1. Introduction

Graphs are beneficial because they summarize and display information in a manner that is easy for most people to comprehend. Graphs are used in many academic disciplines, including math, hard sciences and social sciences. They make appearances in corporate settings, serving as useful tools to convey financial information and facilitate data analysis.

Different graphs are used depending on the information that individuals wish to convey. Many graphs are used to concisely and clearly summarize data; the best type of graph to use depends on the type of data being conveyed (such as nominal, scale-discrete, scale-continuous and ordinal). Data summary graphs are generally nominal or contain data that can be reduced in some way; pie charts and bar charts are common and popular examples.

In the coming lectures, we discuss the graphs in a rectangular coordinate system. Equations can be graphed on a set of coordinate axes. The location of every point on a graph can be determined by two coordinates, written as an ordered pair, (x, y). These are also known as Cartesian coordinates, after the French mathematician Rene Descartes, who is credited with their invention.

Although the use of rectangular coordinates in such geometric applications as surveying and planning has been practiced since ancient times, it was not until the seventeenth century that geometry and algebra were joined to form the branch of mathematics called analytic geometry. French mathematician and philosopher Rene Descartes (1596-1650) devised a simple plan whereby two number lines were intersected at right angles with the position of a point in a plane determined by its distance from each of the lines. This system is called the rectangular coordinate system (or Cartesian coordinate system).

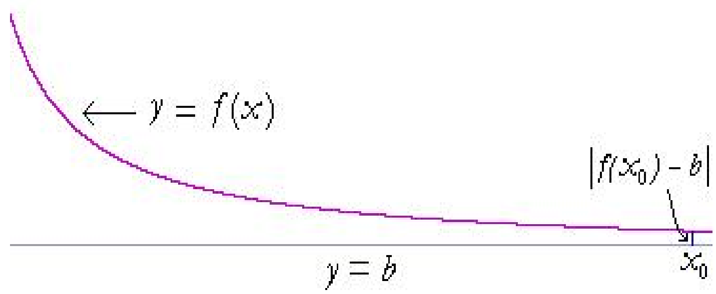
2. Outlines

- The Coordinate Plane.
- Asymptotes.
- .

2.1. **The Coordinate Plane.** The coordinate plane is a plane determined by two perpendicular lines, the x-axis and the y-axis. The x-axis is the horizontal axis, and the y-axis is the vertical axis. Every point in the plane can be stated by a pair of coordinates that express the location of the point in terms of the two axes. The intersection of the x- and y-axes is designated as the origin, and its point is $(0, 0)$.

As you can see from the figure, each of the points on the coordinate plane is expressed by a pair of coordinates: (x, y). The first coordinate in a coordinate pair is called the x-coordinate. The x-coordinate is the point's location along the x-axis and can be determined by the point's distance from the y-axis (where $x = 0$). If the point is to the right of the y-axis, its x-coordinate is positive, and if the point is to the left of the y-axis, its x-coordinate is negative. The second coordinate in a coordinate pair is the y-coordinate. The y-coordinate of a point is its location along the y-axis and can be calculated as the distance from that point to the x-axis. If the point is above the x-axis, its y-coordinate is positive, and if the point is below the x-axis, its y-coordinate is negative.

Definition 2.1. An asymptote is a line to which the graph gets arbitrarily close. That means that for any distance named, no matter how small, the graph will get within that distance and stay within that distance for some section of the graph with infinite length. More precisely



Definition 2.2. The graph of $y = f(x)$ has a horizontal asymptote of $y = b$, if and only if, either

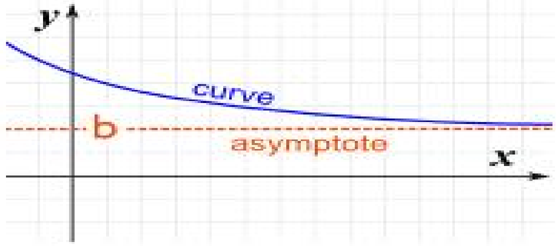
$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b. \tag{2.1}$$

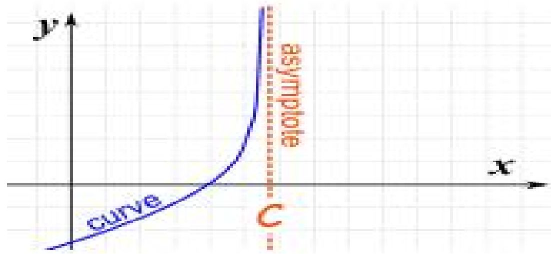
As $x \to \pm\infty$, the curve approaches some constant value b.

Definition 2.3. The graph of $y = f(x)$ has a vertical asymptote of $x = c$, if and only if, either

$$\lim_{x\to c^-} f(x) = \pm\infty \text{ or } \lim_{x\to c^+} f(x) = \pm\infty. \tag{2.2}$$

As $x \to \pm c$, then the curve goes towards $\pm\infty$.





Definition 2.4. Let r(x) be a rational function with polynomial $p(x) = a_n x_n + ... + a_0$ of degree n in the numerator and polynomial $q(x) = b_m x_m + ... + b_0$ of degree m in the denominator

- If $n < m$, then $r(x)$ has a horizontal asymptote of $y = 0$.

- If $n > m$, then $r(x)$ becomes unbounded for large values of x (positive or negative).

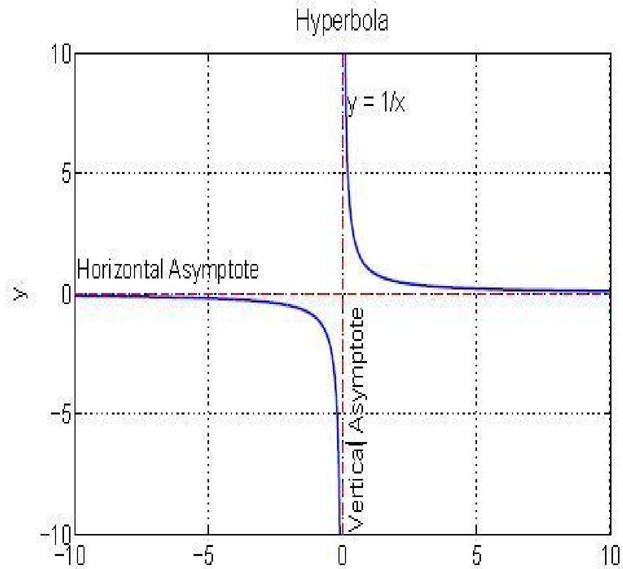
- If $n = m$, then $r(x)$ has a horizontal asymptote of $y = \frac{a_n}{b_n}$.

Example 2.5. Discuss the Asymptotes of the following rational function

$$r(x) = \frac{1}{x}, \tag{2.3}$$

this function is defined $\forall x \neq 0$. Consider the sequence of numbers $x_n = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots, \frac{1}{k}$ for $k = 1, 2, \cdots, n$. These numbers are getting closer and closer to zero. Since $r(x_n) = \frac{1}{x_n} = \frac{1}{\frac{1}{n}} = n$. So, $r(x_n) = 2, 3, 4, \cdots, k, \cdots$ for $n = 2, 3, 3, \cdots, k, \cdots$, which is getting larger and larger, so approaching the vertical line $x = o$. Thus there is a vertical asymptote at $x = 0$.

The graph of $y = \frac{1}{x}$ is



Example 2.6. Discuss the Asymptotes of the following rational function

$$r(x) = \frac{10x}{2+x}, \tag{2.4}$$

defined $\forall x \neq -2$. The numerator and denominator are linear functions (degree of polynomials are the same). Alternatively we can see that as x get large then 2 in the denominator becomes insignificant. Thus $r(x) \approx \frac{10x}{x} = 10$. Thus horizontal asymptote occurs at $x = 10$.

Example 2.7. Discuss the Asymptotes of the following rational function

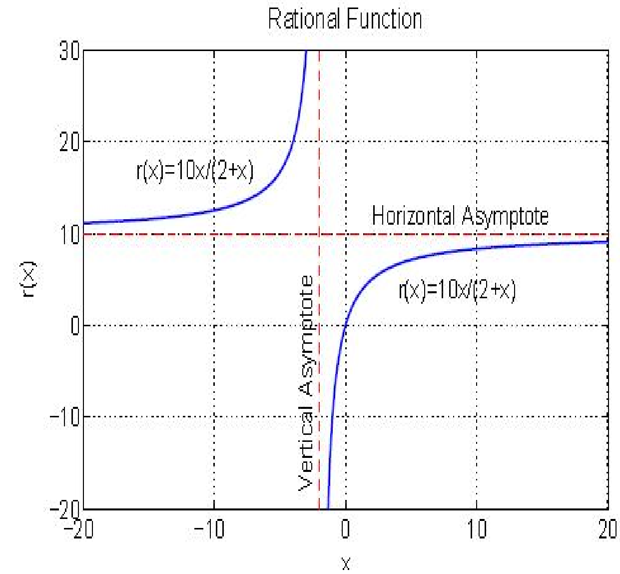
$$r(x) = \frac{4x^2}{4-x^2}, \tag{2.5}$$

defined $\forall x \neq \pm 2$. Clearly the function passes through the origin, so the x and y-intercept is $(x, y) = (0, 0)$. Note that this function is an even function, the edge of the domain is $x = 2$, so we see there are vertical asymptotes at $x = 2$.

