X$$

5. $$X' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix} X$$

6. $$X' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix} X$$

Lecture 30
Non-Homogeneous System

Definition

Consider the system of linear first order differential equations

$$\frac{dx_1}{dt} = a_{11}(t)\ x_1 + a_{12}(t)\ x_2 + \cdots + a_{1n}(t)\ x_n + f_1(t)$$
$$\frac{dx_2}{dt} = a_{21}(t)\ x_1 + a_{22}(t)\ x_2 \cdots + a_{2n}(t)\ x_n + f_2(t)$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$\frac{dx_n}{dt} = a_{n1}(t)\ x_1 + a_{n2}(t)\ x_2 + \cdots + a_{nn}(t)\ x_n + f_n(t)$$

where a_{ij} are coefficients and f_i are continuous on comm on interval I. The system is said to be non-hom ogeneous when $f_i(t) \neq 0, \forall i = 1, 2, \ldots, n$. Otherwise it is called a homogeneous system.

Matrix Notation

In the matrix notation we can write the above system of differential can be written as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) \ldots & a_{2n}(t) \\ \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) \ldots & a_{nn}(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

Or $X' = AX + F(t)$

Method of Solution

To find general solution of the non-hom ogeneous system of linear differential equations, we need to find:

- The complementary function X_c, which is general solu tion of the corresponding homogeneous system $X' = AX$.
- Any particu lar solu tion X_p of the non-hom ogeneous system $X' = AX + F(t)$ by the method of undetermined coefficients and the variation of parameters.

The genera l solution X of the system is then given by sum of the com plementary function and the particular solution.

$$X = X_c + X_p$$

Method of Undetermined Coefficients

The form of $F(t)$

As mentioned earlier in the analogous case of a single *nth* order non-homogeneous linear differential equations. The entries in the matrix $F(t)$ can have one of the following forms:

- Constant functions.
- Polynomial functions
- Exponential functions
- $\sin(\beta x),\ \cos(\beta x)$
- Finite sums and products of these functions.

Otherwise, we cannot apply the method of undetermined coefficients to find a particular solution of the non-homogeneous system.

Duplication of Terms

The assumption for the particular solution X_p has to be based on the prior knowledge of the complementary function X_c to avoid duplication of terms between X_c and X_p.

Example 1

Solve the system on the interval $(-\infty, \quad \infty)$

$$X' = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} X + \begin{pmatrix} -8 \\ 3 \end{pmatrix}$$

Solution

To find X_c, we solve the following homogeneous system

$$X' = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} X$$

We find the determinant

$$\det\ (\mathrm{A} - \lambda \mathrm{I}) = \begin{vmatrix} -1-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix}$$

$$\det\ (\mathrm{A} - \lambda \mathrm{I}) = (-1-\lambda)\ (1-\lambda) + 2$$

$$\det\ (\mathrm{A} - \lambda \mathrm{I}) = \lambda^2 + \lambda - \lambda - 1 + 2 = \lambda^2 + 1$$

The characteristic equation is

$$\det\ (\mathrm{A} - \lambda \mathrm{I}) = 0 = \lambda^2 + 1$$

or $$\lambda^2 = -1 \Rightarrow \lambda = \pm i$$

So that the coefficient matrix of the system has complex eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$ with $\alpha = 0$ and $\beta = \pm 1$.

To find the eigenvector corresponding to λ_1, we must solve the system of linear algebraic equations

$$\begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{aligned} -(1+i)\;k_1 + 2k_2 &= 0 \\ -k_1 + (1-i)\;k_2 &= 0 \end{aligned}$$

Clearly, the second equation of the s ystem is $(1+i)$ times the first equation. So that bo th of the equations can be reduced to the following single equation

$$k_1 = (1-i)k_2$$

Thus, the value of k_2 can be chosen arbitrarily. Choosing $k_2 = 1,$ we get $k_1 = 1-i$. Hence, the eigenvector corresponding to λ_1 is

$$K_1 = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Now we form the matrices B_1 and B_2

$$B_1 = \text{Re}(k_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad B_2 = \text{Im}\;(k_1) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Then, we obtain the following two linearly independent solutions from:

$$X_1 = (B_1 \cos \beta t - B_2 \sin \beta t)e^{\alpha t}$$

$$X_2 = (B_2 \cos \beta t + B_1 \sin \beta t)e^{\alpha t}$$

Therefore $$X_1 = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin t\right]e^{0t}$$

$$X_2 = \left[\cos t\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin t\right]e^{0t}$$

or $$X_1 = \begin{pmatrix} \cos t \\ \cos t \end{pmatrix} + \begin{pmatrix} \sin t \\ 0 \end{pmatrix} = \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix}$$

$$X_2 = \begin{pmatrix} -\cos t \\ 0 \end{pmatrix} + \begin{pmatrix} \sin t \\ \sin t \end{pmatrix} = \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}$$

Thus the complementary function is given by

$$X_c = c_1 X_1 + c_1 X_2$$

or $$X_c = c_1\begin{pmatrix}\cos t + \sin t\\ \cos t\end{pmatrix} + c_2\begin{pmatrix}-\cos t + \sin t\\ \sin t\end{pmatrix}$$

Now since $F(t)$ is a constant vector, we assume a constant particular solution vector

$$X_p = \begin{pmatrix}a_1\\ b_1\end{pmatrix}$$

Substituting this vector into the original system leads to

$$X_p' = \begin{pmatrix}-1 & 2\\ -1 & 1\end{pmatrix}\begin{pmatrix}a_1\\ b_1\end{pmatrix} + \begin{pmatrix}-8\\ 3\end{pmatrix}$$

Since $$X_p' = \begin{pmatrix}0\\ 0\end{pmatrix}$$

Thus $$\begin{pmatrix}0\\ 0\end{pmatrix} = \begin{pmatrix}-a_1 + 2b_1\\ -a_1 + b_1\end{pmatrix} + \begin{pmatrix}-8\\ 3\end{pmatrix}$$

or $$\begin{pmatrix}0\\ 0\end{pmatrix} = \begin{pmatrix}-a_1 + 2b_1 - 8\\ -a_1 + b_1 + 3\end{pmatrix}$$

This leads to the following pair of linear algebraic equations

$$\begin{aligned}-a_1 + 2b_1 - 8 &= 0\\ -a_1 + b_1 + 3 &= 0\end{aligned}$$

Subtracting, we have

$$b_1 - 11 = 0 \Rightarrow b_1 = 11$$

Substituting this value of b_1 into th e s econd equation of the above system of algebraic equations yields

$$a_1 = 11 + 3 = 14$$

Thus our particular solution is

$$X_p = \begin{pmatrix}14\\ 11\end{pmatrix}$$

Hence, the general solution of the non-homogeneous system is

$$X = c_1\begin{pmatrix}\cos t + \sin t \\ \cos t\end{pmatrix} + c_2\begin{pmatrix}-\cos t + \sin t \\ \sin t\end{pmatrix} + \begin{pmatrix}14 \\ 11\end{pmatrix}$$

Note that

- In th e abov e exam ple the entries o f the m atrix $F(t)$ were c onstants a nd the complementary function X_c did not involve any constant vector. Thus there was no duplication of terms between X_c and X_p.
- However, if $F(t)$ were a co nstant vecto r and the coefficient m atrix had an eigenvalue $\lambda = 0$. Then X_c contains a constant vecto r. In such a situation th e assumption for the particular solution X_p would be

$$X_p = \begin{pmatrix}a_2 \\ b_2\end{pmatrix}t + \begin{pmatrix}a_1 \\ b_1\end{pmatrix}$$

instead of

$$X_p = \begin{pmatrix}a_1 \\ b_1\end{pmatrix}$$

Example 2

Solve the system

$$\frac{dx}{dt} = 6x + y + 6t$$

$$\frac{dy}{dt} = 4x + 3y - 10t + 4$$

Solution

In the matrix notation

$$X' = \begin{pmatrix}6 & 1 \\ 4 & 3\end{pmatrix}X + \begin{pmatrix}6 \\ -10\end{pmatrix}t + \begin{pmatrix}0 \\ 4\end{pmatrix}$$

or
$$X' = \begin{pmatrix}6 & 1 \\ 4 & 3\end{pmatrix}X + F(t)$$

Where $F(t) = \begin{pmatrix}6 \\ -10\end{pmatrix}t + \begin{pmatrix}0 \\ 4\end{pmatrix}$

We first solve the homogeneous system

$$X' = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} X$$

Now, we use characteristic equation to find the eigen values

$$\det(\mathrm{A} - \lambda \mathrm{I}) = \begin{vmatrix} 6-\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (6-\lambda)(3-\lambda) - 4 = 0$$

$$\Rightarrow \quad \lambda^2 - 9\lambda + 14 = 0$$

So $\lambda_1 = 2$ and $\lambda_2 = 7$

The eigen vector corresponding to eigen value $\lambda = \lambda_1 = 2$, is obtained from

$$(\mathrm{A} - \lambda \mathrm{I})K_1 = 0, \text{ Where } K_1 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

Or $(\mathrm{A} - 2\mathrm{I})K_1 = 0,$

Therefore

$$\begin{pmatrix} 6-2 & 1 \\ 4 & 3-2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4k_1 + k_2 \\ 4k_1 + k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\left.\begin{array}{l} 4k_1 + k_2 = 0 \\ 4k_1 + k_2 = 0 \end{array}\right\} \Rightarrow \; 4k_1 + k_2 = 0$$

we choose $k_1 = 1$ arbitrarily then $k_2 = -4$

Hence the related corresponding eigen vector is

$$K_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

Now an eigen vector associated with $\lambda = \lambda_2 = 7$ is de termined f rom the f ollowing system

$$(\mathrm{A} - \lambda_2 \mathrm{I})K_2 = 0, \text{ where } K_2 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

or $$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $$\left.\begin{aligned} -k_1 + k_2 &= 0 \\ 4k_1 - 4k_2 &= 0 \end{aligned}\right\} \Rightarrow -k_1 + k_2 = 0$$

Therefore $$K_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Consequently the complementary function is

$$X_c = c_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{7t}$$

Since $$F(t) = \begin{pmatrix} 6 \\ -10 \end{pmatrix}t + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

Now we find a particular solution of the system having the same form.

$$X_p = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

where a_1, a_2, b_1 and b_2 are constants to be determined.

in the matrix terms we must have

$$X'_p = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}X_p + \begin{pmatrix} 6 \\ -10 \end{pmatrix}t + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}\left[\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}\right] + \begin{pmatrix} 6 \\ -10 \end{pmatrix}t + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} a_2t + a_1 \\ b_2t + b_1 \end{pmatrix} + \begin{pmatrix} 6t + 0 \\ -10t + 4 \end{pmatrix}$$

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 6a_2t + 6a_1 + b_2t + b_1 \\ 4a_2t + 4a_1 + 3b_2t + 3b_1 \end{pmatrix} + \begin{pmatrix} 6t + 0 \\ -10t + 4 \end{pmatrix}$$

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 6a_2t + b_2t + 6t + 6a_1 + b_1 \\ 4a_2t + 3b_2t - 10t + 4a_1 + 3b_1 + 4 \end{pmatrix}$$

or
$$\begin{pmatrix}(6a_2 + b_2 + 6)t + (6a_1 + b_1 - a_2) \\ (4a_2 + 3b_2 - 10)t + (4a_1 + 3b_1 - b_2 + 4)\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

from this last identity we conclude that

$$\begin{aligned} 6a_2 + b_2 + 6 &= 0 \\ 4a_2 + 3b_2 - 10 &= 0 \end{aligned} \qquad \text{And} \qquad \begin{aligned} 6a_1 + b_1 - a_2 &= 0 \\ 4a_1 + 3b_1 - b_2 + 4 &= 0 \end{aligned}$$

Solving the first two equations simultaneously yields

$a_2 = -2$ and $b_2 = 6$

Substituting these values into the last two equations and solving for a_1 and b_1 gives

$$a_1 = -\frac{4}{7}$$
$$b_1 = \frac{10}{7}$$

It follows therefore that a particular solution vector is

$$X_p = \begin{pmatrix}-2 \\ 6\end{pmatrix}t + \begin{pmatrix}-4/7 \\ 10/7\end{pmatrix}$$

and so the general solution of the system on $(-\infty, \infty)$ is

$$X = X_c + X_p$$
$$= c_1\begin{pmatrix}1 \\ -4\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1 \\ 1\end{pmatrix}e^{7t} + \begin{pmatrix}-2 \\ 6\end{pmatrix}t + \begin{pmatrix}-4/7 \\ 10/7\end{pmatrix}$$

Example 3

Determine the form of the particular solution vector X_p for

$$\frac{dx}{dt} = 5x + 3y - 2e^{-t} + 1$$
$$\frac{dy}{dt} = -x + y + e^{-t} - 5t + 7$$

Solution

First, we write the system in the matrix form

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ -5 \end{pmatrix} t + \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

or $$X' = \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix} X + F(t)$$

where $$X' = \begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } F(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ -5 \end{pmatrix} t + \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

Now we solve the homogeneous system $X' = \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix} X$ to determine the eigen values, we use the characteristic equation

$$\det(\mathrm{A} - \lambda \mathrm{I}) = 0$$

or $$\begin{vmatrix} 5-\lambda & 3 \\ -1 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda) + 3 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 8 = 0$$

$$\Rightarrow \lambda = 2,\ 4$$

So the eigen values are $\lambda_1 = 2$ and $\lambda_2 = 4$

For $\lambda = \lambda_1 = 2$, an eigen vector corresponding to this eigen value is obtained from

$$(\mathrm{A} - 2\mathrm{I})K_1 = 0$$

Where $$K_1 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

$$\begin{pmatrix} 5-2 & 3 \\ -1 & 1-2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 3k_1 + 3k_2 &= 0 \\ -k_1 - k_2 &= 0 \end{aligned}\right\} \Rightarrow -k_1 - k_2 = 0$$

We choose $k_2 = -1$ then $k_1 = 1$

Therefore $K_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Similarly for $\lambda = \lambda_2 = 4$

$$\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} k_1 + 3k_2 &= 0 \\ -k_1 - 3k_2 &= 0 \end{aligned}\right\} \Rightarrow k_1 + 3k_2 = 0$$

Choosing $k_2 = -1$, we get $k_1 = 3$

Therefore $K_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

Hence the complementary solution is

$$X_c = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 3 \\ -1 \end{pmatrix}e^{4t}$$

Now since

$$F(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}e^{-t} + \begin{pmatrix} 0 \\ -5 \end{pmatrix}t + \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

We assume a particular solution of the form

$$X_p = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}e^{-t} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

Note:

If we replace e^{-t} in $F(t)$ on e^{2t} ($\lambda = 2$ an eigen value), then the correct form of the particular solution is

$$X_p = \begin{pmatrix} a_4 \\ b_4 \end{pmatrix}te^{2t} + \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}e^{2t} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

Variation of Parameters

Variation of parameters is more powerful technique than the method of undetermined coefficients.