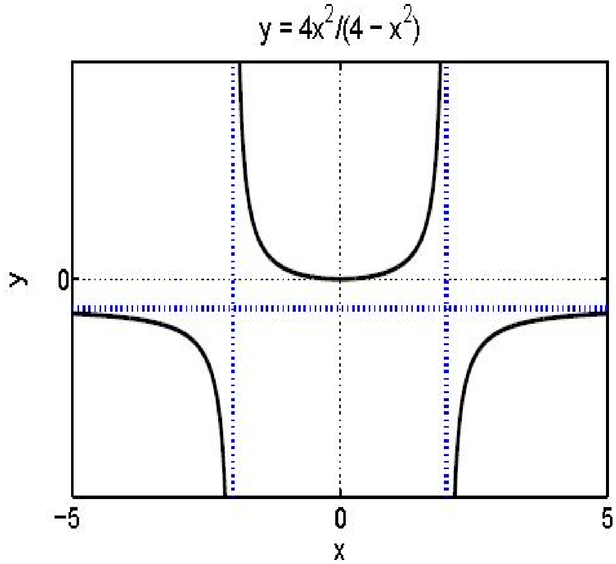
The numerator and denominator are quadratic functions (degree of polynomials are the same). Thus, for x large $r(x) \approx \frac{4x^2}{-x^2} = -4$. Thus, a horizontal asymptote occurs at $y = -4$.

Definition 2.8. We have seen that a rational function $r(x) = \frac{p(x)}{q(x)}$ will have a horizontal asymptote if the degree of the numerator $p(x)$ is less than or equal to the degree of the denominator $q(x)$: In particular, if the degree of $p(x)$ is strictly less than that of $q(x)$; then the x-axis will be the horizontal asymptote-a geometrical condition that can be expressed analytically by saying $r(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$.

The graph of $y = \frac{10x}{2+x}$ is



The graph of $y = \frac{4x^2}{4-x^2}$ is



If the degree of $p(x)$ is greater than or equal to the degree of $q(x)$; then long division can be used to obtain more accurate information about the large scale behavior of the rational function. Recall that $p(x)$ divided by $q(x)$ gives a quotient $f(x)$ and a remainder $g(x)$; provided that $p(x) = q(x) \times f(x) + g(x)$ and provided that the degree of $G(X)$ is strictly less than the degree of $g(x)$

In terms of rational functions, we have

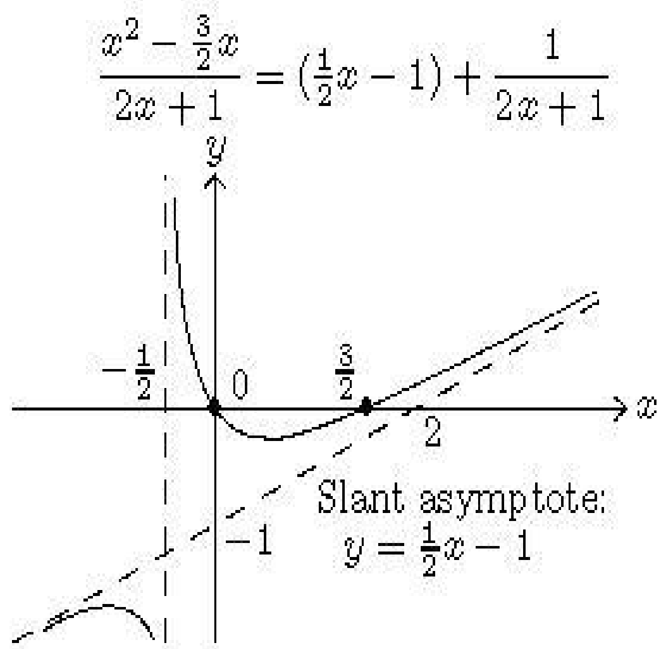
$$
\begin{aligned}
r(x) &= \frac{p(x)}{q(x)} \\
&= f(x) + \frac{g(x)}{q(x)} \qquad (2.6)
\end{aligned}
$$

Because of the degree condition on $g(x)$; it is clear that $\frac{g(x)}{q(x)} \to 0$ as $x \to \pm\infty$ so that $g(x)$ and $q(x)$ are close to each other when $|x|$ is large. Thus the graph of the rational function $r(x)$ is asymptotic to the graph of the polynomial $f(x)$ as $x \to \pm\infty$.In other words, the two graphs are close to each other as $x \to \pm\infty$and as $x \to -\infty$: In the special case where the degree of $p(x)$ is one more than the degree of $q(x)$; the quotient is a linear function, whose graph is a non-horizontal line in the plane. **That line is called an oblique or slant asymptote to the graph of the particular rational function.**

Example 2.9. Discuss the oblique asymptote of the following rational function

$$r(x) = \frac{x^2 - \frac{3}{2}x}{2x+1}, \qquad (2.7)$$

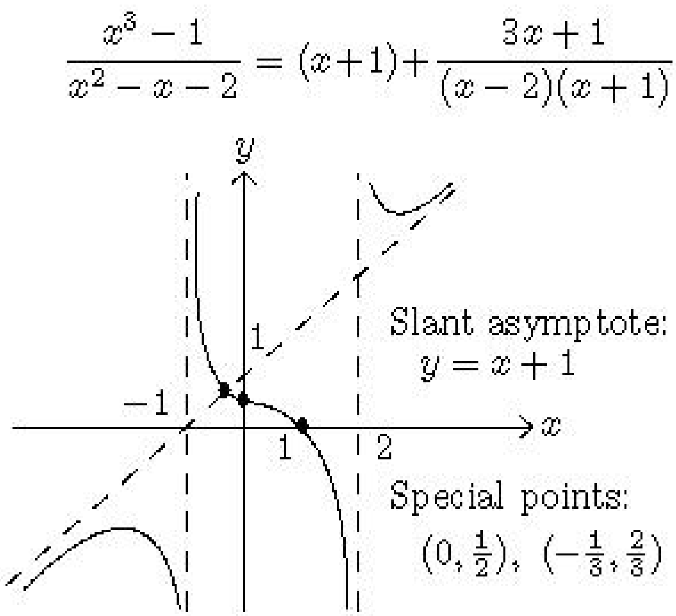
after long division, it becomes $\frac{1}{2}x - 1 + \frac{1}{2x+1}$. Clearly the quotient is a linear function and it is our oblique asymptote according to the Definition (2.8). For more detail see the graph



Example 2.10. Discuss the oblique asymptote of the following rational function

$$r(x) = \frac{x^3 - 1}{x^2 - x - 2}, \tag{2.8}$$

after long division, it becomes $x + 1 + \frac{3x+1}{(x_2)(x+1)}$. Clearly the quotient is a linear function and it is our oblique asymptote according to the Definition (2.8). For more detail see the graph



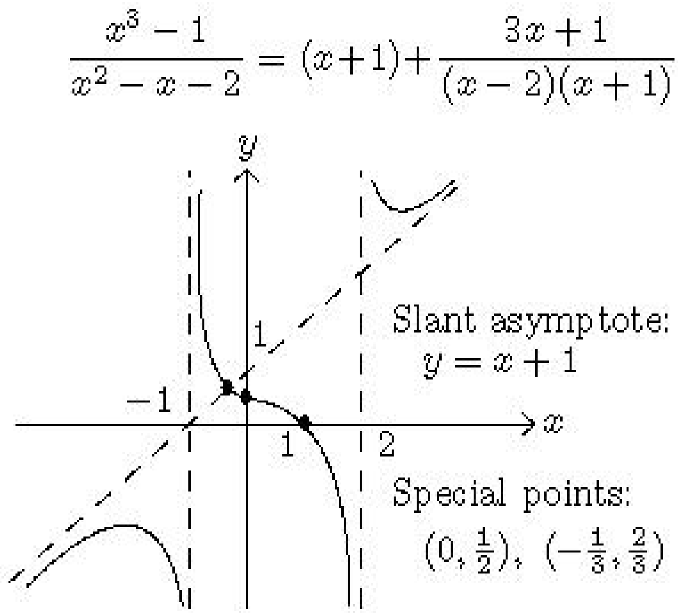
Example 2.11. Discuss the oblique asymptote of the following rational function

$$r(x) = \frac{x^3 - 1}{x^2 - x - 2}, \tag{2.9}$$

after long division, it becomes $x + 1 + \frac{3x+1}{(x_2)(x+1)}$. Clearly the quotient is a linear function and it is our oblique asymptote according to the Definition (2.8). For more detail see the graph

2.2. **Exercise.** Find the equations of the vertical asymptote for the following rational curves

- $\frac{2x+1}{3x-1}$
- $2xy = x^2 + 3$
- $\frac{1}{x^2-1}$
- $\frac{(x-2)^2}{x^2}$
- Find the equations of the horizontal asymptote for the following rational curves
- $\frac{1-6x}{3x+7}$
- $\frac{1+2x^2}{3x^3-4}$
- $\frac{4x^2-3x+1}{2x^2-1}$



- $\frac{x^2-x-6}{x^2-1}$
- Find the equations of the oblique asymptote for the following rational curves
- $\frac{3x^3+2}{x^2-x-1}$
- $\frac{5x^2-3x+1}{x+2}$
- $\frac{x^2+2}{x-2}$
- $\frac{(1-x)^3}{x^2}$
- $\frac{x^3-3x^2}{x^2-1}$
- Find the equations of the asymptotes of the following curves
- $(x-y)^2(x^2+y^2) - 10(x-y)x^2 + 12y^2 + 2x + y = 0$
- $x^2y + xy^2 + xy + y^2 + 3x = 0$
- $(x-y+1)(x-y-2)(x+y) = 8x - 1$

2.3. **Asymptotes in Polar Coordinates.** We first establish the polar equation of straight line.

The polar equation of any line is $p = r\cos(\theta - \alpha)$ where p is the length of the perpendicular from pole to the line and α is the angle which this perpendicular makes with the initial line.

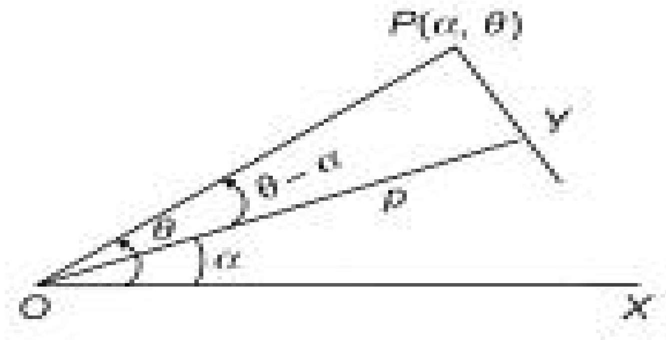
Proof. Let OX be initial line and P be any point on the line whose polar coordinates are (r, θ) and therefore, $OP = r$ and $\angle POX = \theta$.

Let us draw a perpendicular OY on the line from P produced. From $\triangle OPY$

$$\cos POY = \frac{OY}{OP} = \frac{p}{r}$$

$$p = r\cos POY = r\cos(\theta - \alpha), \tag{2.10}$$

which is the required equation of the line. □



Now we determine the asymptotes of the curve $r = s(\theta)$. For this we have to determine the constant p and α so that any line $p = r\cos(\theta - \alpha)$ is the asymptote of the given curve.

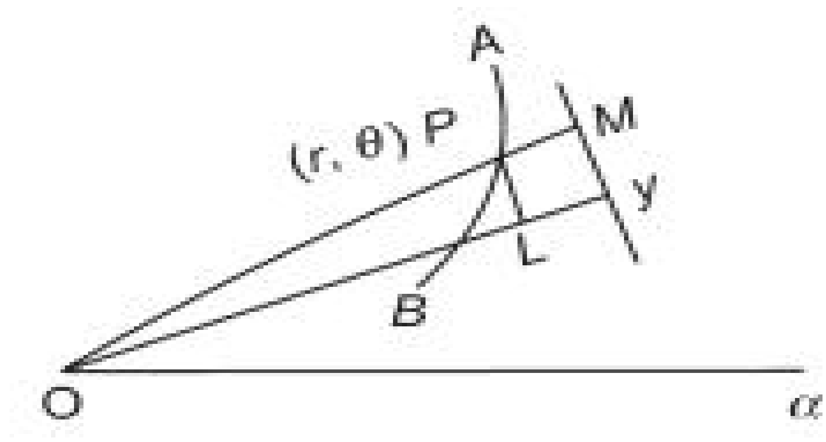
Let $P(r, \theta)$ be any point on the curve AB whose equation is $r = p(\theta)$. Let us draw a perpendicular OY on the line $p = r\cos(\theta - \alpha)$ which is MY. Therefore

$$PM = LY = OY - OL = p - OP\cos(\theta - \alpha)$$

$$= p - r\cos(\theta - \alpha). \tag{2.11}$$

Now $r \to \infty$ as the point recedes to infinity along the curve. Let $\theta \to \theta_1$ where $r \to \infty$. Therefore we have $\frac{PM}{r} = \frac{p}{r} - \cos(\theta - \alpha)$ by (2.11). when $r \to \infty$, $PM \to 0$. So that

$$\frac{PM}{r} = PM.\frac{1}{r} \to 0 \; and \; \frac{p}{r} \to 0 \tag{2.12}$$



Let

$$\begin{aligned}\cos(\theta-\alpha) &= 0\\ \cos(\theta-\alpha) &= \cos(\frac{\pi}{2})\\ \theta_1-\alpha &= \frac{\pi}{2}\ (\because \theta\to\theta_1\ as\ r\to\infty)\\ \theta_1 &= \alpha+\frac{\pi}{2}\\ \alpha &= \theta_1-\frac{\pi}{2}. \qquad (2.13)\end{aligned}$$

Again from (2.11), we have when $r\to\infty$

$$\begin{aligned}p &= \lim_{r\to\infty}[r\cos(\theta-\alpha)]\\ &= \lim_{r\to\infty}[r\cos(\theta-\theta_1+\frac{\pi}{2})]\\ &= \lim_{r\to\infty}[r\sin(\theta_1-\theta)]\\ &= \lim_{\theta\to\theta_1}\left[\frac{\sin(\theta_1-\theta)}{\frac{1}{r}}\right]\cdots\frac{0}{0} \qquad (2.14)\end{aligned}$$

$$\begin{aligned}p &= \lim_{\theta\to\theta_1}\left[\frac{\cos(\theta_1-\theta)}{-\frac{1}{r^2}\frac{dr}{d\theta}}\right]\\ &= \frac{\lim_{\theta\to\theta_1}(\cos(\theta_1-\theta))}{\lim_{\theta\to\theta_1}(-\frac{1}{r^2}\frac{dr}{d\theta})}. \qquad (2.15)\end{aligned}$$

Now let $u = \frac{1}{r}$, this implies $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$. Therefore (2.15) becomes

$$p = \lim_{\theta \to \theta_1} -\frac{d\theta}{du}. \tag{2.16}$$

Hence the asymptotes is

$$\begin{aligned} \lim_{\theta \to \theta_1} -\frac{d\theta}{du} &= r\cos(\theta - \alpha) \\ &= r\cos(\theta - \theta_1 + \frac{\pi}{2}) \\ &= r\sin(\theta_1 - \theta), \end{aligned} \tag{2.17}$$

where θ_1 is the limit of θ as $r \to \infty$.

2.4. **Working Rules For Obtaining Asymptotes to Polar Curve.**

- We have to change r to $\frac{1}{u}$ in the given equation and find out the limit of θ as $u \to 0$.
- Let θ_1 be any one of the angle possible limits of θ.
- We have to determine then $-\frac{d\theta}{du}$ and its limit as $u \to 0$ and $\theta \to \theta_1$.
- Let this limit be p.
- Then $p = r\sin(\theta_1 - \theta)$ is the required asymptote.

2.5. **Examples.**

Example 2.12. Find the asymptotes of the curve

$$r\sin(\theta) = 2\cos(\theta) \tag{2.18}$$

Step1:(2.18) implies

$$\begin{aligned} r &= \frac{2\cos 2\theta}{\sin\theta} \\ \frac{1}{r} &= \frac{\sin\theta}{2\cos 2\theta}, \end{aligned} \tag{2.19}$$

according to (2.4), we can write (2.19) as

$$u = \frac{\sin\theta}{2\cos 2\theta}. \tag{2.20}$$

Step2: As $u \to 0$ then $\theta \to n\pi,\ n \in Z$.
Step3: Now we determine $-\lim_{\theta \to n\pi}(\frac{d\theta}{du})$

$$\begin{aligned} -\lim_{\theta\to n\pi}\left(\frac{du}{d\theta}\right) &= \lim_{\theta\to n\pi}\frac{\cos 2\theta\cos\theta - \sin\theta(-2\sin 2\theta)}{\cos^2 2\theta} \\ &= \frac{\cos^2(2n\pi)}{\cos(2\pi)\cos(\pi) - \sin(\pi)(-2\sin(2\pi)} \\ &= \frac{2}{\cos(n\pi)} \\ &= \frac{2}{(-1)^n}. \end{aligned} \tag{2.21}$$

Step4: This limit be p. That is $p = -\frac{2}{(-1)^n}$. Hence, the required asymptotes are

$$\begin{aligned} r\sin(n\pi - \theta) &= -\frac{2}{(-1)^n} \\ r[(-1)^{n-1}\sin\theta] &= \frac{2}{(-1)^n} \\ r\sin(\theta) &= 2. \end{aligned} \tag{2.22}$$

Example 2.13. Find the asymptotes of the curve

$$r = a(\sec\theta + \tan\theta) \tag{2.23}$$

Step1:(2.23) implies

$$\frac{1}{r} = \frac{1}{a(\sec\theta + \tan\theta)}, \tag{2.24}$$

according to (2.4), we can write (2.24) as

$$\begin{aligned} u &= \frac{1}{a(\sec\theta + \tan\theta)} \\ &= \frac{\cos\theta}{a(1 + \sin\theta)} \end{aligned} \tag{2.25}$$

Step2: As $u \to 0$ then $\theta \to (2n+1)\frac{\pi}{2},\ n \in Z$.