We now develop a systematic produce for finding a solution of the non-homogeneous linear vector differential equation

$$\frac{dX}{dt} = AX + F(t) \qquad (1)$$

Assuming that we know the corresponding homogeneous vector differential equation

$$\frac{dX}{dt} = AX \qquad (2)$$

Let $\phi(t)$ be a fundamental matrix of the homogeneous system (2), then we can express the general solution of (2) in the form

$$X_c = \phi(t)\, C$$

where C is an arbitrary n-rowed constant vector. We replace the constant vector C by a column matrix of functions

$$U(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}$$

so that
$$X_p = \phi(t)\, U(t) \qquad (3)$$

is particular solution of the non-homogeneous system (1).
The derivative of (3) by the product rule is

$$X'_p = \phi(t)\, U'(t) + \phi'(t) U(t) \qquad (4)$$

Now we substitute equation (3) and (4) in the equation (1) then we have

$$\phi(t)\, U'(t) + \phi'(t) U(t) = A\phi(t)\, U(t) + F(t) \qquad (5)$$

Since $\phi'(t) = A\phi(t)$

On substituting this value of $\phi'(t)$ into (5),
We have

$$\phi(t) U'(t) + A\phi(t) U(t) = A\phi(t) U(t) + F(t)$$

Thus, equation (5) becomes
or
$$\phi(t)\, U'(t) = F(t) \qquad (6)$$

Multiplying $\phi^{-1}(t)$ on both sides of equation (6), we get

$$\phi^{-1}(t)\, \phi(t)\, U'(t) = \phi^{-1}(t)\, F(t)$$

or
$$U'(t) = \phi^{-1}(t)\, F(t)$$

or
$$U(t) = \int \phi^{-1}(t)\, F(t) dt$$

Hence by equation (3)

$$X_p = \phi(t) \int \phi^{-1}(t)\, F(t) dt \qquad (7)$$

is particular solution of the non-homogeneous system (1).
To calculate the indefinite integral of the column matrix $\phi^{-1}(t)\ F(t)$ in (7), we integrate each entry. Thus the general solution of the system (1) is

$$X = X_c + X_p$$

or

$$X = \phi(t)C + \phi(t)\int \phi^{-1}(t)\, F(t)dt \qquad (8)$$

Example

Find the general solution of the non-homogeneous system

$$X' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix} X + \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}$$

on the interval $(-\infty, \infty)$

Solution

We first solve the corresponding homogeneous system

$$X' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix} X$$

The characteristic equation of the coefficient matrix is

$$\det(\mathrm{A} - \lambda \mathrm{I}) = \begin{vmatrix} -3-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} = 0$$

or $$(-3-\lambda)(-4-\lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 + 4\lambda + 3\lambda + 12 - 2 = 0$$

$$\Rightarrow \lambda^2 + 7\lambda + 10 = 0$$

$$\Rightarrow \lambda^2 + 5\lambda + 2\lambda + 10 = 0$$

$$\Rightarrow \lambda(\lambda + 5) + 2(\lambda + 5) = 0$$

$$\Rightarrow (\lambda + 5)(\lambda + 2) = 0$$

$$\Rightarrow \lambda_1 = -2, \quad \lambda_2 = -5$$

So the eigen values are $\lambda_1 = -2$ and $\lambda_2 = -5$

Now we find the eigen vectors corresponding to λ_1 and λ_2 respectively,

Therefore

$$(\mathrm{A} - \lambda_1 \mathrm{I}_2)K_1 = 0$$

$$(\mathrm{A} - 2\mathrm{I}_2)K_1 = 0$$

so $$\begin{pmatrix} -3+2 & 1 \\ 2 & -4+2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -k_1 + k_2 \\ 2k_1 - 2k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\left.\begin{aligned} -k_1 + k_2 &= 0 \\ 2k_1 - 2k_2 &= 0 \end{aligned}\right\} \Rightarrow k_1 = k_2$$

We choose $k_2 = 1$ arbitrarily then $k_1 = 1$
Hence the eigen vector is

$$K_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now an eigen vector associated with $\lambda_2 = \lambda = -5$ is dete rmined f rom the f ollowing system

$$(A - \lambda_2 I_2)K_2 = 0$$

or

$$\begin{pmatrix} -3+5 & 1 \\ 2 & -4+5 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2k_1 + k_2 \\ 2k_1 + k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} 2k_1 + k_2 = 0 \\ 2k_1 + k_2 = 0 \end{matrix}\right\} \Rightarrow k_2 = -2k_1$$

We choose arbitrarily $k_1 = 1$ then $k_2 = -2$

Therefore

$$K_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

The solution vectors of the homogeneous system are

$$X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} \text{ And } X_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-5t}$$

X_1 and X_2 can be written as

$$X_1 = \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}, \; X_2 = \begin{pmatrix} e^{-5t} \\ -2e^{-5t} \end{pmatrix}$$

The complementary solution

$$X_c = c_1 X_1 + c_2 X_2$$

$$= c_1 \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-5t} \\ -2e^{-5t} \end{pmatrix}$$

Next, we form the fundamental matrix

$$\phi(t) = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}$$

and the inverse of this fundamental matrix is

$$\phi^{-1}(t)=\begin{pmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & \frac{-1}{3}e^{5t} \end{pmatrix}$$

Now we find X_p by

$$X_p = \phi(t)\int \phi^{-1}(t)\, F(t)dt$$

$$X_p = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\int \begin{pmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & \frac{-1}{3}e^{5t} \end{pmatrix}\begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}dt$$

$$X_p = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\int \begin{pmatrix} 2te^{2t} + \frac{1}{3}e^{t} \\ te^{5t} - \frac{1}{3}e^{4t} \end{pmatrix} dt = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\begin{pmatrix} \int 2te^{2t}dt + \int \frac{1}{3}e^{t}dt \\ \int te^{5t}dt - \int \frac{1}{3}e^{4t}dt \end{pmatrix}$$

$$X_p = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\begin{pmatrix} 2t\frac{e^{2t}}{2} - 2\int \frac{e^{2t}}{2}dt + \frac{1}{3}e^{t} \\ t\frac{e^{5t}}{5} - \int \frac{e^{5t}}{5}dt - \frac{1}{3.4}e^{4t} \end{pmatrix}$$

$$X_p = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\begin{pmatrix} 2t\frac{e^{2t}}{2} - \frac{e^{2t}}{2} + \frac{1}{3}e^{t} \\ \frac{te^{5t}}{5} - \frac{e^{5t}}{25} - \frac{1}{12}e^{4t} \end{pmatrix}$$

$$X_p = \begin{pmatrix} t - \frac{1}{2} + \frac{1}{3}e^{-t} + \frac{t}{5} - \frac{1}{25} - \frac{1}{12}e^{-t} \\ t - \frac{1}{2} + \frac{1}{3}e^{-t} - \frac{2t}{5} + \frac{2}{25} + \frac{1}{6}e^{-t} \end{pmatrix}$$

$$X_p = \begin{pmatrix} \frac{6}{5}t - \frac{27}{50} + \frac{1}{4}e^{-t} \\ \frac{3}{5}t - \frac{21}{50} + \frac{1}{2}e^{-t} \end{pmatrix}$$

Hence the general solution of the non-homogeneous system on the interval $(-\infty, \infty)$ is

$$X = X_c + X_p$$

$$=\phi(t)C+\phi(t)\int\phi^{-1}(t)\,F(t)\,dt$$

or

$$=c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-2t}+c_2\begin{pmatrix}1\\-2\end{pmatrix}e^{-5t}+\begin{pmatrix}\frac{6}{5}t-\frac{27}{50}+\frac{1}{4}e^{-t}\\ \frac{3}{5}t-\frac{21}{50}+\frac{1}{2}e^{-t}\end{pmatrix}$$

Exercise

Use the method of undetermined coefficients to solve the given system on $(-\infty, \infty)$

1. $\dfrac{dx}{dt} = 5x + 9y + 2$

   $\dfrac{dy}{dt} = -x + 11y + 6$

2. $\dfrac{dx}{dt} = x + 3y - 2t^2$

   $\dfrac{dy}{dt} = 3x + y + t + 5$

3. $\dfrac{dx}{dt} = x - 4y + 4t + 9e^{6t}$

   $\dfrac{dy}{dt} = 4x + y - t + e^{6t}$

4. $X' = \begin{pmatrix} 4 & 1/3 \\ 9 & 6 \end{pmatrix} X + \begin{pmatrix} -3 \\ 10 \end{pmatrix} e^t$

5. $X' = \begin{pmatrix} -1 & 5 \\ -1 & 1 \end{pmatrix} X + \begin{pmatrix} \sin t \\ -2\cos t \end{pmatrix}$

Use variation of parameters to solve the given system

6. $\dfrac{dx}{dt} = 3x - 3y + 4$

   $\dfrac{dy}{dt} = 2x - 2y - 1$

7. $X' = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} X + \begin{pmatrix} \sin 2t \\ 2\cos t \end{pmatrix} e^{2t}$

8. $X' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} X + \begin{pmatrix} 2 \\ e^{-3t} \end{pmatrix}$

9. $X' = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} X + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

10. $X' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X + \begin{pmatrix} \sec t \\ 0 \end{pmatrix}$

Lecture 31

Definition of a Partial Differential Equation (PDE)

A partial differential equation (PDE) is an equation that contains the dependent variable (the unknown function), and its partial derivatives. As in the ordinary differential equations (ODEs), the dependent variable $u = u(x)$ depends only on one independent variable x. However in the PDEs, the dependent variable, such as $u = u(x, t)$ or $u = u(x, y, t)$, must depend on more than one independent variable.

If $u = u(x, t)$, then the function u depends on the independent variable x, and on the time variable t. However, if $u = u(x, y, t)$, then the function u depends on the space variables x, y, and on the time variable t.

Examples

1. The heat equation
   $u_t = ku_{xx},$
   $u_t = k(u_{xx} + u_{yy}),$
   in one dimensional space and two dimensional space respectively. The dependent variable $u = u(x, t)$ in first equation depends on the position x and on the time variable t. However, in second equation $u = u(x, y, t)$ depends on three independent variables, the space variables x, y and the time variable t.

2. The wave equations
   $u_{tt} = c^2(u_{xx} + u_{yy}),$
   $u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}),$
   in two dimensional space and three dimensional space respectively.

3. The Laplace equation is
   $u_{xx} + u_{yy} = 0,$
   $u_{xx} + u_{yy} + u_{zz} = 0.$

Order of a PDE

The order of a PDE represents the order of the highest partial derivative that appears in the equation. For example, the following equations

$$u_t = u_{xx},$$
$$u_y - uu_{xxx} = 0,$$

are PDEs of second order, and third order respectively.

Example 1. Find the order of the following PDEs:

a. $u_t = u_{xx} + u_{yy}$

b. $u^4 u_{xx} + u_{xxy} = 0$

Solution

a. The highest partial derivative in this equation is u_{xx} or u_{yy}. The PDE is therefore of order two.
b. The highest partial derivative in this equation is u_{xxy}. The PDE is therefore of order three.

Linear and Nonlinear PDEs

Partial differential equations are classified as linear or nonlinear.

- A partial differential equation is called linear if
  1. the power of the dependent variable u and each partial derivative contained in the equation is one, and
  2. the coefficients of the dependent variable and the coefficients of each partial derivative are constants or independent variables.

- However, if any of these conditions is not satisfied, the equation is called nonlinear PDE.

Example 2. Classify the following PDEs as linear or nonlinear

a. $xu_{xx} + yu_{yy} = 0$

b. $u_x + \sqrt{u} = x$

c. $uu_t + xu_x = 0$

Solution

a. Here the power of each partial derivative u_{xx} and u_{yy} is one. Also, the coefficients of the partial derivatives are the independent variables x and y respectively. Hence, the PDE is linear.
b. The equation is nonlinear because of the term $\sqrt{u}$, as its power is ½.
c. In this case the power of each partial derivative is one, but u_t has the dependent variable u as its coefficient. Therefore, the PDE is nonlinear.

Homogeneous and Inhomogeneous PDEs

A partial differential equation of any order is called homogeneous if every term of the PDE contains the dependent variable u or one of its derivatives, otherwise, it is called an inhomogeneous PDE.

You can also say that a PDE is called homogeneous if the equation does not contain a term independent of the unknown function and its derivatives.

Example 3. Classify the following partial differential equations as homogeneous or inhomogeneous.

a. $u_t + 4u_x = 0$
b. $u_t = u_{xx} + x$
c. $u_x u_{xx} + \left(u_y\right)^2 = 0$

Solution.

a. The terms of the equation contain partial derivatives of u only, therefore it is a linear, homogeneous, 1st order PDE.
b. The equation is an inhomogeneous PDE, as one term contains the independent variable x.
c. This is nonlinear, 2nd order, homogeneous PDE.

Solution of a PDE

A solution of a PDE is a function such that it satisfies the equation under discussion and satisfies the given conditions as well. In other words, for u to satisfy the equation, the left hand side of the PDE and the right hand side should be the same upon substituting the resulting solution.

Example 4. Show that $u(x,y)=\sin x\sin y+x^2$ is a solution of the following PDE

$u_{xx}=u_{yy}+2.$

Solution.

As

$$u_x=-\cos x\sin y+2x$$
$$u_{xx}=-\sin x\sin y+2=L.H.S$$

Now

$$u_y=\sin x\cos y$$
$$u_{yy}=-\sin x\sin y$$
$$u_{yy}+2=-\sin x\sin y+2=R.H.S$$

Hence L.H.S=R.H.S

Example 5. Show that $u(x,\mathrm{t})=\sin xe^{-4t}$ is a solution of the following PDE

$u_t=4u_{xx}.$

Solution.

$$u_t=-4\sin xe^{-4t}=L.H.S$$
$$4u_{xx}=-4\sin xe^{-4t}=R.H.S$$

Boundary Conditions

For a given PDE that controls the mathematical behavior of physical phenomenon in a bounded domain *D*, the dependent variable *u* is usually prescribed at the boundary of the domain *D*. The boundary data is called boundary conditions. There are three types of boundary conditions (BCs) that can occur for heat flow problems. They are

- **Dirichlet Boundary Conditions**
  Consider heat flow problem in a rod ($0\le x\le L$). The specification of the temperatures $u(0, t)$ and $u(L, t)$ at the ends are classified as Dirichlet type BC. In this case, the function *u* is usually prescribed on the boundary of the bounded domain.

- **Neumann Boundary Conditions**
  The specification of the normal derivative (i.e., $\frac{\partial u}{\partial n}$, where *n* is the outward normal to the boundary) on the boundary is classified as Neumann type BCs. For a rod of length *L*,

Neumann boundary conditions are of the form $u_x(0,t)=\alpha, u_x(\mathrm{L},t)=\beta,$ where α and β are constants.

- **Mixed Boundary Conditions**
  If the condition on the boundary is a mixture of both Dirichlet and Neumann types i.e., a linear combination of the dependent variable u and the normal form $\frac{\partial u}{\partial n}$, then it is called Mixed BCs.
  It is not always necessary for the domain to be bounded, however one or more parts of the boundary may be at infinity.

Initial Conditions

PDEs mostly arise to govern physical phenomenon such as heat distribution, wave propagation and quantum mechanics. Most of the PDEs, such as the diffusion equation and the wave equation, depend on the time t. Accordingly, the initial values of the dependent variable u at the starting time $t = 0$ should be prescribed. For the heat case, the initial value u ($t = 0$), that defines the temperature at the starting time, should be prescribed. For the wave equation, the initial conditions u ($t = 0$) and u_t ($t = 0$) should also be prescribed.

Well-posed PDEs

A partial differential equation is said to be well-posed

- if a unique solution that satisfies the equation and the prescribed conditions exists, and
- Provided that the unique solution obtained is stable. The solution of a PDE is said to be stable if a small change in the conditions or the coefficients of the PDE results in a small change in the solution.

Exercises

1. Find the order of the following PDEs.
   a. $u_{xx}+2xu_{xy}+u_{yy}=e^{y}$
   b. $u_{xx}=u_{xxx}+u+1$
   c. $u_{xxy}+xu_{yy}+8u=7y$
   d. $u_t+u_{xxyy}=u$

2. Classify the following PDEs as linear or nonlinear.
   a. $yu_{xx} + 2xyu_{yy} + u = 1$
   b. $u_{xx} = u_{tt} - u^2$
   c. $u_t + uu_x + u_{xxx} = 0$
   d. $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

3. Classify the following PDEs as homogeneous or inhomogeneous.
   a. $u_t = u_{xx} + x$
   b. $u_t + u_{xxy} = u$
   c. $u_x + u_y = u + 4$
   d. $u_{tt} = u_{xx} + u_{yy} + u_{zz}$

4. Verify that the functions
   $u(x, y) = x^2 - y^2,$
   $u(x, y) = e^x \sin y,$
   $u(x, y) = 2xy$
   are the solutions of the equation $u_{xx} + u_{yy} = 0.$

5. Show that $u = f(xy)$, where *f* is an arbitrary differentiable function satisfies $xu_x - yu_y = 0$. Also verify that the functions $\sin(xy), \cos(xy), \log(xy), e^{xy}$ and $(xy)^3$ are solutions.

Classifications of Second-order PDEs

Classification of PDEs is an important concept because the general theory and methods of solution usually apply only to a given class of equations. Let us first discuss the classification of PDEs involving two independent variables.

1. Classification with two independent variables

Consider the following general second order linear partial differential equation in two independent variables

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F_u = G, \qquad (1)$$

where *A, B, C, D, E, F*, and *G* are constants or functions of the independent variables *x* and *y*. The classification of partial differential equations is suggested by the classification of the quadratic equation of conic sections in analytic geometry. The equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

represents hyperbola, parabola, or ellipse accordingly as the discriminant $(B^2 - 4AC)$ is positive, zero, or negative.

Similarly the nature of the second order partial differential equation (1) is determined by the principal part containing the highest partial derivatives, that is,

$$Lu \equiv Au_{xx} + Bu_{xy} + Cu_{yy}$$

Now depending on the sign of the discriminant PDE (1) is usually classified into three basic classes of equations,

1. **Parabolic equation** is an equation which satisfies the property
   $B^2 - 4AC = 0,$
2. **Hyperbolic equation** is an equation which satisfies the property
   $B^2 - 4AC > 0,$
3. **Elliptic equation** is an equation which satisfies the property
   $B^2 - 4AC < 0.$

Note. The classification of eq (1) as parabolic, hyperbolic or elliptic depends only on the coefficients of the second derivatives. It has nothing to do with the first derivative terms, the term in *u*, or the non-homogeneous term.

Example 1. Classify the following equations as parabolic, hyperbolic or elliptic.

a. $u_{xx} + u_{yy} = 0$ (Laplace equation)
b. $u_t = u_{xx}$ (Heat equation)
c. $u_{tt} - u_{xx} = 0$ (Wave equation)
d. $u_{xx} + xu_{yy} = 0;\ x \neq 0$ (Tricomi equation).

Solution.

a. Here $A = 1$, $B = 0$, $C = 1$ and $B^2 - 4AC = -4 < 0.$ Therefore, it is an elliptic equation.

b. Here $A = -1$, $B = 0$, $C = 0$ and $B^2 - 4AC = 0.$ Thus, it is parabolic type equation.

c. Here $A = -1$, $B = 0$, $C = 1$ and $B^2 - 4AC = 4 > 0.$ Hence, it is of hyperbolic type.
d. Here $A = 1$, $B = 0$, $C = x$ and $B^2 - 4AC = -4x.$ Therefore, the equation is parabolic if $x = 0$, hyperbolic if $x < 0$, and elliptic if $x > 0$. This example shows that equations with variable coefficients can change form in the different regions of the domain.

2. Classification with more than two independent variables

Consider the second-order PDE in general form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu + d = 0 \qquad (2)$$

where the coefficients a_{ij}, b_i, c and d are functions of $x = (x_1, x_2, \ldots, x_n)$ alone and $u = u(x_1, x_2, \ldots, x_n)$.
Its principal part is

$$L \equiv \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} \qquad (3)$$

It is enough to assume that $A = [a_{ij}]$ is symmetric if not, let $\overline{a}_{ij} = \frac{1}{2}\left(a_{ij} + a_{ji}\right)$ and rewrite

$$L \equiv \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{a}_{ij}\frac{\partial^2}{\partial x_i \partial x_j} \qquad (4)$$

Note that $\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}$. As in two-space dimension, let us attach a quadratic form P with (4) (i.e., replacing $\frac{\partial u}{\partial x_i}$ by x_i).

$$P(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j \qquad (5)$$

Since A is a real valued symmetric ($a_{ij} = a_{ji}$) matrix, it is diagonalizable with real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. In other words, there exists a corresponding set of orthonormal set of n eigenvectors, say $\sigma_1, \sigma_2, \ldots, \sigma_n$ with $R = [\sigma_1, \sigma_2, \ldots, \sigma_n]$ as column vectors such that

$$R^T AR = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & \bigcirc & \\ & & \cdot & & \\ & & & \cdot & \\ & \bigcirc & & \cdot & \\ & & & & \lambda_n \end{bmatrix} = D \qquad (6)$$

We now classify (2) depending on sign of eigenvalues of A:

- If one or more of the $\lambda_i = 0$ then PDE (2) is of parabolic type.
- If one of the $\lambda_i > 0$ or $\lambda_i < 0$, and all the remaining have opposite sign then PDE (2) is of hyperbolic type.
- If $\lambda_i > 0\ \forall i$ or $\lambda_i < 0\ \forall i$ then PDE (2) is said to be of elliptic type.

Example 2. Classify the following equation as parabolic, hyperbolic or elliptic.

$$u_{x_1x_1} + 2(1 + cx_2)u_{x_2x_3} = 0.$$

Solution. The equation can be rewritten as

$$u_{x_1x_1} + (1 + cx_2)u_{x_2x_3} + (1 + cx_2)u_{x_3x_2} = 0$$

For i, j=1, 2, 3 the eq(3) becomes

$$\begin{aligned} L &\equiv \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} \\ &= a_{11}\frac{\partial^2 u}{\partial x_1\partial x_1} + a_{12}\frac{\partial^2 u}{\partial x_1\partial x_2} + a_{13}\frac{\partial^2 u}{\partial x_1\partial x_3} + a_{21}\frac{\partial^2 u}{\partial x_2\partial x_1} + a_{22}\frac{\partial^2 u}{\partial x_2\partial x_2} + a_{23}\frac{\partial^2 u}{\partial x_2\partial x_3} \\ &\quad + a_{31}\frac{\partial^2 u}{\partial x_3\partial x_1} + a_{32}\frac{\partial^2 u}{\partial x_3\partial x_2} + a_{33}\frac{\partial^2 u}{\partial x_3\partial x_3} \end{aligned} \qquad (7)$$

On comparison of given PDE with eq(7),

$$a_{11}=1, a_{12}=0, a_{13}=0,$$
$$a_{21}=0, a_{22}=0, a_{23}=(1+cx_2),$$
$$a_{31}=0, a_{32}=(1+cx_2), a_{33}=0.$$

Hence $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & (1+cx_2)\\ 0 & (1+cx_2) & 0\end{bmatrix}$

Now the Eigen values of matrix A are

$$\det(A-\lambda I)=0$$
$$\Rightarrow \lambda = 1, (1+cx_2), -(1+cx_2).$$

The corresponding Eigen vector and normalized vectors are $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}, \begin{pmatrix}0\\ -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$ respectively.

$$\Rightarrow R=\begin{bmatrix}0 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$R^T AR=\begin{bmatrix}1 & 0 & 0\\ 0 & (1+cx_2) & 0\\ 0 & 0 & -(1+cx_2)\end{bmatrix}$$

Which is a diagonal matrix. Now we classify the given PDE depending on the sign of Eigen values of matrix A.

For c=0, the equation is hyperbolic type.

For $c \neq 0$, equation is

- parabolic if $x_2 = -\frac{1}{c}$,
- hyperbolic if $x_2 > -\frac{1}{c}$ *and* $x_2 < -\frac{1}{c}$.

Exercises

1. Classify the following equations into hyperbolic, elliptic or parabolic type.
   a. $5u_{xx} - 3u_{yy} + (\cos x)u_x + e^y u_y + u = 0.$
   b. $\sin^2 xu_{xx} + \sin^2 xu_{xy} + \cos^2 xu_{yy} = x.$
   c. $e^x u_{xx} + e^y u_{yy} - u = 0.$
   d. $8u_{xx} + u_{yy} - u_x + [\log(2 + x^2)]u = 0.$
   e. $xu_{xx} + u_{yy} = 0.$

2. Classify the following equations into elliptic, parabolic, or hyperbolic type.
   a. $e^z u_{xy} - u_{xx} = \log[x^2 + y^2 + z^2 + 1].$
   b. $u_{xx} + 2u_{yz} + (\cos x)u_z - e^{y^2} u = \cosh z.$
   c. $u_{xx} + 2u_{xy} + u_{yy} + 2u_{zz} - (1 + xy)u = 0.$

3. Determine the regions where $u_{xx} - 2x^2 u_{xz} + u_{yy} + u_{zz} = 0$ is of hyperbolic, elliptic and parabolic.

Lecture 32

Adomian Decomposition Method

The Adomian decomposition method was introduced and developed by George Adomian and is well addressed in the literature. The Adomian decomposition method has been receiving much attention in recent years in applied mathematics in general, and in the area of series solutions in particular. The method proved to be powerful, effective, and can easily handle a wide class of linear or nonlinear, ordinary or partial differential equations, and linear and nonlinear integral equations. The decomposition method demonstrates fast convergence of the solution and therefore provides several significant advantages.

The Adomian decomposition method consists of decomposing the unknown function *u(x, y)* of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x,y)=\sum_{n=0}^{\infty}u_n(x,y),$$

where the components $u_n(x,y), n\geq 0$ are to be determined in a recursive manner. The decomposition method concerns itself with finding the components $u_0, u_1, u_2, \ldots$ individually.

To have a clear overview of Adomian decomposition method, first consider

$$Fu = g(t)\,,$$

where F is a nonlinear ordinary differential operator with linear and nonlinear terms. We could represent the linear term by $Lu+Ru$ where L is the linear operator. We choose L as the highest ordered derivative, which is assumed to be invertible. The remainder of the linear operator is R. The nonlinear term is represented by $f(u)$. Thus

$$Lu + Ru + f(u) = g \qquad (1)$$

$$Lu = g - Ru - f(u) \qquad (2)$$

After applying the inverse operator L^{-1} to both sides of above equation, we have

$$L^{-1}Lu = u = L^{-1}g - L^{-1}Ru - L^{-1}f(u) \qquad (3)$$

The decomposition method consists in looking for the solution in the series form $u=\sum_{n=0}^{\infty}u_n$. The nonlinear operator is decomposed as

$$f(u)=\sum_{n=0}^{\infty}A_n,$$

Where A_n depends on $u_0, u_1, u_2, \ldots, u_n$, called the Adomian polynomials that are obtained by writing

$$u(\lambda) = \sum_{n=0}^{\infty} u_n \lambda^n; \quad f(u(\lambda)) = \sum_{n=0}^{\infty} A_n \lambda^n, \tag{4}$$

Where λ is a parameter. From eq(4), A_n's are deduced as

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[f\left(\sum_{n=0}^{\infty} u_n \lambda^n \right) \right]_{\lambda=0} \tag{5}$$

The first few Adomian polynomials are

$$\begin{aligned} A_0 &= f(u_0), \\ A_1 &= u_1 f'(u_0), \\ A_2 &= u_2 f'(u_0) + \frac{u_1^2}{2!} f''(u_0), \\ A_3 &= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f^{(3)}(u_0), \\ &\ldots \end{aligned} \tag{6}$$

By substituting eq(4) into (3), The decomposition method consists in identifying the u_n's by means of the formulae

$$\sum_{n=0}^{\infty} u_n = L^{-1} g - L^{-1} R \sum_{n=0}^{\infty} u_n - L^{-1} (\sum_{n=0}^{\infty} A_n)$$

Through using Adomian decomposition method, the components $u_n(x)$ can be determined as

$$\begin{cases} u_0 = L^{-1} g \\ u_{n+1} = -L^{-1} R u_n - L^{-1} A_n \end{cases} \tag{7}$$

Hence the series solution of u(x) can be obtained by using above equations.

For numerical purposes, the n-term approximant

$$\psi_n = \sum_{n=0}^{n-1} u_k$$

can be used to approximate the exact solution.

A REFERENCE LIST OF THE ADOMI.AN POLYNOMIALS

$$A_0 = f(u_0),$$

$$A_1 = u_1 f'(u_0),$$

$$A_2 = u_2 f'(u_0) + \frac{u_1^2}{2!} f''(u_0),$$

$$A_3 = u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f^{(3)}(u_0),$$

$$A_4 = u_4 f'(u_0) + \left[\frac{u_1^2}{2!} + u_1 u_3\right] f''(u_0) + \frac{u_1^2 u_2}{2!} f^{(3)}(\mathrm{u}_0) + \frac{u_1^4}{4!} f^{(4)}(u_0),$$

$$A_5 = u_5 f'(\mathrm{u}_0) + \left[u_2 u_3 + u_1 u_4\right] f''(\mathrm{u}_0) + \left[\frac{u_1 u_2^2}{2!} + \frac{u_1^2 u_3}{2!}\right] f^{(3)}(\mathrm{u}_0)$$

$$+ \frac{u_1^3 u_2}{3!} f^{(4)}(\mathrm{u}_0) + \frac{u_1^5}{5!} f^{(5)}(\mathrm{u}_0),$$

$$A_6 = u_6 f'(\mathrm{u}_0) + \left[\frac{u_3^2}{2!} + u_2 u_4 + u_1 u_5\right] f''(\mathrm{u}_0) + \left[\frac{u_2^3}{3!} + u_1 u_2 u_3 + \frac{u_1^2 u_4}{2!}\right] f^{(3)}(\mathrm{u}_0)$$

$$+ \left[\frac{u_1^2}{2!}\frac{u_2^2}{2!} + \frac{u_1^3 u_3}{3!}\right] f^{(4)}(\mathrm{u}_0) + \frac{u_1^4 u_2}{4!} f^{(5)}(\mathrm{u}_0) + \frac{u_1^6}{6!} f^{(6)}(\mathrm{u}_0),$$

$$A_7 = u_7 f'(\mathrm{u}_0) + \left[u_3 u_4 + u_2 u_5 + u_1 u_6\right] f''(\mathrm{u}_0) + \left[\frac{u_2^2 u_3}{2!} + \frac{u_1 u_3^2}{2!} + u_1 u_2 u_4 + \frac{u_1^2 u_5}{2!}\right] f^{(3)}(\mathrm{u}_0)$$

$$+ \left[\frac{u_1 u_2^3}{3!} + \frac{u_1^2 u_2 u_3}{2!} + \frac{u_1^3 u_4}{3!}\right] f^{(4)}(\mathrm{u}_0) + \left[\frac{u_1^3}{3!}\frac{u_2^2}{2!} + \frac{u_1^4 u_3}{4!}\right] f^{(5)}(\mathrm{u}_0)$$

$$+ \frac{u_1^5 u_2}{5!} f^{(6)}(\mathrm{u}_0) + \frac{u_1^7}{7!} f^{(7)}(\mathrm{u}_0),$$

$$A_8 = u_8 f'(\mathrm{u}_0) + \left[\frac{u_4^2}{2!} + u_3 u_5 + u_2 u_6 + u_1 u_7\right] f''(\mathrm{u}_0)$$

$$+ \left[\frac{u_2 u_3^2}{2!} + \frac{u_2^2 u_4}{2!} + u_1 u_3 u_4 + u_1 u_2 u_5 + \frac{u_1^2 u_6}{2!}\right] f^{(3)}(\mathrm{u}_0)$$

$$+ \left[\frac{u_2^4}{4!} + \frac{u_1 u_2^2 u_3}{2!} + \frac{u_1^2}{2!}\frac{u_3^2}{2!} + \frac{u_1^2 u_2 u_4}{2!} + \frac{u_1^3 u_5}{3!}\right] f^{(4)}(\mathrm{u}_0)$$

$$+ \left[\frac{u_1^2}{2!}\frac{u_2^3}{3!} + \frac{u_1^3 u_2 u_3}{3!} + \frac{u_1^4 u_4}{4!}\right] f^{(5)}(\mathrm{u}_0) + \left[\frac{u_1^4}{4!}\frac{u_2^2}{2!} + \frac{u_1^5 u_3}{5!}\right] f^{(6)}(\mathrm{u}_0) + \frac{u_1^6 u_2}{6!} f^{(7)}(\mathrm{u}_0) + \frac{u_1^8}{8!} f^{(8)}(\mathrm{u}_0),$$