Step3: Now we determine $-\lim_{\theta\to(2n+1)\frac{\pi}{2}}(\frac{d\theta}{du})$

$$
\begin{aligned}
-\lim_{\theta\to(2n+1)\frac{\pi}{2}}\left(\frac{du}{d\theta}\right) &= \lim_{\theta\to(2n+1)\frac{\pi}{2}}\frac{(1+\sin\theta)(-\sin\theta)-\cos^2\theta}{a(1+\sin\theta)^2}\\
&= \lim_{\theta\to(2n+1)\frac{\pi}{2}}\frac{1+\sin\theta[-\sin\theta-1+\sin\theta]}{a(1+\sin\theta)^2}\\
&= -\lim_{\theta\to(2n+1)\frac{\pi}{2}}\frac{1+\sin\theta}{a(1+\sin\theta)^2}\\
&= -\frac{1+\sin((2n+1)\frac{\pi}{2})}{a(1+\sin((2n+1)\frac{\pi}{2}))^2}\\
&= -\frac{(-1)^n+1}{a(1+(-1)^n)^2} \qquad (2.26)
\end{aligned}
$$

Step4: This limit be p. That is $p = \frac{(-1)^n+1}{a(1+(-1)^n)^2}$. Hence, the required asymptotes are

$$
\begin{aligned}
r\sin((2n+1)\frac{\pi}{2}-\theta) &= -\frac{a(1+(-1)^n)^2}{(-1)^n+1}\\
(-1)^n r\sin(\frac{\pi}{2}-\theta) &= -\frac{a(1+(-1)^n)^2}{(-1)^n+1}\\
r\cos(\theta) &= a(1+(-1)^n). \qquad (2.27)
\end{aligned}
$$

2.6. **Exercise.**

- Find the asymptotes of the curve $r = r\tan\theta$.

- Find the asymptotes of the curve $r\sin n\theta = a$.

3. Maxima and Minima

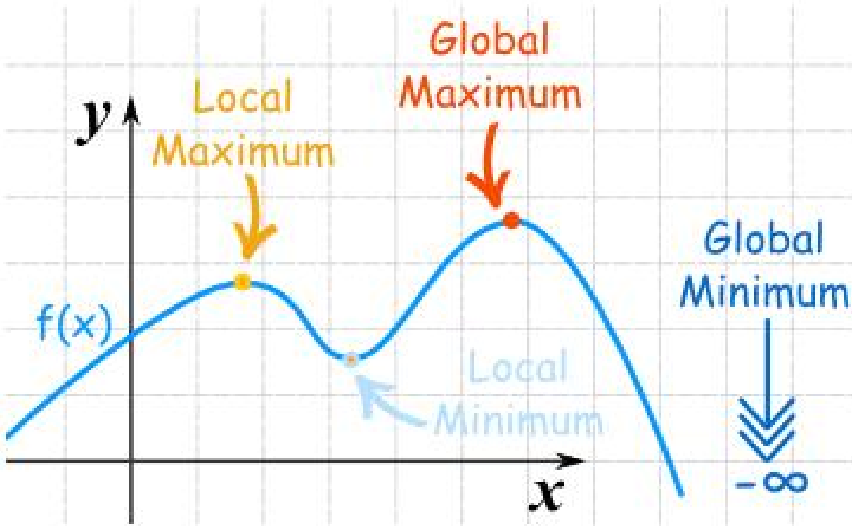
Definition 3.1. A function f has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the maximum value of f on D.

Similarly, f has an absolute minimum (or global minimum) at d if $f(d) \leq f(x)$ for all x in D and the number $f(d)$ is called the minimum value of f on D.

The maximum and minimum values of f are called the extreme values of f.

Definition 3.2. A function f has a local maximum (or relative maximum) at c if $f(c) \geq f(x)$ when x is near c. [This means that $f(c) \geq f(x)$ for all x in some open interval containing c.]

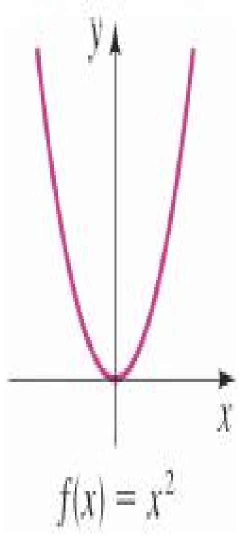
Similarly, f has a local minimum (or relative minimum) at d if $f(d) \leq f(x)$ when x is near c.

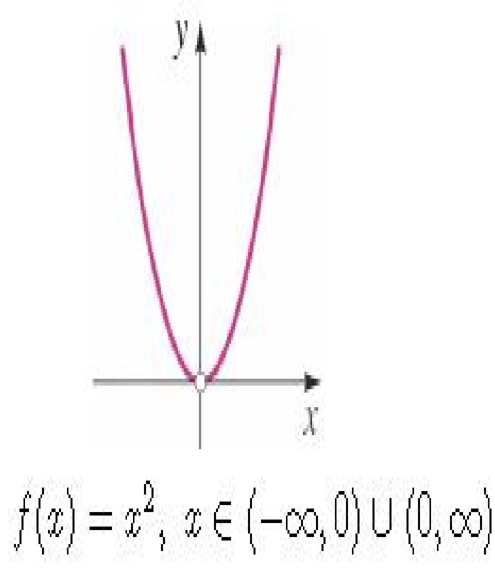


3.1. **Examples.**

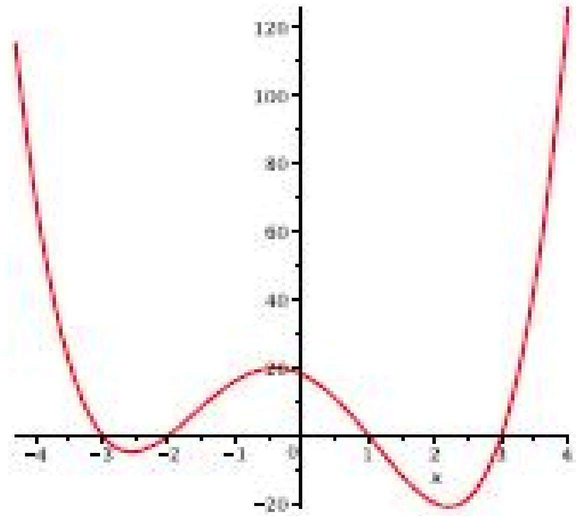
Example 3.3. The function $f(x) = x^2$ has absolute and local minimum at $x = 0$ and has no absolute or local minimum.

Example 3.4. The function $f(x) = x^2,\ x \in (-\infty, 0) \cup (0, \infty)$ has no absolute and local minimum at $x = 0$ and no absolute or local maximum.

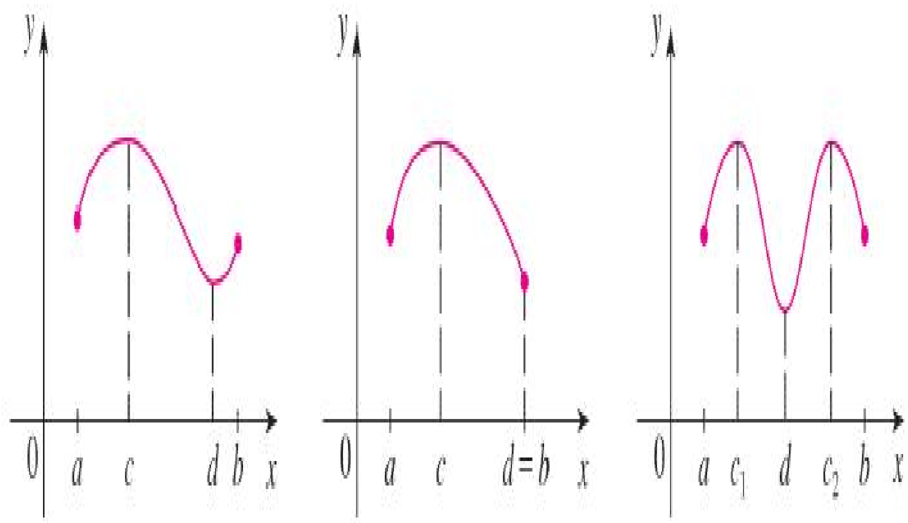




Example 3.5. The function $f(x) = x^4+x^3-11x^2-9x+18 = (x-3)(x-1)(x+2)(x+3)$ has the absolute minimum at $x \approx 2.2$ and has no absolute maximum. It has two local minima at $x \approx -2.6$ and $x \approx 2.2$ and the local maximum at $x \approx -0.4$.