$$A_9 = u_9 f'(\mathrm{u}_0) + \left[u_4u_5 + u_3u_6 + u_2u_7 + u_1u_8\right] f''(\mathrm{u}_0)$$

$$+\left[\frac{u_3^3}{3!} + u_2u_3u_4 + \frac{u_2^2u_5}{2!} + \frac{u_1u_4^2}{2!} + u_1u_3u_5 + u_1u_2u_6 + \frac{u_1^2u_7}{2!}\right] f^{(3)}(\mathrm{u}_0)$$

$$+\left[\frac{u_2^3u_3}{3!} + \frac{u_1u_2u_3^2}{2!} + \frac{u_1u_2^2u_4}{2!} + \frac{u_1^2u_3u_4}{2!} + \frac{u_1^2u_2u_5}{2!} + \frac{u_1^3u_6}{3!}\right] f^{(4)}(\mathrm{u}_0)$$

$$+\left[\frac{u_1u_2^4}{4!} + \frac{u_1^2}{2!}\frac{u_2^2u_3}{2!} + \frac{u_1^3}{3!}\frac{u_3^2}{2!} + \frac{u_1^3u_2u_4}{3!} + \frac{u_1^4u_5}{4!}\right] f^{(5)}(\mathrm{u}_0)$$

$$+\left[\frac{u_1^3}{3!}\frac{u_2^3}{3!} + \frac{u_1^4u_2u_3}{4!} + \frac{u_1^5u_4}{5!}\right] f^{(6)}(\mathrm{u}_0) + \left[\frac{u_1^5}{5!}\frac{u_2^2}{2!} + \frac{u_1^6u_3}{6!}\right] f^{(7)}(\mathrm{u}_0)$$

$$+\frac{u_1^7u_2}{7!} f^{(8)}(\mathrm{u}_0) + \frac{u_1^9}{9!} f^{(9)}(\mathrm{u}_0)$$

$$A_{10} = u_{10} f'(\mathrm{u}_0) + \left[\frac{u_5^2}{2!} + u_4u_6 + u_3u_7 + u_2u_8 + u_1u_9\right] f''(\mathrm{u}_0)$$

$$+\left[\frac{u_3^2u_4}{2!} + \frac{u_4^2u_2}{2!} + u_2u_3u_5 + \frac{u_2^2u_6}{2!} + u_1u_4u_5 + u_1u_3u_6 + u_1u_2u_7 + \frac{u_1^2u_8}{2!}\right] f^{(3)}(\mathrm{u}_0)$$

$$+\left[\frac{u_2^2}{2!}\frac{u_3^2}{2!} + \frac{u_2^3u_4}{3!} + \frac{u_1u_3^3}{3!} + u_1u_2u_3u_4 + \frac{u_1u_2^2u_5}{2!} + \frac{u_1^2}{2!}\frac{u_4^2}{2!} + \frac{u_1^2u_3u_5}{2!} + \frac{u_1^2u_2u_6}{2!} + \frac{u_1^3u_7}{3!}\right] f^{(4)}(\mathrm{u}_0)$$

$$+\left[\frac{u_2^5}{5!} + \frac{u_1u_2^3u_3}{3!} + \frac{u_1^2u_2}{2!}\frac{u_3^2}{2!} + \frac{u_1^2}{2!}\frac{u_2^2u_4}{2!} + \frac{u_1^3u_3u_4}{3!} + \frac{u_1^3u_2u_5}{3!} + \frac{u_1^4u_6}{4!}\right] f^{(5)}(\mathrm{u}_0)$$

$$+\left[\frac{u_1^2}{2!}\frac{u_2^4}{4!} + \frac{u_1^3}{3!}\frac{u_2^2u_3}{2!} + \frac{u_1^4}{4!}\frac{u_3^2}{2!} + \frac{u_1^4u_2u_4}{4!} + \frac{u_1^5u_5}{5!}\right] f^{(6)}(\mathrm{u}_0)$$

$$+\left[\frac{u_1^4}{4!}\frac{u_2^3}{3!} + \frac{u_1^5u_2u_3}{5!} + \frac{u_1^6u_4}{6!}\right] f^{(7)}(\mathrm{u}_0) + \left[\frac{u_1^6}{6!}\frac{u_2^2}{2!} + \frac{u_1^7u_3}{7!}\right] f^{(8)}(\mathrm{u}_0)$$

$$+\frac{u_1^8u_2}{8!} f^{(9)}(\mathrm{u}_0) + \frac{u_1^{10}}{10!} f^{(10)}(\mathrm{u}_0)$$

$$\vdots$$

Examples: Calculate the Adomian polynomials for following non-linear functions

1. $f(u) = u^5$

Solution

The Adomian polynomials are determined by using the above reference list of formulas, as

$$
\begin{aligned}
A_0 &= u_0^5 \\
A_1 &= 5u_0^4u_1 \\
A_2 &= 5u_0^4u_2 + 10u_0^3u_1^2 \\
A_3 &= 5u_0^4u_3 + 20u_0^3u_1u_2 + 10u_0^2u_1^3 \\
A_4 &= 5u_0^4u_4 + 5u_1^4u_0 + 10u_0^3u_2^2 + 20u_0^3u_1u_3 + 30u_0^2u_1^2u_2 \\
A_5 &= u_1^5 + 5u_0^4u_5 + 20u_0^3u_1u_4 + 20u_0^3u_2u_3 + 20u_1^3u_0u_2 + 30u_0^2u_2^2u_1 + 30u_0^2u_1^2u_3 \\
A_6 &= 5u_0^4u_6 + 5u_1^4u_2 + 10u_0^3u_3^2 + 10u_0^2u_2^3 + 20u_0^3u_1u_5 + 20u_0^3u_2u_4 + 20u_1^3u_0u_3 \\
&\quad + 30u_0^2u_1^2u_4 + 30u_1^2u_2^2u_0 + 60u_0^2u_1u_2u_3 \\
A_7 &= 5u_0^4u_7 + 5u_1^4u_3 + 10u_1^3u_2^2 + 20u_0^3u_1u_6 + 20u_0^3u_2u_5 + 20u_0^3u_3u_4 + 20u_2^3u_1u_0 \\
&\quad + 20u_1^3u_0u_4 + 30u_0^2u_2^2u_3 + 30u_0^2u_3^2u_1 + 30u_0^2u_1^2u_5 + 60u_0^2u_1u_2u_4 + 60u_1^2u_0u_2u_3
\end{aligned}
$$

Remark

Notice that for u^m each individual term is the product of m factors. Each term of A_n has five factors--- the sum of superscripts is m (or 5 in this case). The sum of subscripts is n. The second term of A_4, as an example, is $5u_1u_1u_1u_1u_0$ and the sum of subscripts is 4. A very convenient check on the numerical coefficients in each term is the following. Each coefficient is $m!$ divided by the product of factorials of the superscripts for a given term. Thus, the second term of $A_3(u^5)$ has the coefficient $5!(2!)(2!)(1!) = 30$.

2. $f(u) = u^2$

Solution

$$A_0 = u_0^2$$
$$A_1 = 2u_0 u_1$$
$$A_2 = u_1^2 + 2u_0 u_2$$
$$A_3 = 2u_1 u_2 + 2u_0 u_3$$
$$A_4 = u_2^2 + 2u_1 u_3 + 2u_0 u_4$$
$$A_5 = 2u_2 u_3 + 2u_1 u_4 + 2u_0 u_5$$

3. $f(\theta) = \sin\theta$

Solution

$$A_0 = \sin\theta_0$$
$$A_1 = \theta_1 \cos\theta_0$$
$$A_2 = -\left(\theta_1^2 / 2\right)\sin\theta_0 + \theta_2 \cos\theta_0$$
$$A_3 = -\left(\theta_1^3 / 6\right)\cos\theta_0 - \theta_1\theta_2 \sin\theta_0 + \theta_3 \cos\theta_0$$
$$\vdots$$

Remark:

The essential features of the decomposition method for linear and nonlinear equations, homogeneous and inhomogeneous, can be outlined as follows:

- Express the partial differential equation, linear or nonlinear, in an operator form.
- Apply the inverse operator to both sides of the equation written in an operator form.
- Set the unknown function $u(x, y)$ into a decomposition series

  $$u = \sum_{n=0}^{\infty} u_n$$

  whose components are elegantly determined. We next substitute the above series into both sides of the resulting equation.
- Identify the zeroth component $u_0(x, y)$ as the terms arising from the given conditions and from integrating the source term $g(x, y)$, both are assumed to be known.

- Determine the successive components of the series solution u_k, $k \geq 1$ by applying the recursive scheme (6), where each component u_k can be completely determined by using the previous component u_{k-1}.
- Substitute the determined components into $u = \sum_{n=0}^{\infty} u_n$ to obtain the solution in a series form. An exact solution can be easily obtained in many equations if such a closed form solution exists.

It is to be noted that Adomian decomposition method approaches any equation, homogeneous or inhomogeneous, and linear or nonlinear in a straightforward manner without any need to restrictive assumptions such as linearization, discretization or perturbation. There is no need in using this method to convert inhomogeneous conditions to homogeneous conditions as required by other techniques.

Example 4:

Solve the following homogeneous differential equation by using Adomian decomposition method.

$$u'(x) = u(x), \;\; u(0) = \mathrm{A}. \qquad (8)$$

Solution

In an operator form the given equation becomes

$$Lu = u, \qquad (9)$$

where L is the differential operator given by

$$L = \frac{d}{dx}, \qquad (10)$$

and therefore the inverse operator L^{-1} is defined by

$$L^{-1}(.) = \int_0^x (.)dx. \qquad (11)$$

Applying L^{-1} to both sides of (9) and using the initial condition we obtain

$$L^{-1}(Lu) = L^{-1}(u)$$
$$\Rightarrow u(x) - u(0) = L^{-1}(u)$$
$$\Rightarrow u(x) = A + L^{-1}(u) \qquad (12)$$

Substituting the series assumption, $u=\sum_{n=0}^{\infty}u_n$ into both sides of above equation

$$\sum_{n=0}^{\infty}u_n(x)=A+L^{-1}\left(\sum_{n=0}^{\infty}u_n(x)\right) \qquad (13)$$

In view of (13), we have the following recursive relation

$$\begin{aligned} u_{\circ}(x) &= A, \\ u_{k+1}(x) &= L^{-1}(u_k(x)); \quad k\geq 0. \end{aligned} \qquad (14)$$

Consequently, we obtain

$$\begin{aligned} u_{\circ}(\mathrm{x}) &= \mathrm{A}, \\ u_1(\mathrm{x}) &= \mathrm{L}^{-1}(\mathrm{u}_{\circ}(\mathrm{x})) = Ax, \\ u_2(\mathrm{x}) &= \mathrm{L}^{-1}(\mathrm{u}_1(\mathrm{x})) = \frac{Ax^2}{2}, \\ u_3(\mathrm{x}) &= \mathrm{L}^{-1}(\mathrm{u}_2(\mathrm{x})) = \frac{Ax^3}{6}, \\ &\vdots \end{aligned} \qquad (15)$$

Hence $u=\sum_{n=0}^{\infty}u_n$ gives the solution in a series form as

$$\begin{aligned} u(x) &= A\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right) \\ u(x) &= Ae^x. \end{aligned}$$

Example 5:

Solve the following homogeneous differential equation by using Adomian decomposition method.

$$u''(x)=xu; \quad u(0)=A, \ u'(0)=B. \qquad (16)$$

Solution

In an operator form the given equation becomes

$$Lu=xu, \qquad (17)$$

where L is the differential operator given by

$$L(.) = \frac{d^2}{dx^2}(.), \tag{18}$$

and therefore the inverse operator L^{-1} is defined by

$$L^{-1}(.) = \int_0^x \int_0^x (.)dxdx. \tag{19}$$

Applying L^{-1} to both sides of (17) and using the initial condition we obtain

$$L^{-1}(Lu) = L^{-1}(xu)$$

So that

$$u(x) - xu'(0) - u(0) = L^{-1}(xu)$$

$$\Rightarrow u(x) = A + Bx + L^{-1}(xu) \tag{20}$$

Substituting the series assumption, $u = \sum_{n=0}^{\infty} u_n$ into both sides of above equation

$$\sum_{n=0}^{\infty} u_n(x) = A + Bx + L^{-1}\left(x \sum_{n=0}^{\infty} u_n(x) \right) \tag{21}$$

Following the decomposition method we obtain the following recursive relation

$$\begin{aligned} u_\circ(x) &= A + Bx, \\ u_{k+1}(x) &= L^{-1}(xu_k(x)); \quad k \geq 0. \end{aligned} \tag{22}$$

Consequently, we obtain

$$\begin{aligned} u_\circ(x) &= A + Bx, \\ u_1(x) &= L^{-1}(xu_\circ) = \frac{Ax^3}{6} + \frac{Bx^4}{12}, \\ u_2(x) &= L^{-1}(xu_1) = \frac{Ax^6}{180} + \frac{Bx^7}{504}, \\ &\vdots \end{aligned} \tag{23}$$

Thus the solution in series form is

$$u(x) = A\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right) + B\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \cdots\right).$$

Example 6:

Solve the following homogeneous partial differential equation by using Adomian decomposition method.

$$u_x + u_y = x + y, \qquad (24)$$
$$u(x,0) = 0,\ u(0,y) = 0.$$

Solution

In an operator form PDE becomes

$$L_x u(x,y) = x + y - L_y u(x,y), \qquad (25)$$

where the operators are defined as

$$L_x = \frac{\partial}{\partial x},\ L_y = \frac{\partial}{\partial y},$$

and there inverse operators are as

$$L_x^{-1}(.) = \int_0^x (.)dx, \qquad L_y^{-1}(.) = \int_0^y (.)dy.$$

The *x*-solution:

This solution can be obtained by applying L_x^{-1} to both sides of (25),

$$L_x^{-1}L_x u(x,y) = L_x^{-1}(x+y) - L_x^{-1}L_y u(x,y),$$
$$\Rightarrow u(x,y) = u(0,y) + \frac{1}{2}x^2 + xy - L_x^{-1}L_y u,$$
$$\Rightarrow u(x,y) = \frac{1}{2}x^2 + xy - L_x^{-1}L_y u, \qquad (26)$$

obtained on using the given condition $u(0, y) = 0$, by integrating $f\left(x,y\right) = x + y$ with respect to x and using $L_x^{-1}L_x u(x,y) = u(x,y) - u(0,y)$.

Substituting the unknown function $u(x, y)$ as an infinite number of components $u_n(x, y)$, $n \geq 0$ given by $u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$ in (26),

$$\sum_{n=0}^{\infty} u_n(x,y) = \frac{1}{2}x^2 + xy - L_x^{-1}L_y\left[\sum_{n=0}^{\infty} u_n(x,y)\right], \quad (27)$$

$$u_0 + u_1 + u_2 + u_3 + \ldots = \frac{1}{2}x^2 + xy - L_x^{-1}(L_y(u_0 + u_1 + u_2 + \ldots))$$

Consequently, the recursive scheme that will enable us to completely determine the successive components is thus constructed by

$$u_\circ(x,y) = \frac{1}{2}x^2 + xy,$$
$$u_{k+1}(x,y) = -L_x^{-1}(L_y(u_k)); \quad k \geq 0. \quad (28)$$

Which gives

$$u_\circ(x,y) = \frac{1}{2}x^2 + xy,$$
$$u_1(x,y) = -L_x^{-1}(L_y(u_0)) = -L_x^{-1}\left(L_y\left(\frac{1}{2}x^2 + xy\right)\right) = -\frac{1}{2}x^2,$$
$$u_2(x,y) = -L_x^{-1}(L_y(u_1)) = -L_x^{-1}\left(L_y\left(-\frac{1}{2}x^2\right)\right) = 0. \quad (29)$$

Accordingly, $u_k(x, y) = 0$, $k \geq 2$. Having determined the components of $u(x, y)$, we find

$$u = u_0 + u_1 + u_2 + u_3 + \ldots = \frac{1}{2}x^2 + xy - \frac{1}{2}x^2 = xy \quad (30)$$

the exact solution of the equation.

The y-solution:

It is important to note that the exact solution can also be obtained by finding the y-solution. In an operator form we can write the given equation as

$$L_y u = x + y - L_x u, \quad (31)$$

By applying L_y^{-1} to both sides of (31),

$$L_y^{-1}L_y u(x,y) = L_y^{-1}(x+y) - L_y^{-1}L_x u(x,y),$$
$$\Rightarrow u(x,y) = u(x,0) + xy + \frac{1}{2}y^2 - L_y^{-1}L_x u,$$
$$\Rightarrow u(x,y) = xy + \frac{1}{2}y^2 - L_y^{-1}L_x u, \quad (32)$$

Using $u(x,y)=\sum_{n=0}^{\infty}u_n(x,y)$ on both sides of (32),

$$\sum_{n=0}^{\infty}u_n(x,y)=xy+\frac{1}{2}y^2-L_y^{-1}L_x\left[\sum_{n=0}^{\infty}u_n(x,y)\right], \tag{33}$$

$$u_0+u_1+u_2+u_3+\ldots=xy+\frac{1}{2}y^2-L_y^{-1}(L_x(u_0+u_1+u_2+\ldots))$$

The recursive scheme will be defined as

$$u_\circ(x,y)=xy+\frac{1}{2}y^2,$$
$$u_{k+1}(x,y)=-L_y^{-1}(L_x(u_k));\quad k\geq 0. \tag{34}$$

So we have

$$u_\circ(x,y)=xy+\frac{1}{2}y^2,$$
$$u_1(x,y)=-L_y^{-1}(L_x(u_0))=-L_y^{-1}\left(L_x\left(xy+\frac{1}{2}y^2\right)\right)=-\frac{1}{2}y^2,$$
$$u_2(x,y)=-L_y^{-1}(L_x(u_1))=-L_y^{-1}\left(L_x\left(-\frac{1}{2}y^2\right)\right)=0. \tag{35}$$

Consequently, $u_k(x,y)=0$, $k\geq 2$. Having determined the components of $u(x,y)$, we find

$$u=u_0+u_1+u_2+u_3+\ldots=xy+\frac{1}{2}y^2-\frac{1}{2}y^2=xy \tag{36}$$

the exact solution of the equation.

Note:

The exact solution of the given PDE can be obtained by determining the *x*-solution or y-solution only as discussed above, depending upon the given equation.

Exercises

1. Calculate the Adomian polynomials for the following non-linear functions

   a. $f(u) = u^3$,

   b. $f(x) = \sinh\left(\frac{x}{2}\right)$,

   c. $f(u) = u^{-m}; m > 0$.

2. Use the Adomian decomposition method to show that the exact solution can be obtained by determining the *x*-solution or the *y*-solution:

   a. $u_x - u_y = 0;\ \ u(x,0) = x,\ u(0,y) = y.$

   b. $xu_x + u_y = 3u;\ \ u(x,0) = x^2,\ u(0,y) = 0.$

   c. $u_x - yu = 0;\ \ u(0,y) = 1.$

3. Solve the following homogeneous partial differential equation by using Adomian decomposition method

$$u_x + u_y + u_z = u;$$

$$u(0,y,z) = 1 + e^y + e^z,\ u(x,0,z) = 1 + e^x + e^z,\ u(x,y,0) = 1 + e^x + e^y,$$

$$where\ u = u(x,y,z).$$

Lecture 33

Applications of Adomian Decomposition Method

In this lecture the singular initial value problems, linear and nonlinear, homogeneous and nonhomogeneous, generalized Emden-Fowler equation and Bratu-type equations are investigated by using Adomian decomposition method. The solutions are constructed in the form of a convergent series.

ADOMIAN METHOD FOR SINGULAR INITIAL VALUE PROBLEMS IN SECOND-ORDER ODES

The studies of singular initial value problems in the second order ordinary differential equations (ODEs) have attracted the attention of many mathematicians and physicists. One of the equations describing this type is the Lane–Emden-type equations formulated as

$$y'' + \frac{2}{x} y' + f(y) = 0, \quad 0 < x \le 1,$$
$$y(0) = A, \; y'(0) = B. \tag{1}$$

On the other hand, studies have been carried out on another class of singular initial value problems of the form

$$y'' + \frac{2}{x} y' + f(x, y) = g(x), \qquad 0 < x \le 1, \tag{2}$$
$$y(0) = A, \qquad y'(0) = B,$$

where A and B are constants, $f(x, y)$ is a continuous real valued function, and $g(x) \in C[0,1]$. Eq(2) differs from the classical Lane–Emden-type equations (1) for the function $f(x, y)$ and for the inhomogeneous term $g(x)$.

Eq.(1) with specializing $f(y)$ was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and theory of thermionic currents. Due to the significant applications of Lane–Emden-type equations in the scientific community, various forms of $f(y)$ have been investigated in many research works.

In recent years a large amount of literature developed concerning Adomian decomposition method, and the related modification to investigate various scientific models. The Adomian decomposition method provides the solution in a rapidly convergent series with components that are elegantly computed. A reliable part of this approach is how this method can be modified to address the concept of singular points. To properly address this question, we may require slight variation of the decomposition algorithm as described in previous lecture. An alternate framework can be designed to overcome the difficulty of the singular point at x=0.

The Adomian decomposition method usually defines the equation in an operator form by considering the highest-ordered derivative in the problem. To overcome the singularity behavior, we define the differential operator L in terms of the two derivatives contained in the problem. We rewrite (2) in the form

$$Ly = -f(x,y) + g(x), \qquad (3)$$

where the differential operator L in terms of two derivatives, $y'' + \frac{2}{x}y'$, is defined by

$$L = x^{-2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right). \qquad (4)$$

The inverse operator L^{-1} is therefore considered a two-fold integral operator defined by

$$L^{-1}(.) = \int_0^x x^{-2}\int_0^x x^2(.)dxdx. \qquad (5)$$

Operating with L^{-1} on (3), it follows

$$y(x) = A + Bx + L^{-1}g(x) - L^{-1}f(x,y). \qquad (6)$$

As the Adomian decomposition method introduces the solution $y(x)$ by an infinite series of components

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \qquad (7)$$

and the nonlinear function $f(x, y)$ by an infinite series of polynomials

$$f(x,y) = \sum_{n=0}^{\infty} A_n, \qquad (8)$$

where the components $y_n(x)$ of solution $y(x)$ will be determined recurrently, and A_n are Adomian polynomials constructed for non-linear function defined as $A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left((x,y)\lambda\right)\right]_{\lambda=0}$.

Substituting (7) and (8) into (6) gives

$$\sum_{n=0}^{\infty} y_n(x) = A + B(x) + L^{-1}g(x) - L^{-1}\sum_{n=0}^{\infty} A_n. \qquad (9)$$

To determine the components $y_n(x)$, we use Adomian decomposition method that suggests the use of the recursive relation

$$y_0(x) = A + B(x) + L^{-1}g(x),$$
$$y_{k+1}(x) = -L^{-1}(A_k), \quad k \geq 0, \tag{10}$$

which gives

$$y_0(x) = A + Bx + L^{-1}g(x),$$
$$y_1(x) = -L^{-1}(A_0),$$
$$y_2(x) = -L^{-1}(A_1),$$
$$y_3(x) = -L^{-1}(A_2), \tag{11}$$
$$\vdots$$

The series solution of $y(x)$ defined by (7) follows immediately.

The main advantage of using this choice for the operator L is that it tackles the differential equation directly without any need for a transformation formula.

Example 1:

Solve the following linear singular initial value problem by using Adomian decomposition method.

$$y'' + \frac{2}{x}y' + y = 6 + 12x + x^2 + x^3,$$
$$y(0) = 0, \qquad y'(0) = 0. \tag{12}$$

Solution

Firstly we have to re-write the given DE in an operator form as

$$Ly = 6 + 12x + x^2 + x^3 - y. \tag{13}$$

Applying L^{-1} to both sides of (13) and using the initial condition we obtain

$$L^{-1}Ly = L^{-1}(6) + 12L^{-1}(x) + L^{-1}(x^2) + L^{-1}(x^3) - L^{-1}(y). \tag{14}$$

As,

$$L^{-1}L(y) = L^{-1}\left(y'' + \frac{2}{x} y' \right)$$
$$= \int_0^x x^{-2} \int_0^x x^2 \left(y'' + \frac{2}{x} y' \right) dx dx$$
$$= \int_0^x x^{-2} \left(x^2 y' - \int_0^x 2xy' dx + \int_0^x 2xy' dx \right) dx$$
$$= \int_0^x y' dx = y(x) - y(0) = y(x),$$

$$L^{-1}(6) = \int_0^x x^{-2} \int_0^x x^2 \left(6\right) dx dx$$
$$= 6\int_0^x x^{-2} \left(\frac{x^3}{3} \right) dx = 2\int_0^x x dx = x^2,$$

$$12L^{-1}(x) = 12\int_0^x x^{-2} \int_0^x x^2 \left(x\right) dx dx$$
$$= 12\int_0^x x^{-2} \left(\frac{x^4}{4} \right) dx = 3\int_0^x x^2 dx = x^3,$$

$\approx L^{-1}(x^2) = \dfrac{x^4}{20}$ and $L^{-1}(x^3) = \dfrac{x^5}{30}$.

Putting all these values in Eq.(14),

$$y = x^2 + x^3 + \frac{1}{20} x^4 + \frac{1}{30} x^5 - L^{-1} y. \qquad (15)$$

Proceeding as before we obtain the recursive relationship

$$y_0(x) = x^2 + x^3 + \frac{1}{20} x^4 + \frac{1}{30} x^5$$
$$y_{k+1}(x) = -L^{-1}(y_k), \qquad k \geq 0. \qquad (16)$$

Consequently, the first few components are as

$$y_0 = x^2 + x^3 + \frac{1}{20}x^4 + \frac{1}{30}x^5,$$

$$y_1 = -L^{-1}(y_0) = -\frac{1}{20}x^4 - \frac{1}{30}x^5 - \frac{1}{840}x^6 - \frac{1}{1680}x^7,$$

$$y_2 = -L^{-1}(y_1) = -\frac{1}{840}x^6 + \frac{1}{1680}x^7 + \frac{1}{60480}x^8 + \frac{1}{151200}x^9,$$

$$y_3 = -L^{-1}(y_2) = -\frac{1}{60480}x^8 - \frac{1}{151200}x^9 - \dots, \qquad (17)$$

Other components can be evaluated in a similar manner. Substituting these values in Eq.(7) and after cancellation, we have

$$y(\text{x}) = x^2 + x^3, \qquad (18)$$

Which is the exact solution.

Example 2:

Solve the following nonlinear singular initial value problem by using Adomian decomposition method.

$$y'' + \frac{2}{x}y' - 6y = 4y\ln y,$$
$$y(0) = 1, \quad y'(0) = 0. \qquad (19)$$

Solution

Re-write the given DE in an operator form as

$$Ly = 4y\ln y + 6y \qquad (20)$$

Applying L^{-1} to both sides of (20) and using the initial condition we obtain

$$L^{-1}Ly = 4L^{-1}\left(y\ln y\right) + 6L^{-1}\left(y\ \right)$$
$$y(x) = 1 + 4L^{-1}\left(y\ln y\right) + 6L^{-1}\left(y\ \right) \qquad (21)$$

Proceeding as previous example we obtain the recursive relationship

$$y_0(x) = 1,$$
$$y_{k+1}(x) = 6L^{-1}(y_k) + 4L^{-1}(A_k), \qquad k \geq 0. \qquad (22)$$

The Adomian polynomials for the nonlinear term *F*(*y*)= *y lny* are computed as follows:

$$
\begin{aligned}
A_0 &= y_0 \ln(y_0),\\
A_1 &= y_1 F'(y_0) = y_1(1+\ln y_0),\\
A_2 &= y_2 F'(y_0) + \frac{y_1^2}{2} F''(y_0) = y_2(1+\ln y_0) + \frac{y_1^2}{2y_0},\\
A_3 &= y_3 F'(y_0) + y_1 y_2 F''(y_0) + \frac{y_1^3}{6} F'''(y_0) = y_3(1+\ln y_0) + \frac{y_1 y_2}{y_0} - \frac{y_1^3}{6y_0^2}, \qquad (23)
\end{aligned}
$$

which are obtained by using reference list of the Adomian polynomials given in lecture 32.

By putting (23) into (22), we get the following components

$$
\begin{aligned}
y_0 &= 1,\\
y_1 &= 6L^{-1}(y_0) + 4L^{-1}(A_0) = x^2,\\
y_2 &= 6L^{-1}(y_1) + 4L^{-1}(A_1) = \frac{1}{2!}x^4,\\
y_3 &= 6L^{-1}(y_2) + 4L^{-1}(A_2) = \frac{1}{3!}x^6,\\
y_4 &= 6L^{-1}(y_3) + 4L^{-1}(A_3) = \frac{1}{4!}x^8,\\
y_5 &= 6L^{-1}(y_4) + 4L^{-1}(A_4) = \frac{1}{5!}x^{10}, \qquad (24)
\end{aligned}
$$

and so on. In view of above equation, the solution in a series form is given by

$$y(x) = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \frac{1}{5!}x^{10} + ..., \qquad (25)$$

and in the closed form

$$y(x) = e^{x^2}. \qquad (26)$$

GENERALIZATION:

Replace the standard coefficient of *y*' in (2) by *n*/*x*, for real *n*; *n*≥0. In other words, a general equation

$$y''+\frac{n}{x}y'+f(x,y)=g(x), \qquad n\geq 0, \tag{27}$$

with initial conditions

$$y(0)=A, \qquad y'(0)=B, \tag{28}$$

can be formulated.

Here, the differential operator is defined as

$$L_n=x^{-n}\frac{d}{dx}\left(x^n\frac{d}{dx}\right), \tag{29}$$

for which the inverse operator L^{-1} is expressed by

$$L_n^{-1}(.)=\int_0^x x^{-n}\int_0^x x^n(.)dxdx. \tag{30}$$

Applying L_n^{-1} to both sides of (27) yields

$$y(x)=A+Bx+L_n^{-1}g(x)-L_n^{-1}f(x,y). \tag{31}$$

Proceeding as before we obtain

$$\begin{aligned} y_0(x)&=A+Bx+L_n^{-1}g(x),\\ y_{k+1}(x)&=-L_n^{-1}A_k, \qquad k\geq 0, \end{aligned} \tag{32}$$

where A_k are Adomian polynomials that represent the nonlinear term *f*(*x*, *y*). In view of (32), the components of the function *y(x)* can be elegantly determined. The slight change we imposed on defining the operator L_n in (29), in terms of the first two derivatives, was successful to overcome the singularity issue for *n*≥0. To illustrate the generalization discussed above, we discuss an example:

Example 3:

Solve the following nonlinear singular initial value problem by using Adomian decomposition method.

$$y''+\frac{6}{x}y'+14y=-4y\ln y,$$
$$y(0)=1,\quad y'(0)=0. \qquad (33)$$

Solution

In an operator form, given differential equation becomes

$$L_n(y)=-14y-4y\ln y. \qquad (34)$$

Recall that the operator L_n is defined by

$$L_n=x^{-6}\frac{d}{dx}\left(x^6\frac{d}{dx}\right), \qquad (35)$$

for which the inverse operator L_n^{-1} is expressed by

$$L_n^{-1}(.)=\int_0^x x^{-6}\int_0^x x^6(.)dxdx. \qquad (36)$$

Operating L_n^{-1} on both sides of (34) we have

$$y=1-14L_n^{-1}(y)-4L_n^{-1}(y\ln y). \qquad (37)$$

Proceeding as before we obtain the recursive relationship

$$y_0(x)=1,$$
$$y_{k+1}(x)=-14L_n^{-1}(y_k)-4L_n^{-1}(A_k),\qquad k\geq 0. \qquad (38)$$

The Adomian polynomials for the nonlinear term $F(y)= y\ lny$ are computed before in (23). Substituting (23) into (38) gives the components

$$\begin{aligned}
y_0 &= 1,\\
y_1 &= -14L_n^{-1}(y_0) - 4L_n^{-1}(A_0) = -x^2,\\
y_2 &= -14L_n^{-1}(y_1) - 4L_n^{-1}(A_1) = \frac{1}{2!}x^4,\\
y_3 &= -14L_n^{-1}(y_2) - 4L_n^{-1}(A_2) = -\frac{1}{3!}x^6,\\
y_4 &= -14L_n^{-1}(y_3) - 4L_n^{-1}(A_3) = \frac{1}{4!}x^8,\\
y_5 &= -14L_n^{-1}(y_4) - 4L_n^{-1}(A_4) = -\frac{1}{5!}x^{10},
\end{aligned} \tag{39}$$

so that other components can be evaluated in a similar manner. In view of (39), the solution in a series form is given by

$$y(x) = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 - \frac{1}{5!}x^{10} + \ldots\ldots, \tag{40}$$

and in closed form

$$y(x) = e^{-x^2}. \tag{41}$$

ADOMIAN DECOMPOSITION METHOD FOR EMDEN–FOWLER EQUATION

Many problems in the literature of mathematical physics can be distinctively formulated as equations of Emden–Fowler type defined in the form

$$y'' + \frac{2}{x}y' + af(x)g(y) = 0, \qquad y(0) = y_0, \qquad y'(0) = 0, \tag{42}$$

where $f(x)$ and $g(y)$ are some given functions of x and y respectively. For $f(x)=1$ and $g(y)=y^n$, Eq.(42) becomes the standard Lane–Emden equation.