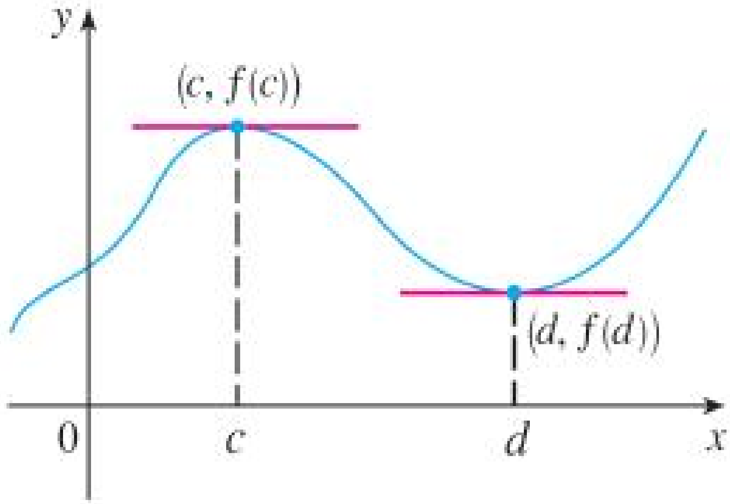


Theorem 3.6. *(The Extreme Value Theorem): If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.*

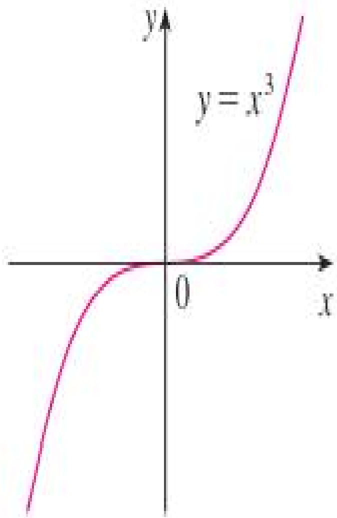


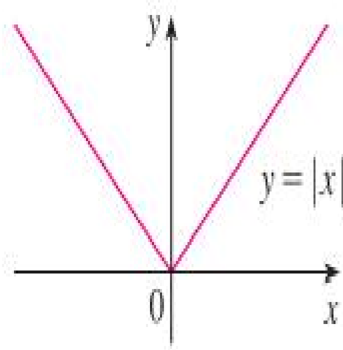
Theorem 3.7. *(Fermats Theorem): If f has a local maximum or minimum at c, and if $f^{'}(c)$ exists, then $f'(c) = 0$.*

Remark 3.8. The converse of this theorem is not true. In other words, when $f'(c) = 0$, f does not necessarily have a local maximum or minimum. For example, if $f(x) = x^3$, then $f^{'}(x) = 3x^2$ equals 0 at $x = 0$, but $x = 0$ is not a point of a local minimum or maximum.



Remark 3.9. Sometimes $f'(c)$ does not exist, but $x = c$ is a point of a local maximum or minimum. For example, if $f(x) = |x|$, then $f'(0)$ does not exist. But $f(x)$ has its local (and absolute) minimum at $x = 0$.





A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

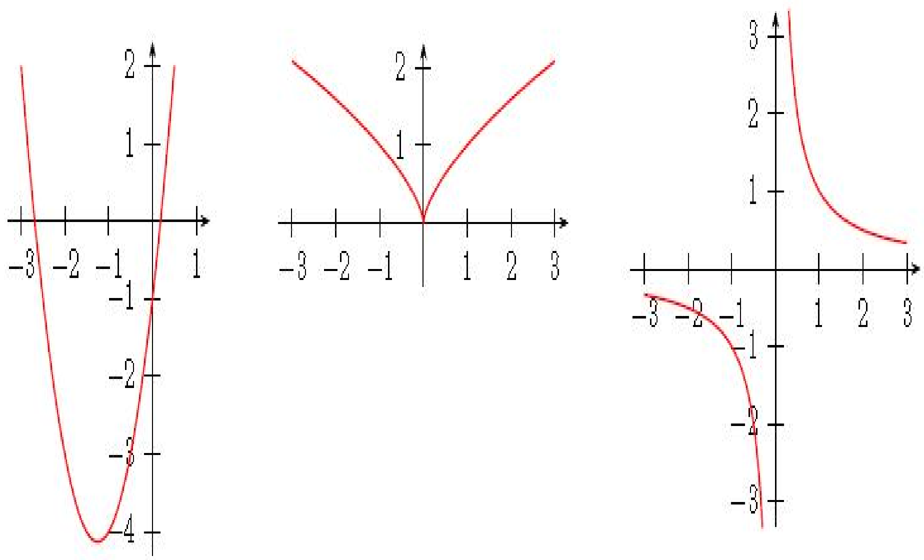
Remark 3.10. From Theorem 3.7 it follows that if f has a local maximum or minimum at c, then c is a critical number of f.

3.2. **Examples.**

Example 3.11. If $f(x) = 2x^2 + 5x - 1$, then $f^{'}(x) = 4x + 5$. Hence the only critical number of f is $x = -\frac{5}{4}$.

Example 3.12. If $f(x) = \sqrt[3]{x^2}$, then $f^{'}(x) = \frac{2}{3}x^{\frac{-1}{3}}$. Hence the only critical number of f is $x = 0$.

Example 3.13. If $f(x) = \frac{1}{x}$, then $f^{'}(x) = -\frac{1}{x^2}$. Since $x = 0$ is not in the domain, f has no critical numbers.



3.3. **Exercise.**

- Find the critical numbers of $f(x) = 2x^3 - 9x^2 + 12x - 5$.

- Find the critical numbers of $f(x) = 2x + 3\sqrt[3]{x^2}$.

3.4. **The Closed Interval Method.** To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

- Find the values of f at the critical numbers of f in (a, b).
- Find the values of f at the endpoints of the interval.
- The largest of the values from Step 1 and 2 is the absolute maximum value; the smallest value of these values is the absolute minimum value.

Example 3.14. Find the absolute maximum and minimum values of $f(x) = 2x^3 - 15x^2 + 36x$ on the interval $[1, 5]$ and determine where these values occur.

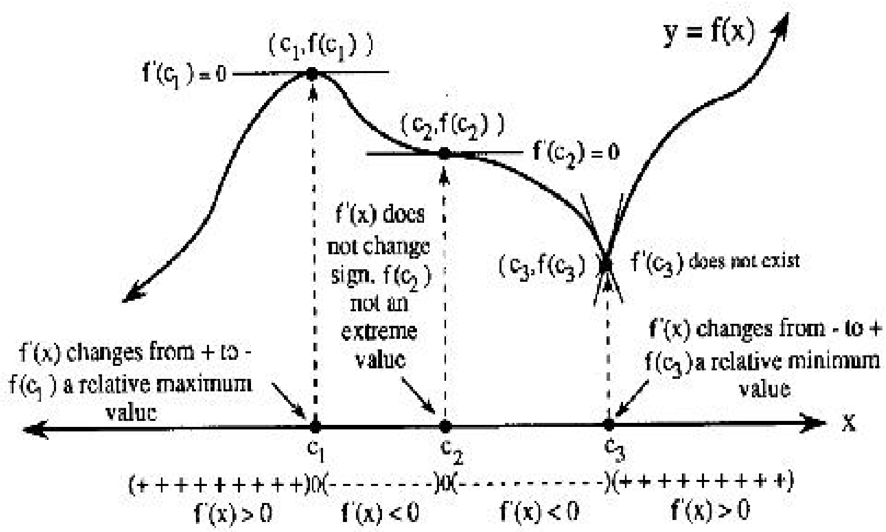
Step1: Since

$$f^{'}(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x-2)(x-3), \tag{3.1}$$

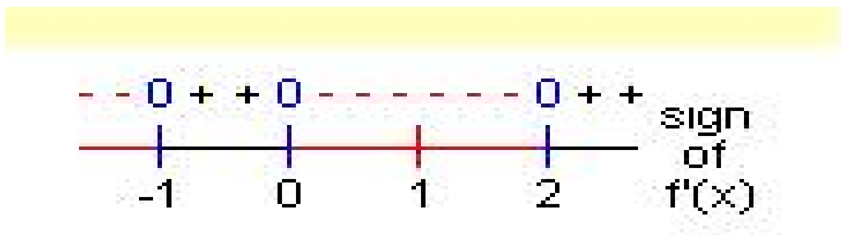
there are two critical numbers $x = 2$ and $x = 3$.
Step2: We now evaluate f at these critical numbers and at the endpoints $x = 1$ and $x = 5$. We have $f(1) = 23, f(2) = 28, f(3) = 27, f(5) = 55$.
Step3 The largest value is 55 and the smallest value is 23. Therefore the absolute maximum of f on $[1, 5]$ is 55, occurring at $x = 5$ and the absolute minimum of f on $[1, 5]$ is 23, occurring at $x = 1$.