The standard coefficient of y' in Emden–Fowler equation is $2/x$. However, if we replace $2/x$ by r/x, for real r, $r \geq 0$, then we write down Emden–Fowler equation in general as

$$y'' + \frac{r}{x}y' + af(x)g(y) = 0, \qquad r \geq 0 \tag{43}$$

with boundary conditions given by

$$y(0) = \alpha, \qquad y'(0) = 0. \tag{44}$$

Introducing the differential operator

$$L = x^{-r} \frac{d}{dx}\left(x^r \frac{d}{dx} \right), \qquad (45)$$

for which the inverse operator L^{-1} is expressed by

$$L^{-1}(.) = \int_0^x x^{-r} \int_0^x x^r (.) dx dx. \qquad (46)$$

In an operator form, Eq. (43) may be rewritten as

$$Ly = -af(x)g(y). \qquad (47)$$

Operating with L^{-1} on (47), we have

$$y = \alpha - aL^{-1}\left(f(x)g(y) \right) \qquad (48)$$

The slight change we imposed in defining the operator L in (45), in terms of the first two derivatives, was successful to overcome the singularity issue for $r\neq 0$.

As discussed above, Adomian decomposition method introduces the decomposition series

$y(x) = \sum_{n=0}^{\infty} y_n(x)$ and the infinite series of polynomials

$$f(y) = \sum_{n=0}^{\infty} A_n(y_0, y_1, \ldots, y_n), \qquad (49)$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently, and A_n are Adomian polynomials. Substituting the value of $y(x)$ and (49) into (48) gives

$$\sum_{n=0}^{\infty} y_n(x) = \alpha - aL^{-1}\left(f(x) \sum_{n=0}^{\infty} A_n(y_0, y_1, \ldots, y_n) \right). \qquad (50)$$

Identifying $y_0(x)=\alpha$, the recursive relation

$$\begin{aligned} &y_0(x) = \alpha, \\ &y_{k+1}(x) = -aL^{-1}\left(f(x)A_k \right), \quad k \geq 0 \end{aligned} \qquad (51)$$

or equivalently

$$\begin{aligned} &y_0(x) = \alpha, \\ &y_{k+1}(x) = -a\int_0^x x^{-r} \int_0^x x^r \left(f(x)A_k \right) dx dx, \quad k \geq 0 \end{aligned} \qquad (52)$$

will lead to the complete determination of the components $y_n(x)$ of $y(x)$. The series solution of $y(x)$ follows immediately.

Example 4:

Solve the following equation by using Adomian decomposition method.

$$y'' + \frac{8}{x} y' + 18ay = -4ay \ln y,$$

where the boundary conditions are given by

$$y(0) = 1, \qquad y'(0) = 0.$$

Solution

Using the recursive relation (52) yields

$$\begin{aligned} &y_0(x) = 1 \\ &y_{k+1}(x) = -18aL^{-1}(y_k) - 4aL^{-1}(A_k), \qquad k \geq 0 \end{aligned} \tag{53}$$

The first few Adomian polynomials for $g(y)=y\ln y$ are given by

$$\begin{aligned} A_0 &= y_0 \ln y_0, \\ A_1 &= y_1(1 + \ln y_0), \\ A_2 &= y_2(1 + \ln y_0) + \frac{y_1^2}{2y_0}. \end{aligned} \tag{54}$$

Using (53) yields

$$\begin{aligned} y_0 &= 1, \\ y_1 &= -18aL^{-1}(y_0) - 4aL^{-1}(A_0) = -ax^2, \\ y_2 &= -18aL^{-1}(y_1) - 4aL^{-1}(A_1) = \frac{a^2}{2} x^4, \\ y_3 &= -18aL^{-1}(y_2) - 4aL^{-1}(A_2) = \frac{a^3}{6} x^6. \end{aligned} \tag{55}$$

Consequently, the series solution is

$$y(x) = 1 - ax^2 + \frac{a^2}{2!} x^4 - \frac{a^3}{3!} x^3 + \ldots \tag{56}$$

and in a closed form by $y(x) = e^{-ax^2}$.

ADOMIAN DECOMPOSITION METHOD FOR BRATU-TYPE EQUATIONS

The standard Bratu's boundary value problem in one-dimensional planar coordinates is of the form

$$u'' + \lambda e^{u} = 0, \qquad 0 < x < 1,$$
$$u(0) = u(1) = 0.$$

The Bratu model appears in a number of applications such as the fuel ignition of the thermal combustion theory. It stimulates a thermal reaction process in a rigid material where the process depends on the balance between chemically generated heat and heat transfer by conduction.

Example 5:

Solve the following Bratu-type model equation by using Adomian decomposition method.

$$u'' - \pi^2 e^{u} = 0, \qquad 0 < x < 1, \tag{57}$$
$$u(0) = u(1) = 0.$$

Solution

The given problem can be written in an operator form as

$$Lu = \pi^2 e^{u} = 0, \qquad 0 < x < 1, \tag{58}$$
$$u(0) = u(1) = 0,$$

where L is the differential operator given by

$$L = \frac{\partial^2}{\partial x^2}.$$

The inverse L^{-1} is assumed to be a two-fold integral operator given by

$$L^{-1}(.) = \int_0^x \int_0^x (.)dxdx.$$

Applying the inverse operator L^{-1} on both sides of (58) and using the initial condition $u(0) = 0$, we find

$$u(x) = ax + L^{-1}(\pi^2 e^{u}), \tag{59}$$

where $a = u'(0)$. Substituting (7) and (8) into the functional equation (59) gives

$$\sum_{n=0}^{\infty} u_n(x) = ax + L^{-1}\left(\pi^2 \sum_{n=0}^{\infty} A_n\right), \tag{60}$$

where A_n are the so-called Adomian polynomials. Identifying the zeroth component $u_0(x)$ by ax, the remaining components $u_n(x)$, $n \geq 1$ can be determined by using the recurrence relation

$$\begin{aligned} &u_0(x) = ax, \\ &u_{k+1}(x) = \pi^2 L^{-1}(A_k), \qquad k \geq 0 \end{aligned} \tag{61}$$

where A_k are Adomian polynomials that represent the nonlinear term e^u and given by

$$\begin{aligned} &A_0 = e^{u_o}, \\ &A_1 = u_1 e^{u_o}, \\ &A_2 = \left(u_2 + \frac{1}{2}u_1^2\right)e^{u_o}, \\ &A_3 = \left(u_3 + u_1 u_2 + \frac{1}{6}u_1^3\right)e^{u_o}, \\ &A_4 = \left(u_4 + u_1 u_3 + \frac{1}{2}u_2^2 + \frac{1}{2}u_1^2 u_2 + \frac{1}{24}u_1^4\right)e^{u_o}, \\ &\ldots \end{aligned} \tag{62}$$

Other polynomials can be generated in a similar way to enhance the accuracy of approximation. Combining (61) and (62) yields

$$\begin{aligned} &u_0(x) = ax, \\ &u_1(x) = -\frac{\pi^2}{a^2}(-e^{ax} + ax + 1), \\ &u_2(x) = -\frac{\pi^4}{4a^4}(-e^{2ax} + 4axe^{ax} - 4e^{ax} + 2ax + 5), \\ &u_3(x) = \frac{\pi^6}{12a^6}(e^{3ax} + 6e^{2ax}(1 - ax) + 3e^{ax}(2a^2x^2 - 6ax + 5) - 6ax - 22), \\ &\ldots \end{aligned} \tag{63}$$

In view of (63), the solution $u(x)$ is readily obtained in a series form by

$$\begin{aligned} u(x) = ax &- \frac{\pi^2}{a^2}(-e^{ax} + ax + 1) - \frac{\pi^4}{4a^4}(-e^{2ax} + 4axe^{ax} - 4e^{ax} + 2ax + 5) \\ &+ \frac{\pi^6}{12a^6}(e^{3ax} + 6e^{2ax}(1 - ax) + 3e^{ax}(2a^2x^2 - 6ax + 5) - 6ax - 22) \\ &+ \ldots, \end{aligned}$$

or equivalently

$$u(x) = ax + \frac{\pi^2}{2!}x^2 + \frac{\pi^2 a}{3!}x^3 + \left(\frac{\pi^2 a^2 + \pi^4}{4!}\right)x^4 + \left(\frac{\pi^2 a^3 + 4\pi^4 a}{5!}\right)x^5$$
$$+\left(\frac{11\pi^4 a^2 + \pi^2 a^4 + 4\pi^6}{6!}\right)x^6 + \left(\frac{26\pi^4 a^3 + \pi^2 a^5 + 34\pi^6 a}{6!}\right)x^7$$
$$+\ldots,$$
$$\Rightarrow u(x) = -\ln\left(1 + \cos\left(\left(\frac{1}{2} + x\right)\pi\right)\right).$$

Exercises

Solve the following problems by using Adomian decomposition method.

1. $y'' + \frac{2}{x}y' = 2(2x^2 + 3)y,$
   $y(0) = 1, \quad y'(0) = 0.$

2. $y'' + \frac{5}{x}y' + 8a(e^y + 2e^{y/2}) = 0,$
   $y(0) = 0, \quad y'(0) = 0.$

3. $u'' + \pi^2 e^{-u} = 0, \qquad 0 < x < 1,$
   $u(0) = u(1) = 0.$

Lecture 34

Convergence of Adomian Decomposition Method

In this lecture the rate of convergence of Adomian decomposition method will be discussed. Cherruault, proposed a new definition of the technique to prove the convergence, of this method, under suitable and reasonable hypothesis. We used Cherruault's definition and consider the order of convergence of the method.

THE ADOMIAN DECOMPOSITION METHOD FOR FUNCTIONAL EQUATIONS

Consider the functional equation

$$y - Ny = f, \tag{1}$$

where N is a non-linear operator from a Hilbert space H into H, f is a given function in H and we are looking for $y \epsilon H$ satisfying (1).

As the initial Adomian technique considers of representing y as a series

$$y = \sum_{i=0}^{\infty} y_i, \tag{2}$$

and the non-linear operator as the sum of the series

$$Ny = \sum_{n=0}^{\infty} A_n,$$

The method consist of the scheme:

$$\begin{cases} y_0 = f, \\ y_{k+1} = A_n(y_0, y_1, \ldots, y_n), \end{cases} \tag{3}$$

where A_n's are polynomials in $y_0, y_1, \ldots, y_n$ called Adomian polynomials, obtained by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0} ; n = 0, 1, 2, \ldots$$

The Adomian technique is equivalent to determining the sequence

$$S_n = y_1 + y_2 + \ldots + y_n,$$

by using the iterative scheme

$$\begin{cases} S_0 = 0, \\ S_{n+1} = N(y_0 + S_n). \end{cases} \qquad (4)$$

Associated with the functional equation

$$S = N(y_0 + S). \qquad (5)$$

For the study of the numerical resolution of (4) Cherruault used fixed point theorem.

<u>Theorem</u>.

Let N be an operator from a Hilbert space H in to H and y be the exact solution of (1). $\sum_{i=0}^{\infty} y_i$, which is obtained by (3), converges to y when $\exists 0 \le \alpha < 1, \ \|y_{k+1}\| \le \alpha \|y_k\|, \ \ \forall k \in N \cup \{0\}$.

Proof.

We have

$$\begin{aligned} &S_0 = 0, \\ &S_1 = y_1, \\ &S_2 = y_1 + y_2, \\ &\vdots \\ &S_n = y_1 + y_2 + \ldots + y_n, \end{aligned}$$

and we show that, $\{S_n\}_{n=0}^{+\infty}$ is a Cauchy sequence in the Hilbert space H. For this reason, consider,

$$\|S_{n+1} - S_n\| = \|y_{n+1}\| \le \alpha \|y_n\| \le \alpha^2 \|y_{n-1}\| \le \ldots \le \alpha^{n+1} \|y_0\|.$$

But for every $n, m \in N, n \ge m,$ we have

$$\begin{aligned} \|S_n - S_m\| &= \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \ldots + (S_{m+1} - S_m)\| \\ &\le \|(S_n - S_{n-1})\| + \|(S_{n-1} - S_{n-2})\| + \ldots + \|(S_{m+1} - S_m)\| \le \alpha^n \|y_o\| + \alpha^{n-1} \|y_o\| + \ldots + \alpha^{m+1} \|y_o\| \\ &\le \left(\alpha^{m+1} + \alpha^{m+2} + \ldots\right) \|y_o\| = \frac{\alpha^{m+1}}{1-\alpha} \|y_o\|. \end{aligned}$$

Hence, $\lim\limits_{n,m \to +\infty} \|S_n - S_m\| = 0,$ i.e., $\{S_n\}_{n=0}^{+\infty}$ is a Cauchy sequence in the Hilbert space H and it implies that $\exists S, S \in H, \quad \lim\limits_{n \to +\infty} S_n = S,$

i.e., $S=\sum_{n=0}^{+\infty} y_n$. But, to solve Eq. (1) is equivalent to solving Eq. (5) and it implies that if N be a continuous operator then

$$N(y_0+S)=N\left(\lim_{n\to+\infty}(y_0+S_n)\right)=\lim_{n\to+\infty}N(y_0+S_n)=\lim_{n\to+\infty}S_{n+1}=S,$$

i.e., S is a solution of Eq. (1), too.

Definition. For every $i\in N\cup\{0\}$ we define

$$\alpha_i=\begin{cases}\dfrac{\|y_{i+1}\|}{\|y_i\|}, & \|y_i\|\neq 0,\\ 0, & \|y_i\|=0.\end{cases} \tag{6}$$

Corollary.

In above theorem, $\sum_{i=0}^{\infty} y_i$ converges to exact solution y, when $0\le\alpha_i<1,\ i=1,2,3,\ldots$ □

The standard Adomian decomposition method usually defines the equation in an operator form by considering the highest-ordered derivative in the problem

$$L=\frac{d^n}{dx^n},$$

So

$$L^{-1}(.)=\int_0^x\int_0^x\ldots\int_0^x(.)dxdx\ldots dx.$$

Example 1.

Consider the initial value problem

$$\begin{aligned}&y'+(1+x^2)y^2=x^4+2x^3+2x^2+2x+2,\\&y(0)=1.\end{aligned} \tag{7}$$

Solution:

In an operator form, (7) becomes

$$Ly = \left(x^4 + 2x^3 + 2x^2 + 2x + 2\right) - (1 + x^2)y^2, \tag{8}$$

where

$$L = \frac{d}{dx},$$

and

$$L^{-1}(.) = \int_0^x (.)dx.$$

By applying L^{-1} on the both sides of (8), we obtain

$$y = L^{-1}\left(x^4 + 2x^3 + 2x^2 + 2x + 2\right) - L^{-1}\left((1 + x^2)y^2\right) + y(0). \tag{9}$$

So, we have

$$\begin{cases} y_0 = L^{-1}\left(x^4 + 2x^3 + 2x^2 + 2x + 2\right) + y(0), \\ y_{n+1} = -L^{-1}\left((1 + x^2)A_n\right), \quad n \geq 0 \end{cases} \tag{10}$$

where A_n*'s* are Adomian polynomials for the nonlinear term y^2, as

$$\begin{aligned} A_0 &= y_0^2, \\ A_1 &= 2y_0y_1, \\ A_2 &= 2y_0y_2 + y_1^2, \\ A_3 &= 2y_0y_3 + 2y_1y_2, \\ &\vdots \end{aligned}$$

which are obtained by using a reference list of the Adomian polynomials given in lecture 32.

Hence

$$\begin{aligned} y_0 &= \frac{x^5}{5} + \frac{x^4}{2} + \frac{2x^3}{3} + x^2 + 2x + 1, \\ y_1 &= -L^{-1}\left((1 + x^2)y_0^2\right) = -L^{-1}\left((1 + x^2)y_0^2\right) \\ &= -\frac{x^{13}}{325} - \frac{x^{12}}{60} - \frac{167x^{11}}{3300} - \frac{19x^{10}}{150} - \frac{497x^9}{1620} - \frac{3x^8}{5} - \frac{311x^7}{315} - \frac{68x^6}{45} - \frac{32x^5}{15} - \frac{7x^4}{3} - \frac{7x^3}{3} - 2x^2 - x, \\ &\vdots \end{aligned}$$

By computing α_i's for this problem, we have

$$\alpha_0 = \frac{\|y_1\|}{\|y_0\|} = 1.5750915 > 1,$$
$$\alpha_1 = \frac{\|y_2\|}{\|y_1\|} = 2.9244392 > 1,$$
$$\vdots$$

Here, α_i's are not less than one and thus standard Adomian decomposition method is not convergent. So, the standard Adomian decomposition method may be divergent, even, to solve a simple nonsingular initial value problem.

Now we focus on singular ODEs. For this reason, consider the Lane–Emden equation formulated as

$$y'' + \frac{2}{x} y' + F(x, y) = g(x), \qquad 0 < x \le 1, \tag{11}$$
$$y(0) = A, \qquad y'(0) = B,$$

where A and B are constants, $F(x, y)$ is a continuous real valued function, and $g(x) \in C[0,1]$. Usually, the standard Adomian decomposition method may be divergent to solve singular Lane–Emden equations. To overcome the singularity behavior, Wazwaz defined the differential operator L in terms of two derivatives contained in the problem. He rewrote (11) in the form

$$Ly = -F(x, y) + g(x),$$

where the differential operator L is defined by

$$L = x^{-2} \frac{d}{dx}\left(x^2 \frac{d}{dx} \right).$$

There is an example of the form (11) that both standard and modified Adomian decomposition methods are convergent.

Example 2.

Consider the linear singular initial value problem

$$y'' + \frac{2}{x} y' + y = x^5 + 30x^3,$$
$$y(0) = 0, \qquad y'(0) = 0. \tag{12}$$

Solution:

Standard Adomian method:

In the operator form, (12) becomes

$$Ly = x^5 + 30x^3 - y - \frac{2}{x}y', \tag{13}$$

where

$$L = \frac{d^2}{dx^2},$$

so

$$L^{-1}(.) = \int_0^x \int_0^x (.)dxdx.$$

By applying L^{-1} on the both sides of (13), we obtain

$$y = L^{-1}\left(x^5 + 30x^3\right) + y(0) + xy'(0) - L^{-1}\left(\frac{2}{x}y' + \mathrm{y}\right).$$

We obtain the recursive relationship

$$\begin{cases} y_0 = L^{-1}\left(x^5 + 30x^3\right) + y(0) + xy'(0), \\ y_{n+1} = -L^{-1}\left(\frac{2}{x}y' + \mathrm{y}\right), \quad n \ge 0 \end{cases}$$

Consequently, the first few components are as

$$\begin{aligned} y_0 &= \frac{3x^5}{2} + \frac{x^7}{42}, \\ y_1 &= -\frac{3x^5}{4} - \frac{11x^7}{252} - \frac{x^9}{3024}, \\ y_2 &= \frac{3x^5}{8} + \frac{7x^7}{216} + \frac{25x^9}{36288} + \frac{x^{11}}{332640}, \\ &\vdots \end{aligned} \tag{14}$$

and

$$\alpha_0 = 0.5178432480 < 1,$$
$$\alpha_1 = 0.5118817363 < 1,$$
$$\alpha_2 = 0.5079364636 < 1,$$
$$\vdots$$

Hence the standard Adomian decomposition method is convergent.

Modified Adomian method:

In an operator form Eq. (12) becomes

$$Ly = x^5 + 30x^3 - y, \tag{15}$$

where

$$L = x^{-2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right).$$

The inverse operator L^{-1} is therefore expresses as

$$L^{-1}(.) = \int_0^x x^{-2}\int_0^x x^2(.)dxdx.$$

Operating with L^{-1} on (15), it follows

$$y(x) = y(0) + L^{-1}\left(x^5 + 30x^3\right) - L^{-1}(y),$$

Proceeding as in previous lecture, we obtain

$$y_0(x) = y(0) + L^{-1}\left(x^5 + 30x^3\right),$$
$$y_{n+1}(x) = -L^{-1}(y_n), \qquad n \geq 0.$$

This gives the first few components

$$y_0 = x^5 + \frac{x^7}{56},$$
$$y_1 = -\frac{x^7}{56} - \frac{x^9}{5040},$$
$$y_2 = \frac{x^9}{5040} + \frac{x^{11}}{665280}, \tag{16}$$
$$\vdots$$

and

$$\begin{aligned} &\alpha_0 = 0.01521198194 < 1, \\ &\alpha_1 = 0.009843639085 < 1, \qquad (17) \\ &\vdots \end{aligned}$$

The obtained results in (14) and (16) show that the rate of convergence of modified Adomian method is higher than standard Adomian method for this problem.

Exercises

1. Check the convergence of linear nonsingular initial value problem,

$$\begin{aligned} &y'' + y' = 2x + 2, \\ &y(0) = 0, \; y'(0) = 0. \end{aligned}$$

2. Check the convergence of linear singular initial value problem

$$\begin{aligned} &y'' + \frac{2}{x} y' + y = 6 + 12x + x^2 + x^3, \\ &y(0) = 0, \qquad y'(0) = 0. \end{aligned}$$

Lecture 35

Adomian Method for Higher-Order Ordinary Differential Equations

It is possible to model many of the physical events that take place in nature using linear and nonlinear differential equations. This modelling enables us to understand and interpret the particular event in a much better manner. Thus, finding the analytical and approximate solutions of such models with initial and boundary conditions gain importance. Differential equations have had an important place in engineering since many years. Scientists and engineers generally examine systems that undergo changes.

Many methods have been developed to determine the analytical and approximate solutions of linear and nonlinear differential equations with initial and boundary value conditions and among these methods, the Adomian decomposition method (ADM), homotopy perturbation method, variational iteration method, and homotopy analysis method can be listed.

Recall that in previous lectures solving differential equations, solutions are usually obtained as exact solutions defined in closed form expressions, or as series solutions normally obtained from concrete problems.

To apply the Adomian decomposition method for solving nonlinear ordinary differential equations, we consider the equation

$$Ly + R(y) + F(y) = g(x), \qquad (1)$$

where the differential operator L may be considered as the highest order derivative in the equation, R is the remainder of the differential operator, *F(y)* expresses the nonlinear terms, and *g(x)* is an inhomogeneous term. If L is a first order operator defined by

$$L = \frac{d}{dx},$$

then, we assume that L is invertible and the inverse operator L^{-1} is given by

$$L^{-1}(.) = \int_0^x (.)dx.$$

$$\Rightarrow L^{-1}(Ly) = y(x) - y(0).$$

However, if L is a second order differential operator given by

$$L(.) = \frac{d^2}{dx^2}(.),$$

and therefore the inverse operator L^{-1} is defined by

$$L^{-1}(.) = \int_0^x \int_0^x (.)dxdx$$

$$\Rightarrow L^{-1}L(y) = y(x) - y(0) - xy'(0)$$

In a parallel manner, if L is a third order differential operator, we can easily show that

$$L^{-1}L(y) = y(x) - y(0) - xy'(0) - \frac{1}{2!}x^2 y''(0)$$

For higher order operators we can easily define the related inverse operators in a similar way.

Applying L^{-1} to both sides of (1) gives

$$y(x) = \psi_0 + L^{-1}g(x) - L^{-1}R(y) - L^{-1}F(y), \tag{2}$$

Where

$$\psi_0 = \begin{cases} y(0), & \text{for } L = \frac{d}{dx}, \\ y(0) + xy'(0), & \text{for } L = \frac{d^2}{dx^2}, \\ y(0) + xy'(0) + \frac{1}{2!}x^2 y''(0), & \text{for } L = \frac{d^3}{dx^3}, \\ y(0) + xy'(0) + \frac{1}{2!}x^2 y''(0) + \frac{1}{3!}x^3 y'''(0), & \text{for } L = \frac{d^4}{dx^4}, \\ y(0) + xy'(0) + \frac{1}{2!}x^2 y''(0) + \frac{1}{3!}x^3 y'''(0) + \frac{1}{4!}x^4 y^{(4)}(0), & \text{for } L = \frac{d^5}{dx^5}, \end{cases} \tag{3}$$

and so on.

The Adomian decomposition method admits the decomposition of y into an infinite series of components

$$y(x) = \sum_{n=0}^{\infty} y_n, \tag{4}$$

and the nonlinear term *F(y)* be equated to an infinite series of polynomials

$$F(y) = \sum_{n=0}^{\infty} A_n, \tag{5}$$

where A_n are the Adomian polynomials. Substituting (4) and (5) into (2) gives

$$\sum_{n=0}^{\infty} y_n = \psi_0 + L^{-1}g(x) - L^{-1}R\left(\sum_{n=0}^{\infty} y_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{6}$$

The various components y_n of the solution y can be easily determined by using the recursive relation

$$\begin{cases} y_0 = \psi_0 + L^{-1}g(x), \\ y_{n+1} = -L^{-1}R\left(y_n\right) - L^{-1}\left(A_n\right), \ n \geq 0. \end{cases} \tag{7}$$

Consequently, the first few components can be written as

$$\begin{cases} y_0 = \psi_0 + L^{-1}g(x), \\ y_1 = -L^{-1}R\left(y_0\right) - L^{-1}\left(A_0\right), \\ y_2 = -L^{-1}R\left(y_1\right) - L^{-1}\left(A_1\right), \\ y_3 = -L^{-1}R\left(y_2\right) - L^{-1}\left(A_2\right). \end{cases} \tag{8}$$

Having determined the components y_n, $n \geq 0$, the solution y in a series form follows immediately. As stated before, the series may be summed to provide the solution in a closed form. However, for concrete problems, the n−term partial sum

$$\phi_n = \sum_{k=0}^{n-1} y_k,$$

may be used to give the approximate solution.

Example 1:

Use the Adomian decomposition method to find the solution of the following second order nonlinear ordinary differential equation

$$y'' + (y')^2 + y^2 = 1 - \sin x; \quad y(0) = 0, y'(0) = 1. \qquad (9)$$

Solution

In an operator form, the given equation can be written as

$$Ly = 1 - \sin x - (y')^2 - y^2, \qquad (10)$$

where L is a second order differential operator. It is clear that L^{-1} is invertible and given by

$$L^{-1}(.) = \int_0^x \int_0^x (.)dxdx.$$

Applying L^{-1} to both sides of (10) and using the initial condition we obtain

$$L^{-1}Ly = L^{-1}(1 - \sin x) - L^{-1}\left((y')^2 + y^2\right), \qquad (11)$$

$$\Rightarrow y(x) = y(0) + xy'(0) + \frac{x^2}{2} + \sin x - x - L^{-1}\left((y')^2 + y^2\right),$$

$$= x + \frac{x^2}{2} + \sin x - x - L^{-1}\left((y')^2 + y^2\right),$$

$$y(x) = \frac{x^2}{2} + \sin x - L^{-1}\left((y')^2 + y^2\right). \qquad (12)$$

The decomposition method suggests that the solution $y(x)$ be expressed by the decomposition series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \qquad (13)$$

and the nonlinear terms $(y')^2 + y^2$ be equated to

$$(y')^2 + y^2 = \sum_{n=0}^{\infty} A_n, \qquad (14)$$

where $y_n(x)$, $n \geq 0$ are the components of $y(x)$ that will be determined recursively, and A_n, $n \geq 0$ are the Adomian polynomials that represent the nonlinear term $(y')^2 + y^2$.

Inserting equations (13) and (14) into (12), gives

$$\sum_{n=0}^{\infty} y_n(x) = \frac{x^2}{2} + \sin x - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right)$$

The zeroth component y_0 is usually defined by all terms that are not included under the operator L^{-1}. The remaining components can be determined recurrently such that each term is determined by us ing t he pr evious component. Consequently, t he components of $y(x)$ can be el egantly determined by using the recursive relation

$$\begin{aligned} y_0(x) &= \frac{x^2}{2} + \sin x, \\ y_{k+1}(x) &= -L^{-1}(A_k), \quad k \geq 0, \end{aligned} \tag{15}$$

Note that the Adomian polynomials A_n for the nonlinear term $(y')^2 + y^2$ were determined before by using Adomian algorithm and are calculated as

$$\begin{aligned} A_0 &= (y_0')^2 + y_0^2, \\ A_1 &= 2\left(y_0' y_1' + y_0 y_1\right), \\ A_2 &= 2\left(y_0' y_2' + y_0 y_2\right) + (y_1')^2 + y_1^2, \end{aligned} \tag{16}$$

and s o on. U sing t hese pol ynomials i nto (15), the f irst f ew c omponents c an be determined recursively by

$$\begin{aligned} y_0 &= \frac{x^2}{2} + \sin x, \\ y_1 &= -L^{-1}\left((y_0')^2 + y_0^2\right) = -\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^6}{120}, \\ y_2 &= -L^{-1}\left(2\left(y_0' y_1' + y_0 y_1\right)\right) = \frac{x^3}{3} + \frac{x^4}{3} + \frac{2x^5}{15} + \ldots, \end{aligned} \tag{17}$$

Consequently, the solution in a series form is given by

$$y(x) = \sin x + \frac{x^2}{2} - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^6}{120} + \frac{x^3}{3} + \frac{x^4}{3} + \frac{2x^5}{15} + \ldots$$

After cancellation of terms, we have

$$y(x) = \sin x.$$

Example 2:

Find the solution of the following third order nonlinear ordinary differential equation by using the Adomian decomposition method.

$$y^{(3)}(x)+\left(y''(x)\right)^2+\left(y'(x)\right)^2=2+\cos x;$$
$$y(0)=0, y'(0)=2, y''(0)=0. \qquad (18)$$

Solution

In an operator form, the given equation becomes

$$Ly=2+\cos x-\left(y'\right)^2-\left(y''\right)^2, \qquad (19)$$

where L is a third order differential operator. It is clear that L^{-1} is invertible and given by

$$L^{-1}(.)=\int_0^x\int_0^x\int_0^x(.)dxdxdx$$
$$L^{-1}L(y)=y(x)-y(0)-xy'(0)-\frac{1}{2!}x^2y''(0).$$