3.5. **First Derivative Test.** Suppose f is continuous at a critical point c.

- If $f^{'}(x) > 0$ on an open interval extending left from c and $f^{'}(x) < 0$ on an open interval extending right from c, then f has a relative maximum at c.
- If $f^{'}(x) < 0$ on an open interval extending left from c and $f^{'}(x) > 0$ on an open interval extending right from c, then f has a relative minimum at c.
- If $f^{'}(x)$ has the same sign on both an open interval extending left from c and an open interval extending right from c, then f does not have a relative extremum at c.



Example 3.15. The function $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$ is differentiable everywhere on $[-2.3]$, with $f^{'}(x) = 0$ for $x = -1, 0, 2$. These are the three critical points of f on $[-2, 3]$. By Test 3.5, f has a relative maximum at $x = 0$ and relative minima at $x = -1$ and $x = 2$.



3.6. **Concavity.**

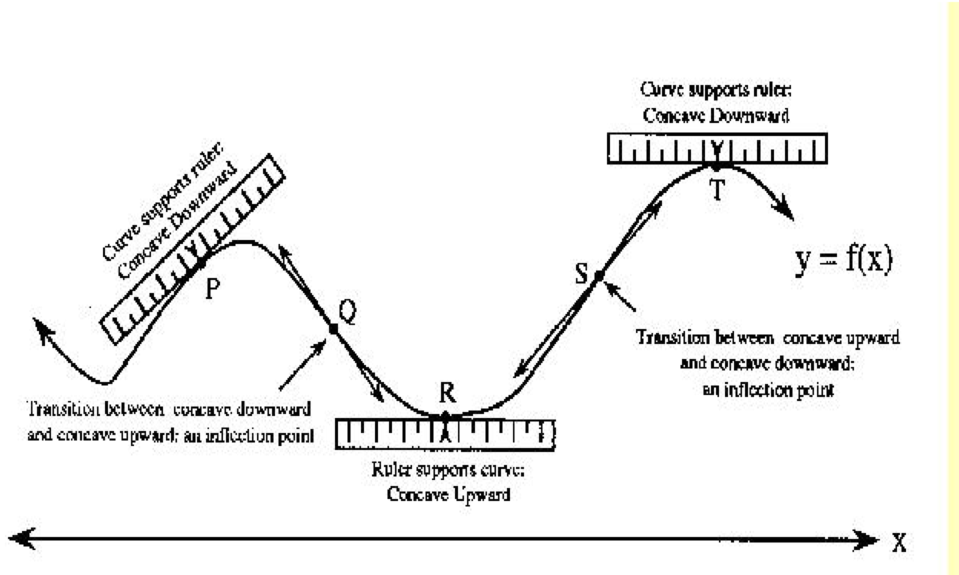
Definition 3.16. The graph of a function f is concave upward at the point $(c, f(c))$ if $f^{'}(c)$ exists and if for all x in some open interval containing c, the point $(x, f(x))$ on the graph of f lies above the corresponding point on the graph of the tangent line to f at c. This is expressed by the inequality $f(x) > [f(c) + f^{'}(c)(x - c)]$ for all x in some open interval containing c. Imagine holding a ruler along the tangent line through the point $(c, f(c))$: if the ruler supports the graph of f near $(c, f(c))$, then the graph of the function is concave upward.

Similarly, the graph of a function f is concave downward at the point $(c, f(c))$ if $f^{'}(c)$ exists and if for all x in some open interval containing c, the point $(x, f(x))$ on the graph of f lies below the corresponding point on the graph of the tangent line to f at c. This is expressed by the inequality $f(x) < [f(c) + f^{'}(c)(x - c)]$ for all x in some open interval containing c. In this situation the graph of f supports the ruler. This is pictured below:

Theorem 3.17. *If the function f is twice differentiable at $x = c$, then the graph of f is concave upward at $(c, f(c))$ if $f^{''}(c) > 0$ and concave downward if $f^{''}(c) < 0$.*

Definition 3.18. If $f^{'}(c)$ exists and $f^{''}(c)$ changes sign at $x = c$, then the point $(c, f(c))$ is an inflection point of the graph of f. If $f^{''}(c)$ exists at the inflection point, then $f^{''}(c) = 0$.

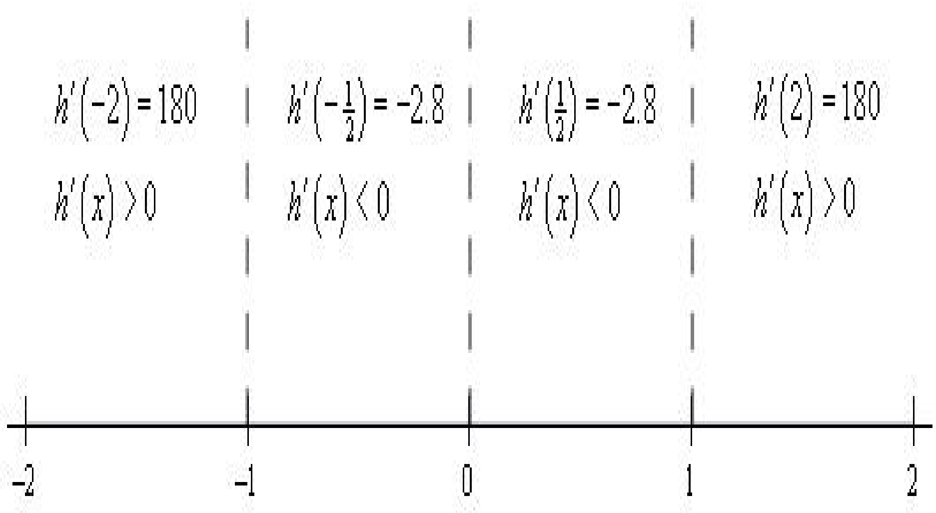
Theorem 3.19. *Second Derivative Test:*
Suppose that c is a critical point at which $f^{'}(c) = 0$, that $f^{'}(x)$ exists in a neighborhood of c, and that $f^{''}(c)$ exists. Then f has a relative maximum value at c if $f^{''}(c) < 0$ and a relative minimum value at c if $f^{''}(c) > 0$. If $f^{''}(c) = 0$, the test is not informative.



Example 3.20. For the function $f(x) = 3x^5 - 5x^3 + 3$ identify the intervals where the function is increasing and decreasing and the intervals where the function is concave up and concave down. Here

$$\begin{aligned} f^{'}(x) &= 15x^2(x-1)(x+1) \\ f^{''}(x) &= 30x(2x-1). \end{aligned} \tag{3.2}$$

$x = -1, 1$ and 0 are the critical points. Look at the graphical picture So, we got the following

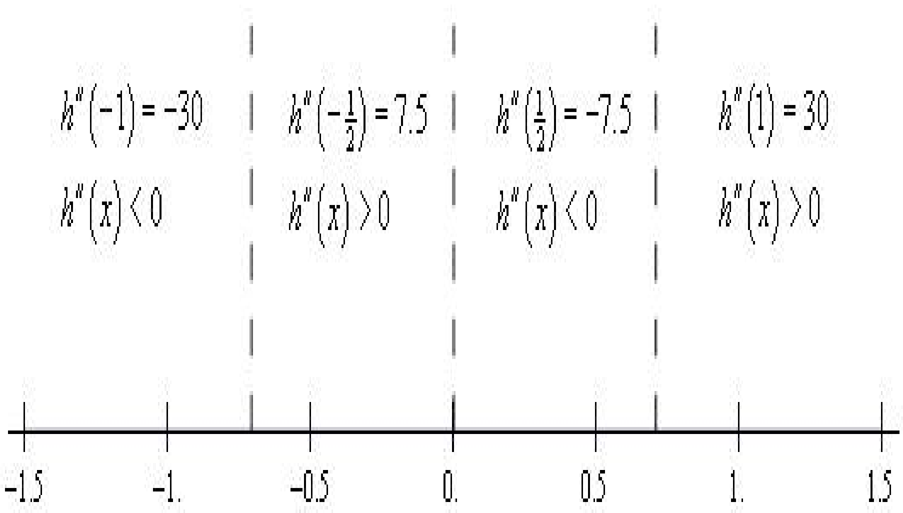


intervals for increasing and decreasing

$$(3.3)\qquad \begin{aligned} &Increasing: \ -\infty < x < -1 \ and \ 1 < x < \infty \\ &Decreasing: \ -1 < x < 0 \ and \ 0 < x < 1 \end{aligned}$$

Note that from the first derivative test we can also say that $x = -1$ is a relative maximum and that $x = 1$ is a relative minimum. Also $x = 0$ is neither a relative minimum or maximum.