So by applying L^{-1} on both sides of (19), we have

$$y(x)-y(0)-xy'(0)-\frac{1}{2!}x^2y''(0)=L^{-1}\left(2+\cos x\right)-L^{-1}\left(\left(y'\right)^2+\left(y''\right)^2\right),$$
$$y(x)-2x=\frac{x^3}{3}+x+\sin x-L^{-1}\left(\left(y'\right)^2+\left(y''\right)^2\right),$$
$$y(x)=\frac{x^3}{3}+3x+\sin x-L^{-1}\left(\left(y'\right)^2+\left(y''\right)^2\right), \qquad (20)$$

We next represent the linear term $y(x)$ by the decomposition series of components $y_n(x)$, $n\geq 0$, equate the nonlinear term y'^2 by the Adomian polynomials A_n, $n\geq 0$, and equate the nonlinear term y''^2 by the series of Adomian polynomials B_n, $n\geq 0$, to find

$$\sum_{n=0}^{\infty}y_n(x)=\frac{x^3}{3}+3x+\sin x-L^{-1}\left(\sum_{n=0}^{\infty}A_n+\sum_{n=0}^{\infty}B_n\right),$$

Identifying the zeroth component y_0, and following the decomposition method we set the recursive relation

$$y_0(x) = \frac{x^3}{3} + 3x + \sin x,$$
$$y_{k+1}(x) = -L^{-1}\left(A_k + B_k\right), \quad k \geq 0, \tag{21}$$

Consequently, for finding the first few components of the solution proceeds as

$$y_1(x) = -L^{-1}\left(A_0 + B_0\right)$$

As

$$A_0 = y_0'^2 = \left(x^2 + 3 - \cos x\right)^2$$
$$A_0 = x^4 + 6x^2 + 9 - 6\cos x - 2x^2\cos x + \cos^2 x$$

and

$$B_0 = y_0''^2 = \left(2x + \sin x\right)^2 = 4x^2 + 4x\sin x + \sin^2 x$$
$$y_1 = -L^{-1}\left(A_0 + B_0\right) = -L^{-1}\left(x^4 + 10x^2 + 10 - 6\cos x - 2x^2\cos x + 4x\sin x\right)$$
$$y_1 = -\frac{2}{3}x^3 - \frac{2}{15}x^5 - \frac{1}{210}x^7$$

Similarly you can find A_1 and B_1 by using Adomian algorithm, as

$$A_1 = 2y_0'y_1'$$
$$A_1 = -\frac{1}{15}x^8 - \frac{23}{15}x^6 - 8x^4 - 12x^2 + \frac{1}{15}x^6\cos x + \frac{4}{3}x^4\cos x + 4x^2\cos x$$

and

$$B_1 = 2y_0''y_1'' = -\frac{4}{5}x^6 - \frac{32}{3}x^4 - 16x^2 - 8x\sin x - \frac{2}{5}x^5\sin x - \frac{16}{3}x^3\sin x$$
$$y_2 = -L^{-1}\left(A_1 + B_1\right)$$
$$y_2 = \frac{2}{5}x^5 + \frac{26}{315}x^7 + \frac{17}{3780}x^9 + \frac{1}{14850}x^{11}$$

Consequently, the solution in a series form is given by

$$y(x) = \frac{x^3}{3} + 3x + \sin x - \frac{2}{3}x^3 - \frac{2}{15}x^5 - \frac{1}{210}x^7 + \frac{2}{5}x^5 + \frac{26}{315}x^7 + \frac{17}{3780}x^9 + \frac{1}{14850}x^{11} + \ldots$$
$$= \sin x + 3x - \frac{1}{3}x^3 + \frac{4}{15}x^5 + \ldots$$

Example 3:

Consider the linear boundary value problem

$$y^{(3)} = y - 3e^{x},$$
$$y'(0) = 0,\ y(1) = 0,\ y(0) = 1. \qquad (22)$$

Find its approximate solution by using Adomian decomposition method.

Solution

In operator form the given differential equation becomes

$$Ly = y - 3e^{x},$$

where L is a third order differential operator. It is clear that L^{-1} is invertible and given by

$$L^{-1}(.) = \int_0^x \int_0^x \int_0^x (.)dxdxdx$$

So by applying L^{-1} on both sides of above equation, we have

$$y = 1 + \frac{1}{2}Ax^{2} - L^{-1}(3e^{x}) + L^{-1}(y), \qquad (23)$$

where $A = y''(0)$. Using $y(x) = \sum_{n=0}^{\infty} y_n(x),$ in (23)

$$\sum_{n=0}^{\infty} y_n(x) = 1 + \frac{1}{2}Ax^{2} - L^{-1}(3e^{x}) + L^{-1}\left(\sum_{n=0}^{\infty} y_n(x)\right),$$

$$\Rightarrow y_0(x) = 1 + \frac{1}{2}Ax^{2} - L^{-1}(3e^{x}),$$
$$y_{n+1}(x) = L^{-1}\left(y_n\right);\ n \geq 0$$

The constant A in the reduction formula will be determined using boundary conditions (22) after finding the decomposition series. From the reduction relation, we can obtain the solution terms of the decomposition series as

$$y_0(x) = 4 + 3x - 3e^x + \frac{1}{2}(A+3)x^2,$$
$$y_1(x) = L^{-1}\left(y_0\right) = 3 + 3x - 3e^x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{x^4}{8} + \frac{(A+3)}{120}x^5,$$
$$y_2(x) = L^{-1}\left(y_1\right) = 1 + 3x - 3e^x + \frac{3}{2}x^2 + \frac{x^3}{2} + \frac{x^4}{8} + \frac{x^5}{40} + \frac{x^6}{180}$$
$$+ \frac{x^7}{1680} + \frac{(A+3)}{40320}x^8,$$
$$\vdots$$

Consequently, the approximate solution obtained by using Adomian decomposition method using the first three terms of the given problem in a series form is given by

$$y(x) = 8 + 9x - 9e^x + \frac{1}{2}(A+9)x^2 + \frac{7}{6}x^3 + \frac{x^4}{4} + \frac{(A+6)}{120}x^5 + \frac{x^6}{180} + \frac{x^7}{1680} + \frac{(A+3)}{40320}x^8.$$

Example 4:

Consider the fourth order linear nonhomogeneous differential equation

$$y^{(4)} - 2y'' + y = -8e^x,$$

with the boundary conditions

$$y(0) = y''(0) = 0,\ y'(1) = y''(1) = -e. \qquad (24)$$

Solution

In operator form the given differential equation becomes

$$Ly = 2y'' - y - 8e^x.$$

Here,

$$L = \frac{d^4}{dx^4}, \qquad L^{-1}(.) = \int_0^x\int_0^x\int_0^x\int_0^x (.)dxdxdxdx,$$

are the derivative and integral operators.

Apply the inverse operator and initial conditions are taken, we find

$$y(x) = Ax + \frac{1}{3!}Bx^3 - L^{-1}\left(8e^x\right) + 2L^{-1}\left(y''\right) - L^{-1}\left(y\right), \tag{25}$$

where $A = y'(0)$ *and* $B = y'''(0)$. Using $y(x) = \sum_{n=0}^{\infty} y_n(x),$ in (25)

$$\sum_{n=0}^{\infty} y_n(x) = Ax + \frac{1}{3!}Bx^3 - L^{-1}\left(8e^x\right) + 2L^{-1}\left(\sum_{n=0}^{\infty} y_n''(x)\right) - L^{-1}\left(\sum_{n=0}^{\infty} y_n(x)\right), \tag{26}$$

whereas the reduction formula given below can be written using (26),

$$y_0(x) = Ax + \frac{1}{3!}Bx^3 - L^{-1}\left(8e^x\right),$$
$$y_{n+1}(x) = 2L^{-1}\left(y_n''\right) - L^{-1}\left(y_n\right); n \geq 0$$

The A and B constants in the reduction formula will be determined using boundary conditions (24) after finding the decomposition series. From the reduction relation, we can obtain the solution terms of the decomposition series as

$$y_0(x) = 8 - 8e^x + (8 + A)x + 4x^2 + \frac{1}{6}(8 + B)x^3,$$
$$y_1(x) = 2L^{-1}\left(y_0''\right) - L^{-1}\left(y_0\right)$$
$$= 8 - 8e^x + 8x + 4x^2 + \frac{4x^3}{3} + \frac{x^4}{3} - \frac{1}{120}(-8 + A - 2B)x^5$$
$$- \frac{x^6}{90} - \frac{(8 + B)}{5040}x^7,$$
$$y_2(x) = 2L^{-1}\left(y_1''\right) - L^{-1}\left(y_1\right)$$
$$= 8 - 8e^x + 8x + 4x^2 + \frac{4x^3}{3} + \frac{x^4}{3} - \frac{x^5}{15} + \frac{x^6}{90} - \frac{1}{2520}(A - 2B - 4)x^7$$
$$- \frac{x^8}{1680} + \frac{(A - 4B - 24)}{362880}x^9 + \frac{x^{10}}{453600} - \frac{(8 + B)}{39916800}x^{11},$$
$$\vdots$$

Consequently, the approximate solution obtained by using Adomian decomposition method using the first three terms of the given problem in a series form is given by

$$y(x) = 24 - 24e^{x} + (24 + A)x + 12x^{2} + \frac{1}{6}(24 + B)x^{3} + \frac{2x^{4}}{3} + \frac{1}{120}(16 - A + 2B)x^{5}$$
$$- \frac{(2\text{A} - 3B)}{5040}x^{7} - \frac{x^{8}}{1680} + \frac{(\text{A} - 4B - 24)}{362880}x^{9} + \frac{x^{10}}{453600} - \frac{(8 + B)}{39916800}x^{11}.$$

Example 5:

Consider the linear boundary value problem

$$y^{(5)}(x) = y - 15e^{x} - 10xe^{x}, \quad 0 < x < 1$$

subject to the boundary conditions

$$y(0) = y''(0) = 0, \quad y'(0) = 1, \quad y(1) = 0, \quad y'(1) = -e. \tag{27}$$

Find out the recursive relation for it, by using Adomian decomposition method from which the various components y_n of the solution y can be determined.

Solution

In operator form the given differential equation becomes

$$Ly = y - 15e^{x} - 10xe^{x}.$$

Here,

$$L = \frac{d^{5}}{dx^{5}}, \qquad L^{-1}(.) = \int_0^x\int_0^x\int_0^x\int_0^x\int_0^x (.)\, dxdxdxdxdx,$$

are the derivative and integral operators. Apply the inverse operator and initial conditions are taken, we find

$$L^{-1}Ly(x) = -L^{-1}\left(15e^{x}\right) - L^{-1}\left(10xe^{x}\right) + L^{-1}\left(y\right)$$
$$\Rightarrow y(x) = -35 - 24x - \frac{15}{2}x^{2} + \left(\frac{A}{6} - \frac{5}{6}\right)x^{3} + \left(\frac{B}{24} + \frac{5}{24}\right)x^{4}$$
$$+ (35 - 10x)e^{x} + L^{-1}\left(y\right), \tag{28}$$

where the constants $A = y'''(0)$ *and* $B = y^{(4)}(0)$. Using $y(x) = \sum_{n=0}^{\infty} y_n(x)$, in (28)

$$\sum_{n=0}^{\infty} y_n(x) = -35 - 24x - \frac{15}{2}x^2 + \left(\frac{A}{6} - \frac{5}{6}\right)x^3 + \left(\frac{B}{24} + \frac{5}{24}\right)x^4$$
$$+(35-10x)e^x + L^{-1}\left(\sum_{n=0}^{\infty} y_n(x)\right), \qquad (29)$$

whereas the recursive relation given below can be written using (29),

$$y_0(x) = -35 - 24x - \frac{15}{2}x^2 + \left(\frac{A}{6} - \frac{5}{6}\right)x^3 + \left(\frac{B}{24} + \frac{5}{24}\right)x^4 + (35-10x)e^x,$$
$$y_{n+1}(x) = L^{-1}\left(y_n\right); n \ge 0$$

The A and B constants in the reduction formula will be determined using boundary conditions (27) after finding the decomposition series.

Example 6:

Consider the nonlinear boundary value problem

$$y^{(6)}(x) = e^x y^2(x), \quad 0 < x < 1$$

subject to the boundary conditions

$$y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 1,$$
$$y(1) = e^{-1}, \quad y'(1) = -e^{-1}, \quad y''(1) = e^{-1}. \qquad (30)$$

Find out the recursive relation for it, by using Adomian decomposition method from which the various components y_n of the solution y can be determined.

Solution

In operator form the given differential equation becomes

$$Ly = e^x y^2(x).$$

Here,

$$L = \frac{d^6}{dx^6}, \qquad L^{-1}(.) = \int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x (.)\, dxdxdxdxdxdx,$$

are the derivative and integral operators.

Apply the inverse operator and initial conditions are taken, we find

$$L^{-1}Ly(x) = L^{-1}\left(e^x y^2(x)\right)$$

$$\Rightarrow y(x) = 1 - x + \frac{1}{2}x^2 + \frac{A}{6}x^3 + \frac{B}{24}x^4 + \frac{C}{120}x^5 + L^{-1}\left(e^x y^2(x)\right), \tag{31}$$

where $A = y'''(0), B = y^{(4)}(0)$ *and* $C = y^{(5)}(0)$ are the constants that will be determined later. Using $y(x) = \sum_{n=0}^{\infty} y_n(x)$ and $y^2(x) = \sum_{n=0}^{\infty} A_n$ in (31)

$$\sum_{n=0}^{\infty} y_n(x) = 1 - x + \frac{1}{2}x^2 + \frac{A}{6}x^3 + \frac{B}{24}x^4 + \frac{C}{120}x^5 + L^{-1}\left(e^x\left(\sum_{n=0}^{\infty} A_n\right)\right), \tag{32}$$

where A_n are Adomian polynomials. Hence the recursive relation is as

$$y_0(x) = 1 - x + \frac{1}{2}x^2 + \frac{A}{6}x^3 + \frac{B}{24}x^4 + \frac{C}{120}x^5,$$

$$y_{n+1}(x) = L^{-1}\left(e^x\left(\sum_{n=0}^{\infty} A_n\right)\right); n \geq 0$$

Exercises

Use the Adomian decomposition method to find the series solution of the following nonlinear ordinary differential equations:

1. $y''(x) - y^3(x) = 0; \quad y(0) = 1,\ y'(0) = 0.$

2. $y''(x) - ye^y = 0; \quad y(0) = 1,\ y'(0) = 0.$

3. $y^{(3)}(x) + \left(y''(x)\right)^2 + \left(y'(x)\right)^2 = 1 - \sin x;$
   $y(0) = y'(0) = 0,\ y''(0) = 1.$

4. $y^{(3)}(x) - \left(y''(x)\right)^2 + \left(y'(x)\right)^2 = 1 + \cosh x;$
   $y(0) = 0,\ y'(0) = 1,\ y''(0) = 0.$

5. $y^{(4)}(x) - 18y''(x) + 81y(x) = 0;$
   $y(0) = 0,\ y'(0) = -1,\ y''(0) = y'''(0) = 0.$

Find out the recursive relation for the following ordinary differential equations, by using Adomian decomposition method from which the various components y_n of the solution y can be determined.

6. $y^{(5)}(x) = e^x \left(y\right)^4; \ 0 < x < 1$
   $y(0) = 1,\ y'(0) = -\frac{1}{3},\ y''(0) = \frac{1}{9},\ y(1) = e^{-1/3},\ y'(1) = \left(-\frac{1}{3}\right)e^{-1/3}.$

7. $y^{(6)}(x) = e^{-x} y^2(x), \quad 0 < x < 1$

   $y(0) = y''(0) = y^{(4)}(0) = 1,$
   $y(1) = y''(1) = y^{(4)}(1) = e.$