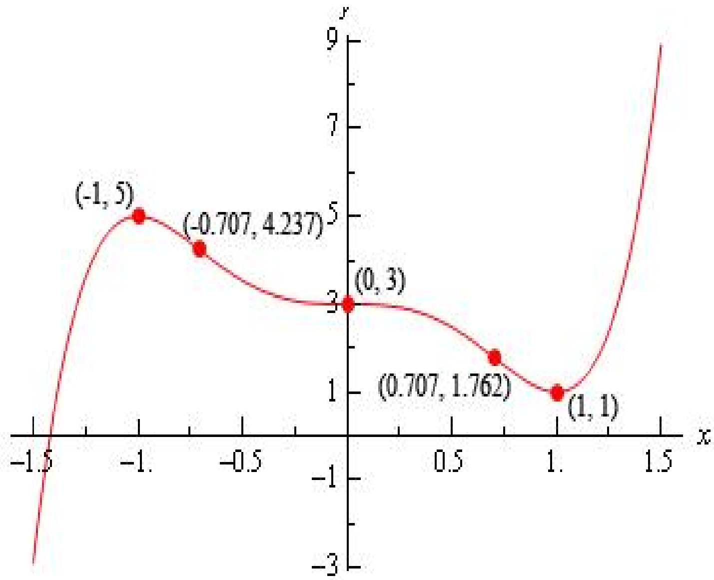
Now lets get the intervals where the function is concave up and concave down. The first thing that we need to do is identify the possible inflection points. These will be where the second derivative is zero or does not exist. The second derivative in this case is a polynomial and so will exist everywhere. It will be zero at $x = 0, \pm\frac{1}{\sqrt{2}}$. Look at the graphical picture for the second derivative So, we got the following intervals for the concavity



$$(3.4)\qquad \begin{aligned} &Concave\ up \ -\frac{1}{\sqrt{2}} < x < 0 \ and \ \frac{1}{\sqrt{2}} < x < \infty \\ &Concave\ down \ -\infty < x < -\frac{1}{\sqrt{2}} \ and \ 0 < x < \frac{1}{\sqrt{2}}. \end{aligned}$$

3.7. **Exercise.**

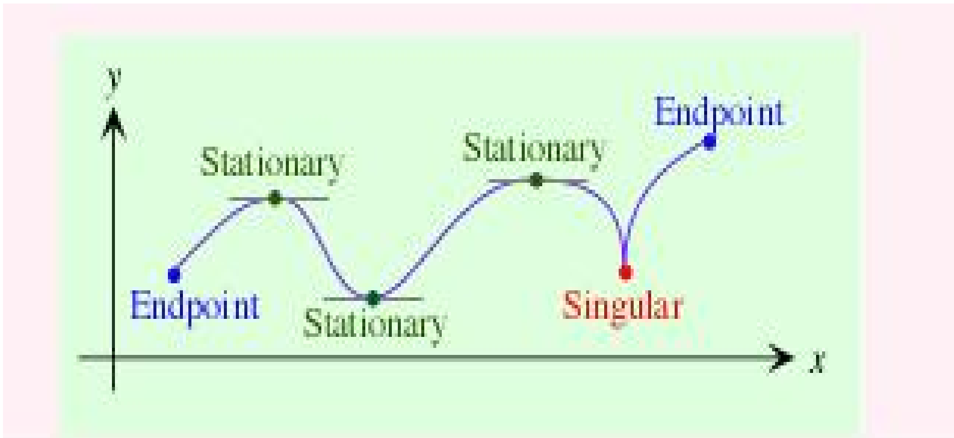
- Discuss the concavity of the graph of $f(x) = 2\sin^2 x - x^2$.
- Find α and β so that the function $f(x) = \alpha x^3 + \beta x^2 + 1$ has a point of inflection at $(-1, 2)$.



- Discuss the concavity of the graph of $f(x) = 2x^3 + 9x^2 - 24x - 10$.
- Discuss the concavity of the graph of $f(x) = x^4 - 6x^2 + 1$.
- Determine the concavity of $f(x) = x^4$ and locate any inflection points.
- Use the second derivative test to find the local extrema of $f(x) = x^4 - 8x^2 + 10$.
- Use the second derivative test to try to classify any local extrema for $f(x) = x^3$, $g(x) = x4$ and $h(x) = -x^4$.
- Discuss the concavity of $f(x) = x + \frac{25}{x}$.
- Find the maximum and minimum values of the function $f(x) = 7 + |x - 2|$ between 1 and 4 inclusive.
- Find all local maxima and minima for $f(x) = x^3 - x$, and determine whether there is a global maximum or minimum on the open interval $(-2, 2)$.
- You want to sell a certain number n of items in order to maximize your profit. Market research tells you that if you set the price at §1.50, you will be able to sell 5000 items, and for every 10 cents you lower the price below §1.50 you will be able to sell another 1000 items. Suppose that your fixed costs ("start-up costs") total §2000, and the per item cost of production ("marginal cost") is §0.50. Find the price to set per item and the number of items sold in order to maximize profit, and also determine the maximum profit you can get.

4. Singular Points

Definition 4.1. A point on the curve at which the curve exhibits an extra-ordinary behavior is called a Singular Point. That is where $f^{'}(x)$ is not defined but $f(x)$ is defined. Points of inflection and multiple points are the types of singular points.



Example 4.2. Find the singular point of $f(x) = (x-1)^{\frac{2}{3}} - 3(x-1)$ on $[0, \infty]$.
To locate singular points, we look for values x where $f^{'}(x)$ is not defined, but $f(x)$ is defined. Now $f^{'}(x) = \frac{2}{3(x-1)^{\frac{1}{3}}} - 3$. Notice that the denominator is zero when $x = 1$, so that $f^{'}(x)$ is not defined when $x = 1$, even though $f(x)$ is defined when $x = 1$. Thus, we have a singular point at $x = 1$.

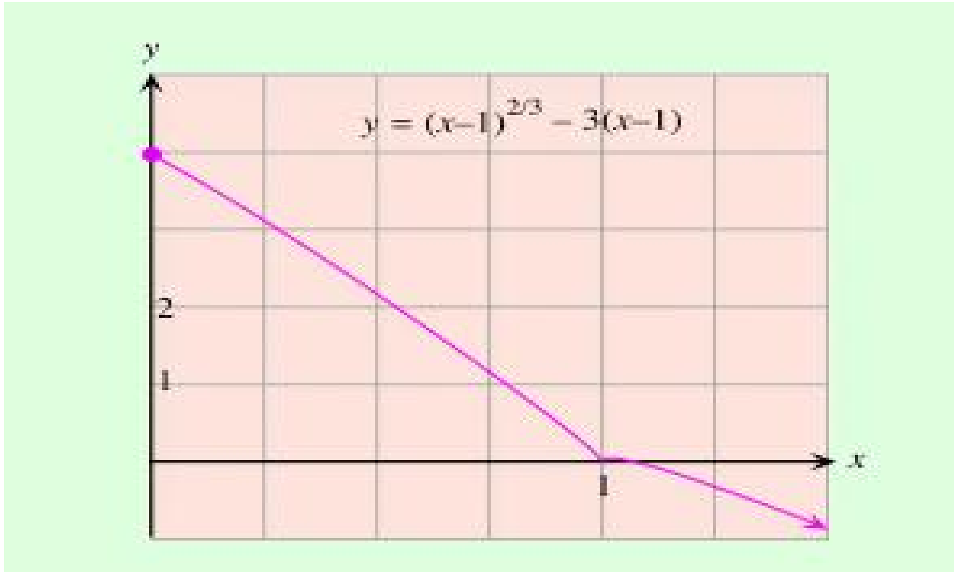
Definition 4.3. A Point on the curve through which more than one branch of the curve pass is called Multiple Point.

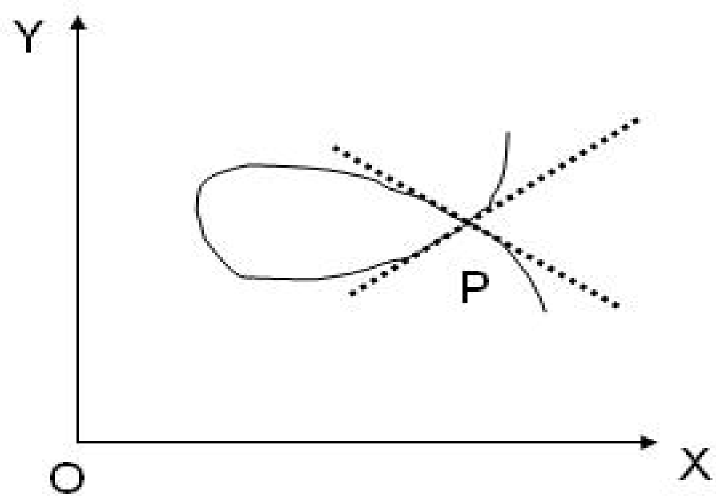
Definition 4.4. A Point on the curve through which two branches of the curve pass is called Double Point.

Definition 4.5. A Point on the curve through three branches of the curve pass is called Triple Point.

Definition 4.6. A Point on the curve through which r branch of the curve pass is called Multiple Point of rth order.

Definition 4.7. A Double Point P on a curve is called a Node if two real branches of a curve pass through P and two tangents at which are real and different. Thus the point P shown is a Node.



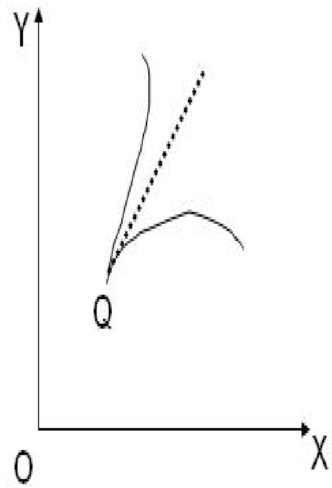


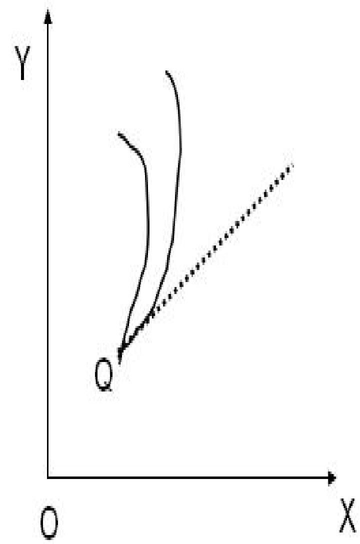
Definition 4.8. A Double Point Q on a curve is called a Cusp if two real branches of a curve pass through Q and two tangents at which are real and coincident. Thus the point Q shown in the adjoining two figures is a Cusp.

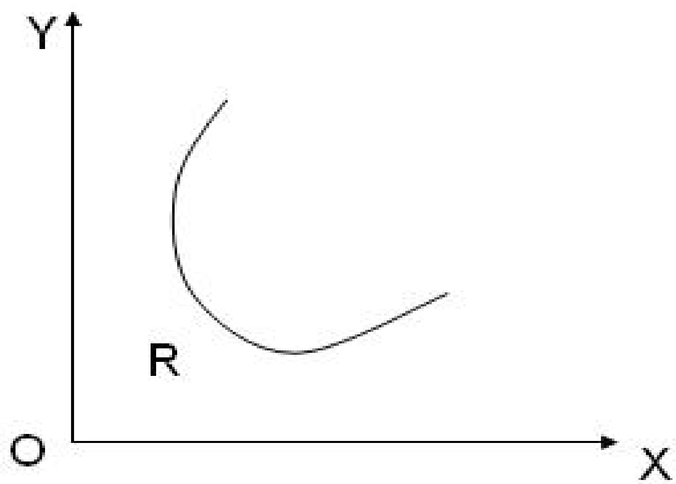
Definition 4.9. A Double Point R on a curve is called a Conjugate Point or Isolated point if there exists no real points of the curve in the neighborhood of R.

Definition 4.10. The tangents at the origin are obtained by equating to zero the lowest degree terms present in the equation of the given curve.

Remark 4.11. Working rule for investigating the nature of the double point at the origin.







- Find the tangents at the origin by equating to zero the lowest degree terms present in the equation of the curve. If origin is a double point, then we shall get tangents real and imaginary.

- If the two tangents at the origin are imaginary, then the origin is a conjugate point.

- If the two tangents at the origin are real and different, then the tangent is a node or a conjugate point.

- If the two tangents at the origin are real and coincident, then the origin is a cusp or a conjugate point.

Remark 4.12. To study the nature of the curve near origin

- If the tangents at the origin are $y^2 = 0$, solve the equation of the curve for y, neglecting all terms of y having powers above second. If for small non zero values of x, the values of y are real, then the branches of the curve through the origin are also real, otherwise they are imaginary.

- If the tangents at the origin are $x^2 = 0$, solve the given equation of the curve for x instead of y and proceed as mentioned in the above point.

- In other cases, solve for y or x, whichever is convenient.

Remark 4.13. Conditions for the existence of Multiple Points

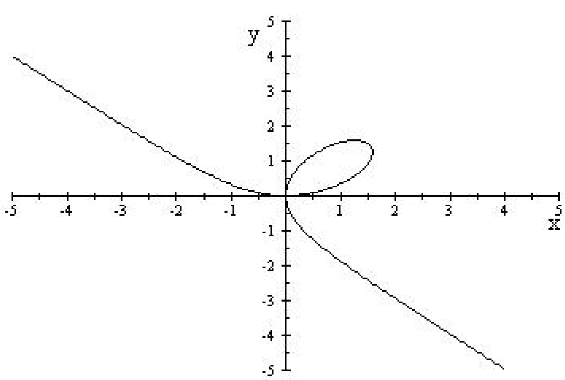
- For Multiple Points of the curve $f(x, y) = 0, \frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

4.1. **Examples.**

Example 4.14. Determine the nature of the singular point $(0, 0)$ of $f(x, y) = x^3 + y^3 - 3xy$.

According to Definition 4.10, the tangents at the origin are $x = 0$ and $y = 0$. Hence the origin is either a node or isolated point by Definitions 4.7 and 4.9.

When x is so small, the equation of the curve is $y^2 = 3x$ which represent two real branches through the origin. Hence the origin is a node by Definition 4.7. See the graphical picture



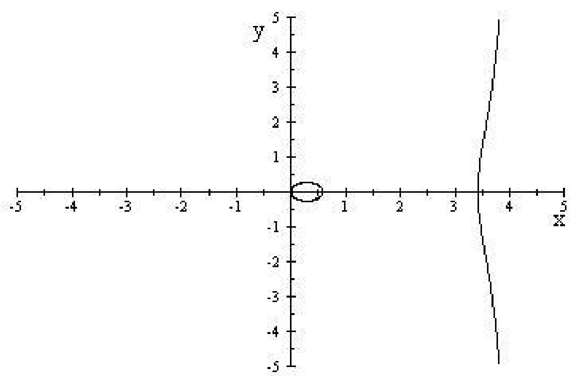
Example 4.15. Determine the nature of the singular point $(0, 0)$ of $f(x, y) = (x^2 + y^2)(2a - x) - b^2 x$.

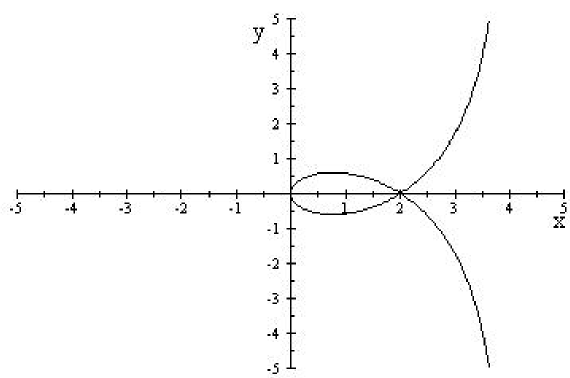
According to Definition 4.10, the tangents at the origin are $x = 0$. Hence the origin is either a cusp or isolated point by Definitions 4.7 and 4.9.

To see whether the origin is a cusp or an isolated point. Solve the equation of a curve for x. By neglecting y^2, the equation of the curve can also be written as

$$-x^3 - b^2x + 2ax^2 = 0$$
$$x^2 - 2ax + b^2 =$$
$$x = \frac{2a \pm \sqrt{4a^2 - 4(1)(b^2)}}{2}$$
$$x = a \pm \sqrt{a^2 - b^2}.$$

Hence origin is a cusp if $a^2 - b^2 > 0$.





4.2. **Exercise.**

- Discuss the nature of singular point $(0, 0)$ of the following curves

- $x^4 + y^4 + 2x^2y^2 - 4a^2xy = f(x, y)$.

- $y^2(a - x^2) = x^2(b - x)^2$.

- $x^5 - ax^2y + axy^2 + a^2y^2 = f(x, y)$

4.3. **Working rules to calculate singular points and its nature using partial derivatives.**

- If $f(x,y)$ be a curve then its singular points are the simultaneous solutions of $f(x,y) = 0$, $f_x(x,y) = 0$ and $f_y(x,y) = 0$.

- The values of tangents at singular points are the roots of $f_{yy}(y^{'}(x))^2+2f_{xy}y^{'}(x)+f_{xx} = 0$.

- If f_{xx}, f_{xy} and f_{yy} are not all zero then the point (x,y) will be a double point.

- If $(f_{xy})^2 - f_{xx}f_{yy} > 0$ the point (x,y) would be a node.

- If $(f_{xy})^2 - f_{xx}f_{yy} = 0$ the point (x,y) would be a cusp.

- If $(f_{xy})^2 - f_{xx}f_{yy} < 0$ the point (x,y) would be an isolated point.